Partial Differential Equations (PDE) for Engineers:
Solution by Separation of Variables
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Lecture – 18
Example of Generalized 3 Dimensional Problem

Welcome to the session. Now in the last class, we have we were solving the three dimensional problem in spherical polar coordinate system and we have completely, we are using the separation of variables and we have formulated the eigenvalue problem in the phi direction because there is there is no phi symmetric in this particular problem. We have seen the Eigen function is constituted by the sin function by a combination of linear combination of sin function and cosine function and the corresponding eigenvalues are natural are the natural number including zero.

So, then we formulated the problem into the divided the problem into radial direction and the theta direction. And we have formulated the eigenvalue problem in the theta direction and we are solving in the half y of the eigenvalue problem in the theta direction. Now in this class we will be solving the problem completely. So, we have seen that in the last class that the governing equation in the theta direction is nothing, but d d theta of sin theta d theta d theta plus m square sine square theta theta plus lambda square theta is equal to 0 is the governing equation.

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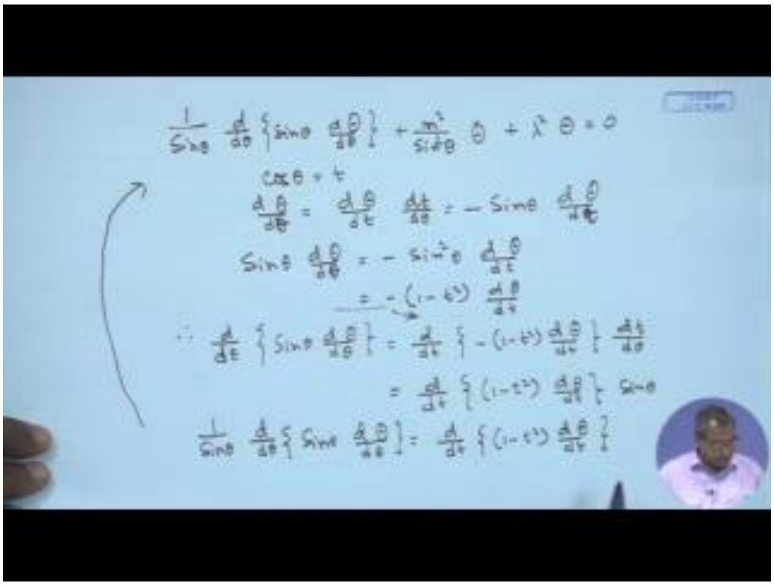

Then we substitute cosine theta is equal to t. Let us see what d theta d t in terms of theta so, in terms of t. So, d theta d dt what is d theta d theta, d theta dt dt d theta. So, dt d theta will be nothing, but minus cosine theta minus sine theta. So, it will be minus sine theta d theta d theta d theta d t. So, I am expressing in terms of the transformed coordinate system t. So, sine theta d theta dt will be nothing, but minus sine square theta d theta dt is minus 1 minus t square d theta dt. So, therefore, d d theta of sine theta d theta d theta so, it will be d theta d theta.

This will be nothing, but d dt of minus 1 minus d square, I am just substituting this here; 1 minus d square d capital theta dt and d d theta. So, I excused by a d d t. So, it will be a dt d theta here. And dt d theta will be nothing, but minus sine theta as per it is definition. So, this becomes d dt of 1 minus t square, the minus will be consumed. d theta dt and this will be sine theta. So, we will be having 1 over sine theta d d theta sine theta d theta d theta is equal to d d t of 1 minus t square d theta d t. Then I substitute these equations into the governing equation here and let us see what we get. If you really do that then will be getting 1 over d dt of 1 minus t square d theta d t plus m square 1 minus t square theta plus lambda square theta is equal to 0.

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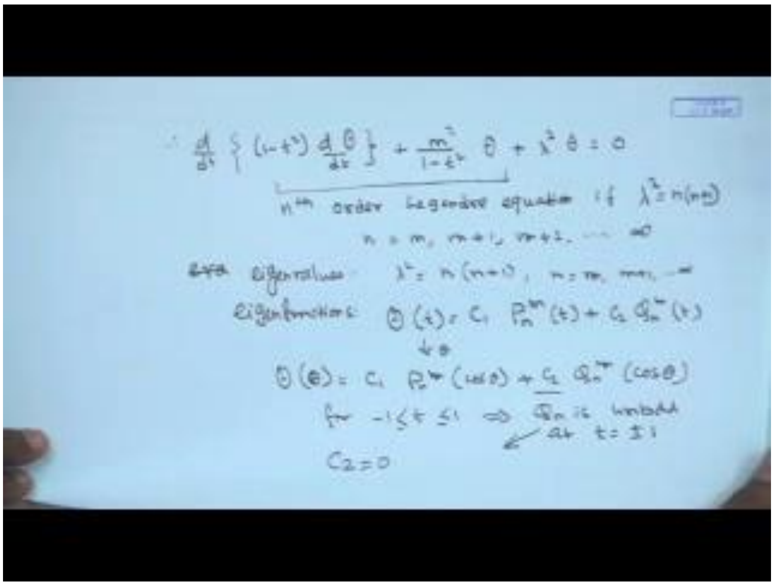

Now, this is nothing, but the Legendre equation of n’th order. So, this will be nothing, but the n’th order Legendre equation. If lambda square is equal to n into n plus 1, if we put lambda square is equal to n into n plus 1, then this will be nothing, but the n’th order Legendre polynomial and n is nothing, but m m plus 1 m plus 2 up to infinite terms where m is basically the eigenvalues that we have already talked about in the whenever we are solving about the phi direction. So, n was running from 0 to infinity. So, eigenvalues where lambda square is equal to n into n plus 1 where n rm m plus 1 up to infinity and what are the Eigen functions? Eigen functions will be the combination of Legendre function, Legendre polynomial and Legendre functions.

So, theta of t will be nothing, but C1 Pnm t plus C2 Qnm t. And in terms of theta so, t is equal to we put theta, in terms of small theta is nothing, but c1 Pnm cosine theta plus C2 Qnm cosine theta. And we already know that for t lying between minus 1 to plus 1 Qn is unbounded, unbounded at t is equal to plus minus 1. So, therefore, the whole solution is bounded. Therefore, the associated constant has to be equal to 0 in order to satisfy this boundary condition C2 has to be equal to 0. So, therefore, the Eigen function will be nothing, but the Legendre polynomial.

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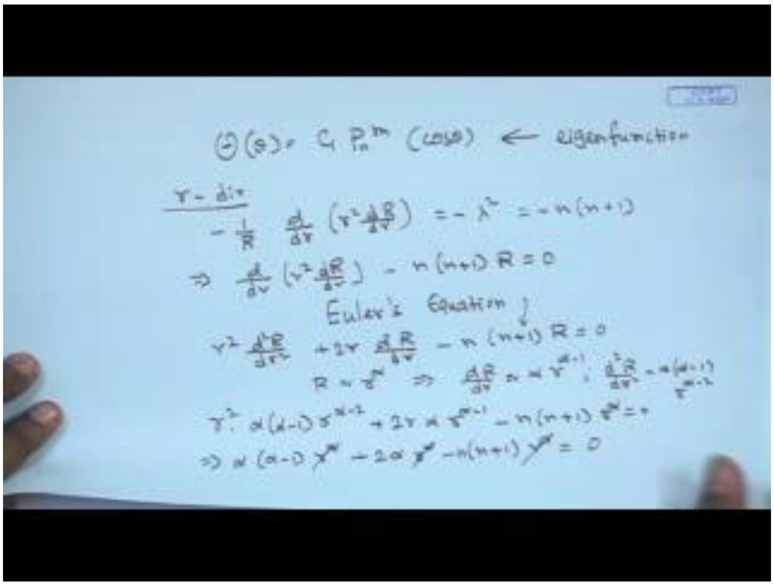

So, c theta as a function of theta will be is equal to C1 Pnm cosine theta. These are the Eigen function in the theta deduction.

Now we have already almost solved the problem. Next we will be solving the r direction. So, if you look in to the r direction, the r varying part will be nothing, but minus 1 over R d dr of r square dR dr is nothing, but minus lambda square is n into n plus 1. So, we have d dr of r square dR dr minus n into n plus 1 R is equal to 0. So, this becomes the governing equation in the r deduction. And if you really look into this particular governing equation then we can identify this is nothing, but Euler's equation. So, this will be nothing, but Euler's equation.

So, now if you just open the differentiation, you will be getting r square d square R dr square plus 2r dR dr minus n into n plus 1 R is equal to 0. So, this is the Euler's equation and the solution will be in the form of r to the power alpha. So, if you just put r square dR dr will be nothing, but alpha r to the power alpha minus one and d square R d r square will be nothing, but alpha into alpha minus 1 r to the power alpha minus 2. So, it becomes r square into alpha alpha minus 1 r to the power alpha minus 2 plus 2r dR dr will be alpha r to the power alpha minus 1 n into n plus 1 r to the power alpha is equal to 0. So, this becomes alpha into alpha minus 1 r to the power alpha plus 2 alpha r to the power alpha minus n into n plus 1 times r to the power alpha is equal to 0. So, r to the power alpha will be canceled from all the sides.

So, we will be getting a characteristic equation in alpha which will be nothing, but a quadratic in alpha.

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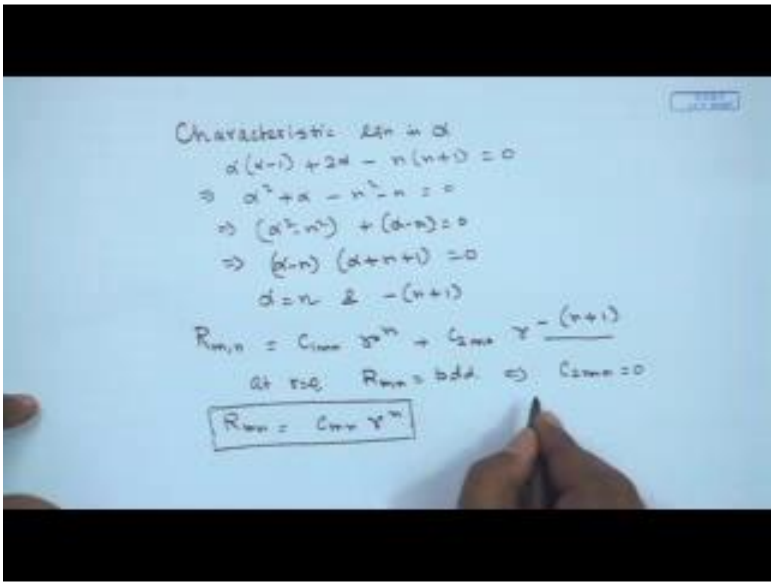

So, I will be getting a characteristic equation in alpha. So, it is a nothing, but a quadratic alpha into alpha minus 1 plus 2 alpha minus n into n plus 1 is equal to 0. So, you will be getting alpha square plus alpha minus n square minus n is equal to 0. So, alpha square minus n square plus alpha minus n is equal to 0. So, alpha minus n into alpha plus n plus 1 is equal to 0. So, we have two roots of alpha, alpha is equal to n and alpha is equal to minus n plus 1. So, I can get the solution of this Euler's equation and Rmn will be nothing, but C1 mn r to the power n plus C2 mn r to the power minus n plus 1.

And if you look in to the boundary condition that r is equal to 0, my solution is bounded. So, therefore, if I put that condition here in this governing in the solution, then the associated this term become unbounded for. r at r is equal to zero Rmn is bounded or it will be having a finite value. Therefore, this term becomes infinite or unbounded. So, the associated constant must be equal to 0. So, the solution becomes Rnm is equal to Cnm r to the power m. So, there is the solution of radial varying part.

Now we are in the position to construct the complete solution and let us construct the complete solution. So, we will be having two standard eigenvalue problems in this particular situation. One is in the phi direction that we already seen. The Eigen functions are a m sine cosine phi plus b m sine phi sine m phi where the m transform 0 to infinity and in the theta direction, we had the Legendre polynomial is the is the Eigen function and the eigenvalues are m n into n plus 1 where n transform m m plus 1 up to infinity.

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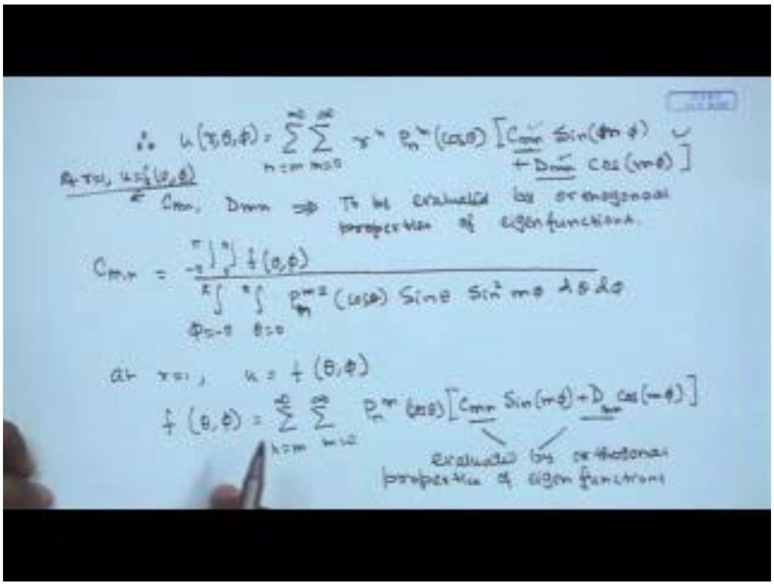

So, the complete solution it will be in the form of r theta phi is equal to, there will be two summation because we are now dealing with two independent eigenvalue problem. So, one is m is equal to 0 to infinity, another is n is equal to m to infinity r to the power n Pnm cosine theta multiplied by Cnm sine m phi plus Dnm cosine m phi and the constant associated with r that will be incorporated into the constants of phi varying part. So, we will be having the new constants of integration Cnm and Dnm. Now these two constants will be evaluated by using the orthogonal property of the Eigen functions. To be evaluated by exploiting orthogonal properties of Eigen functions.

So, I am just writing the final solution. We have done this exercise several number of times. So, this will be nothing, but phi is equal to minus pi to plus pi theta is equal to from 0 to pi Pnm square of that of cosine theta sine theta sine square m phi d theta d phi and again from minus pi to plus pi 0 to pi f theta phi, f theta phi is the condition at r is equal to 0 r equal to 1. So, we will be utilizing, we will be getting this from the boundary conditions using the unutilized condition that at r is equal to 1, your u is equal to f of theta phi some initial condition some prescribed function of known function of theta or phi or in general it will be a function of theta and phi or it will be it may be function of theta alone, may be function of phi alone, it may be function it may be a constant.

So, let us put it at as a general function of theta and phi at boundary condition at r is equal to 1. So, utilizing that will be and then we will be, so we will be putting the value. So, i omitted this step in in between. So, it will be f of theta phi should be is equal to this. Let me put it that way. So, at r is equal to so, i will be evaluating the constant Cnm Dnm by putting at r is equal to 1 u is equal to f of theta and phi in general. So, if I put that in governing equation, then this becomes f of theta and phi is equal to double summation n is equal to m to infinity m is equal to 0 to infinity and r is equal to 1. So, that will be a gone. So, Pnm cosine theta multiplied by Cnm sine m phi plus Dnm cosine m phi.

So, now the constants Cnm and Dnm will be evaluated from the by exploiting the orthogonal property of the sine functions and cosine functions. So, these will be now evaluated by orthogonal property, properties of Eigen functions. So, I will be directly now writing the governing you know final solution. So, you have done in (Refer Time: 17:24) number of times.

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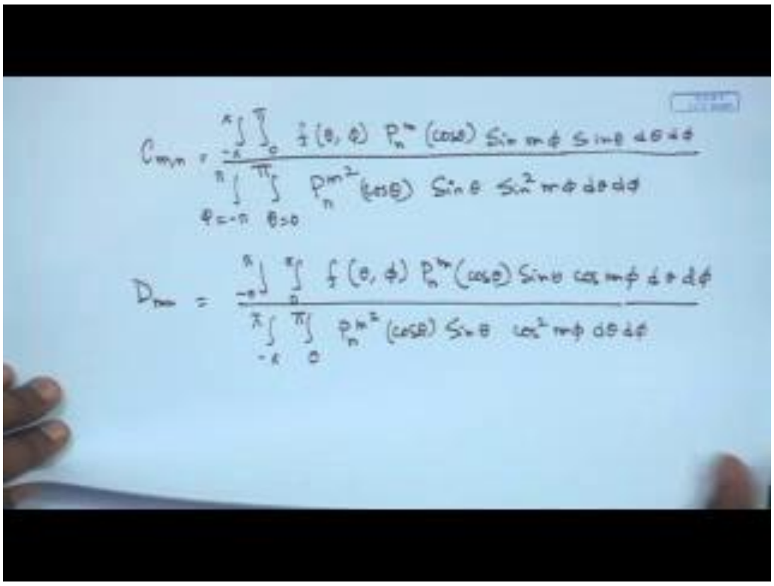

So, Cnm now becomes phi is equal to minus pi to plus pi theta is equal to 0 to pi Pnm square cosine theta sine theta sine square m phi d theta d phi and in the numerator, we will be having minus pi to plus pi 0 to pi f of theta phi Pnm cos theta sine m phi sine theta d theta d phi where sine theta is the weight function appearing in both denominator and numerator and Dnm becomes minus pi to plus pi 0 to pi Pnm square of that cos theta sine theta cos square m phi d theta d phi and here it will be minus pi to plus pi 0 to pi f of theta phi Pnm cos theta sine theta cosine m phi d theta d phi.

So, that gives the complete solution of spherical polar coordinate problem in three dimensional. And since we have there are two independent eigenvalue problems existing in this physical situation. We will be having a double summation and there will be two index m and n corresponding to each eigenvalue problem. So, we have really come down to the end of our course and now I will be so, let us summarize whatever we have done. We have looked into the operator and they are certain definitions and then we have looked into the properties of Sturm-Liouville problem and how to obtain the adjont operator.

And then we have looked into the various property of Sturm-Liouville problem and establish that what is the definition of standard eigenvalue problem having infinite number of Eigen values and Eigen functions will be orthogonal to each other. We have proofed these specific properties of the Sturm-Liouville problem or the standard eigenvalue problem. Then we have earlier we have defined the classifications of the partial differential equations. How to evaluate the parabolic you know hyperbolic, elliptical different types. And then the different kinds of boundary conditions we have discussed with and also we have looked into the system that how to you know break down the problem into sub problems, if we have the number of non-homogeneities in the boundary conditions and in the initial condition.

Then we have looked into the all kinds of problem in rectangular polar rectangular coordinate system. Parabolic, hyperbolic and elliptical and also two dimensional problems, three dimensional problems and four dimensional problem in rectangular polar rectangular coordinate system. Then we moved diode with the cylindrical polar coordinate system and solved a you know two dimensional problem as well as three dimensional problem, then we have moved over to the spherical polar coordinate system and looked into the solution of two dimensional and three dimensional problems which are quite common in various engineering applications.

Now, we what will be doing? We will be taking two big example of an actual real problem which will be effectively a three dimensional problem and we will be going ahead with a complete solution of that. So, let us look into a three dimensional problem heat conduction problem which will be a real problem, actual life problem three dimensional, heat conduction problem in a rectangular coordinate system.

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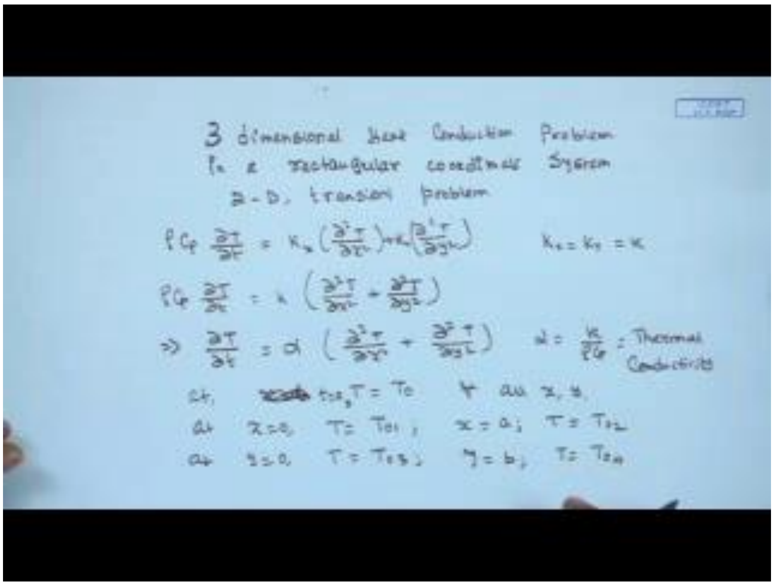

So, there will be, it is a two dimensional transient problem. So, one two dimensional in space, one dimensional in time so, if you look into the governing equation this becomes rho Cp del T del t will be nothing, but k times del square T. In fact, it will be k x del square T del x square plus k y del square T del y square and we assume that thermal conductivity in is isotropic, they it is same as all directions. So, it will be rho Cp del T del t will be nothing, but k del square T del x square plus del square T del y square. So, if I divide by rho Cp this becomes thermal diffusivity. So, del T del t will be nothing, but alpha del square T del x square plus del square T del y square. So, alpha is known as the thermal conductivity.

Now let us put the boundary condition at x is equal to 0. Let us have the Dirichlet boundary conditions for all the directions. So, T is initially T naught, at t is equal sorry that t is equal to 0 t is equal to t naught for all x and y. Now at x is equal to 0, let say T is equal to T 01. At x is equal to a; let us say T is equal to T 02 at y is equal to 0. We have T is equal to T 03 and at y is equal to b, we have t is equal to T 04. So, let us have a general problem which will be having all boundary conditions. They are we are maintaining at different temperatures located at x is equal to 0 and x is equal to a, y is equal to 0 and y equal to b. All the four boundaries we are maintaining four different temperatures which is basically nothing as special form of which will be nothing, but the Dirichlet boundary conditions. And at time t is equal to 0, we had some initial temperature and on all the four boundaries by maintaining four different temperatures.

Now, we will be looking into the solution of this particular problem. Now as we have seen earlier, let us count how many non-homogeneities we are having in this particular problem. So, number of non-homogeneities will be, there will be four non homogeneities appearing in the four boundary conditions and one non-homogeneity appearing in the initial conditions. There are five sources of non-homogeneities we have in this particular problem. So, we will be now what we decided earlier that we can reduce the number of non-homogeneities by defining the non-dimensional you know independent variable. Independent variable is temperature in these cases. So, we can define a non-dimensional temperature such that like t minus theta is equal to t minus 1 01 by T 01.

So, these will be basically a non-dimension, in this non-dimensional temperature. We can reduce the at least one non homogeneity in our system. So, we can then talk about instead of five non homogeneities, we will be having four non homogeneities in this particular problem. So, we will be reducing one non-homogeneity and that will be decreasing the rigger of the solution. So, if we do a proper non dimensionalization.

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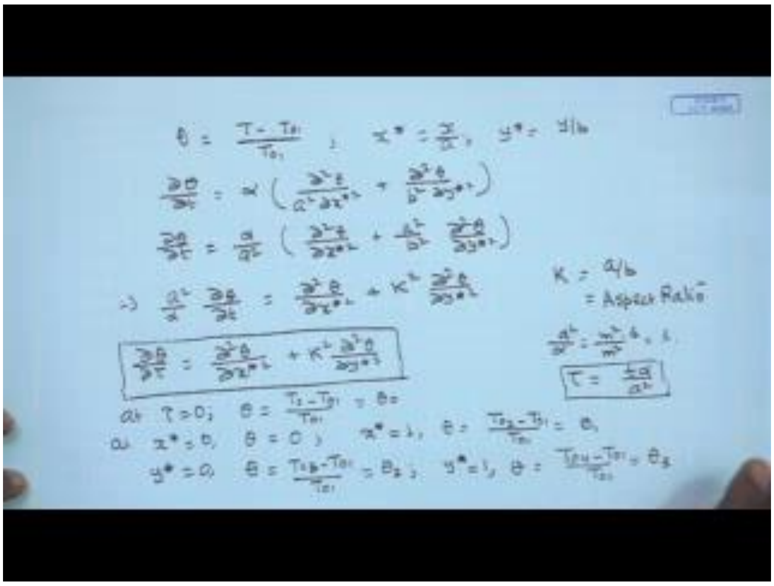

For example if we define theta is equal to T minus T 01 divided by let say T 01, and we define let say x star is equal to x by a and y star is equal to y by b then x star will be varying basically 0 to 1 and y star will be varying from 0 to 1.

So, if we do that. So, it will be getting del theta del t is equal to alpha del square theta a square del x star square plus del square theta b square del y star square. So, we will be having del theta del t is equal to we take alpha a square outside, alpha a square del square theta del x star square plus a square by b square del square theta del y star square. So, this becomes a square by alpha del theta del del t is equal to del square theta del x star square plus k square del square theta del y star square. Now what is k square? k is nothing, but a by b, this is sort of aspect ratio. This is a non-dimensional aspect ratio. So, left hand side all the two terms, the right hand side the term is non-dimensional the second term, the first term is non-dimensional. On the left hand side theta is non-dimensional.

So, therefore, t alpha by a square will be non-dimensional; that means, x square by alpha will be having a unit of time. So, if you really look into that a square by alpha a square is meter square and alpha will be having a unit meter square per unit time. So, it will be having a unit of time. So, there will be identical. So, tau will be non-define as T alpha over a square is the non-dimensional time. So, we make the system non dimensional by del theta by del tau is equal to del square theta del x star square plus k square del square theta del y star square.

So, this is the entirely the non-dimensional version of the governing equation. Now our work is next is to make the boundary conditions to be non-dimensional. So, at T is equal to 0 means at tau is equal to 0 my T is equal to T naught. So, my theta is equal to T naught. So, this will be T naught minus T 01 divided by T 01. So, I call this as theta naught. At x is equal to 0 means at x star is equal to 0 T is equal to T 01. So, theta will be is equal to 0 and at x star is equal to a; that x is equal to a; that means, x star is equal to 1, theta is equal to T 02. So, T 02 minus T 01 divided by T 01. So, this will be theta 1 at y star is y is equal to 0 means y star is equal to 0 theta is equal to T 03. So, T 03 minus T 01 divided by T 01. So, I call this as theta two and at y star y is equal to b means y star is equal to 1, my theta is equal to T 04. So, T 04 minus T 01 divided by T 01. So, this will be equal to theta 3.

Now, if you look into this system, now let us identify how many non-homogeneities are present. Non homogeneities are 1, 2, 3 and 4. So, instead of five non-homogeneities in the original system, now after non dimensional non dimensionalization we have reduced one non-homogeneity. Now we will be having four non homogeneities in our system. So, next as we have discussed earlier, we have to divide this problem into four sub problems considering one non-homogeneity at a time.

So, I stop you in this class. We will take up this problem in the next class and solve this problem completely.

Thank you very much.