Partial Differential Equations (PDE) for Engineers:
Solution by Separation of Variables
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Lecture - 17
Spherical Polar Coordinate System (Contd.)

Very good morning to everyone, so, we are discussing about the solution of spherical polar coordinate system, partial differential equations in spherical polar coordinate system by using separation of variable. And in the last class, we have seen that today how to solve a two dimensional problem. Now we will be looking into a three dimensional problem today and many of the engineering applications we deal with the spherical coordinate system.

For example if you are talking about the you know heat transfer in a ball or in a got metallic ingot then one has to take request to the spherical polar coordinate system. The heat transfer problem from a sphere is very very important in various engineering applications starting from covering mechanical, aerospace, chemical engineering and other engineering disciplines. Similarly in refinery engineering there do we have to you know store the oil in hot and spheres and gases and oils etcetera, in the hot and spheres petrochemical products. So, there the spherical polar coordinate system is very very important for you know quantifying the temperature distribution and other things.

So, therefore, spherical polar coordinate system, the solution of partial differential equations in spherical polar coordinates system is an integral part of any it is application for engineering problems. So, in the last class we have looked into the two dimensional problems. In today's class, we will be looking into a more generalized problem which is nothing, but a three dimensional one.

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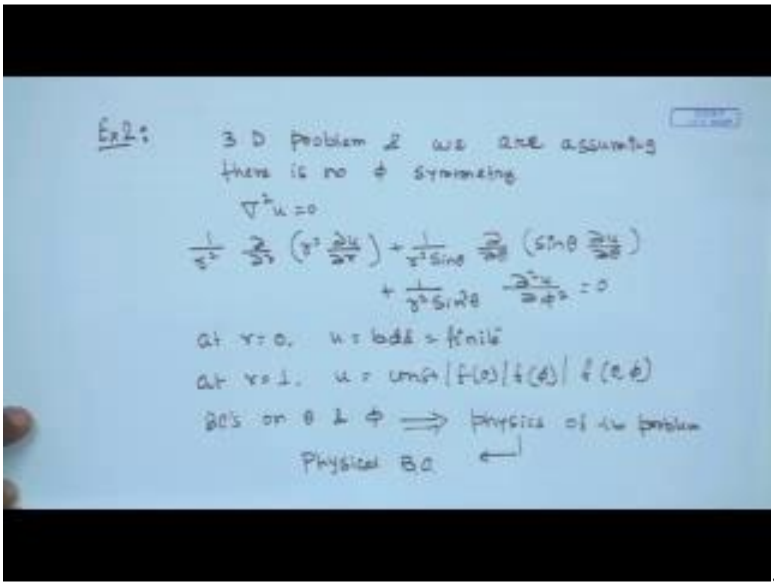

So, it has look into the example of spherical polar coordinate system. It is a three dimensional problem and we are assuming, there is no phi symmetry.

So, thus the problem formulation will be grad square u will be equal 0; that means, if you write down in this complete form, it will be 1 over r square del del r r square del u del r plus 1 over r square sin theta del del theta sin theta del u del theta plus 1 over r square sin square theta del square u del phi square is equal to 0. So, this is a complete problem in a three dimensional symmetry, three dimensional domain and the boundary conditions are at r is equal to 0, you have u is equal to finite value. So, this is bounded, it will be assuming a finite value and at r is equal to 1.

Here u may be a constant, may be a function of theta, may be a function of phi, may be a function of theta and phi both. And the boundary conditions on theta and phi will be evolving out of physics of the problem as we will see later on physics of the problem. So, they will be the physical boundary condition. Thus let us as we will be going hide with the solution, we will be looking into the, you know boundary conditions in more a prepaid version.

Now we will be looking for a separation of variable type of solution.

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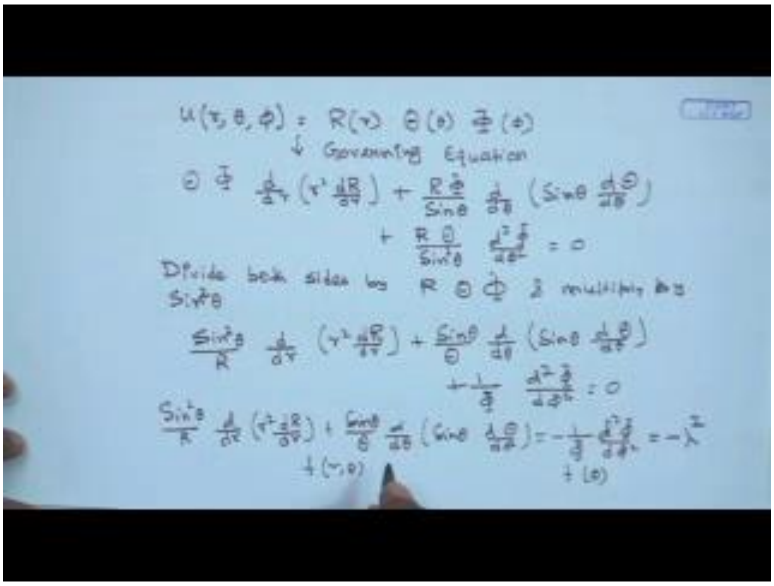

So, u r theta phi should be is equal to R of r capital theta of theta phi of phi. So, it will be considered to be a product of three functions. One will be function of r alone, another will be a function of theta alone, and the other one will be function of phi alone. Now if I replace these in the governing equation, governing equation then it becomes theta phi d d r of r square dR dr plus R phi divided by sin theta d d theta of sin theta d capital theta d theta plus R capital theta divided by sin square theta d square phi d phi square is equal to 0. Now I divide both sides by R theta and phi. So, let see what we get.

What we get is that and then we I multiply it by sin square theta as well. So, sin square theta divided by R d dr of r square dR dr plus sin theta divided by theta d d theta of sin theta d theta d theta plus 1 over phi d square phi by d square phi d phi square. So, that will constitute this equation then we will be formulating a standard eigenvalue problem in the phi direction; taking the physical boundary conditions on phi which are nothing, but the periodic boundary conditions as we have discussed earlier.

So, let us write it in a special form that sin square theta by R d dr. So, that we can formulate a standard eigenvalue problem dR dr plus sin theta by theta d d theta sin theta d theta d theta is equal to minus 1 over phi d square phi d phi square and left end side is a function of r and theta only, the right end side the function of phi only. So, they are equal and they will be equal to some constant and this constant has to be a negative constant in order to have a non-trivial solution.

So, in order to have a non trivial solution, we will be having a negative constant out there. Now let us formulate the standard eigenvalue problem in the phi direction and let us formulate the boundary conditions on phi which are nothing, but the physical boundary conditions

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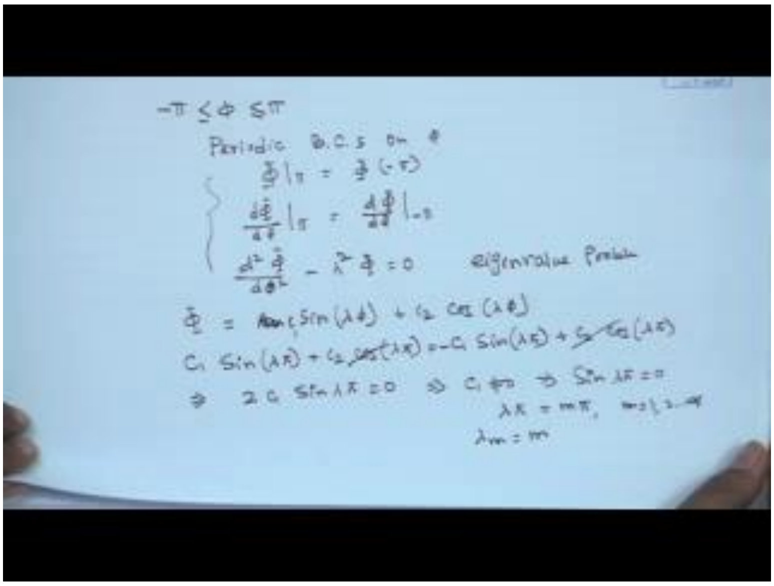

So, if you remember that phi is varying from minus pi to plus pi and we will be having the periodic boundary conditions. Periodic boundary conditions are only (Refer Time: 09:12) on phi are phi at pi is equal to phi at minus pi, d phi d phi at pi will be nothing, but d phi d phi at minus pi. So, therefore, we will be having d square phi d phi square minus lambda square phi should be equal to 0. This is the eigenvalue problem with this boundary condition and we have seen earlier that if we have this boundary condition, let us solve this equation.

So, phi is equal to, this will be solution will be composed of sin functions and cosine functions. So, Am sin lambda phi, let say C1 sin lambda plus C2 cosine lambda phi. Now phi at pi becomes C1 sin lambda pi plus C2 cosine lambda pi is equal to phi at minus pi c1 minus sin pi lambda pi plus C2 cosine lambda pi. So, minus lambda pi cosine minus lambda pi will same as cosine plus lambda pi. So, these two will be canceled out. So, what you will be getting is 2 C1 sin lambda pi is equal to 0. So, C1 for C1 not equal to 0. We have sin lambda pi is equal to 0. So, lambda pi is equal to m pi, this is a general solution when m transforms 1 to infinity. So, lambda m is equal to nothing, but m the natural numbers.

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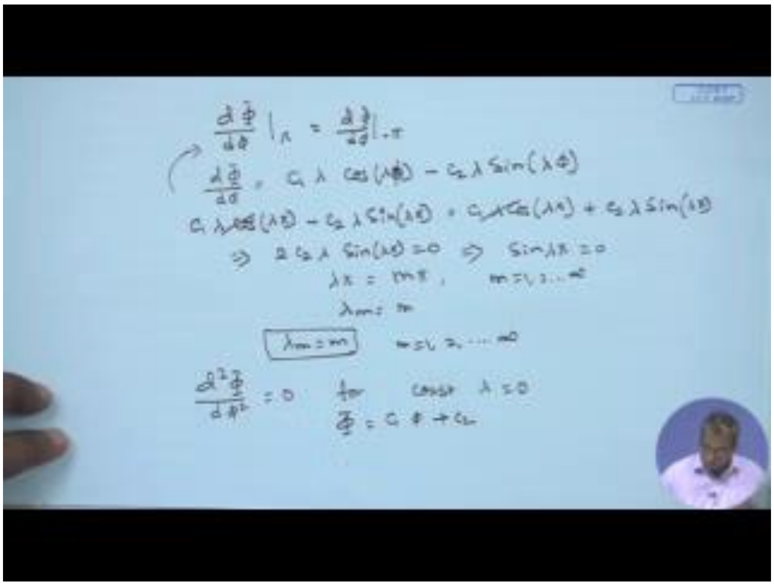

Now in the next, what we will be utilizing the other boundary condition that d phi d phi at pi is equal to d phi d phi at minus pi. So, if you use this boundary condition so, what is d phi? d phi d phi is nothing, but c1 lambda cosine lambda pi lambda phi minus c2 lambda sin lambda phi. So, now if I utilize this boundary condition, this becomes c1 lambda cosine lambda pi minus C2 lambda sin lambda pi is equal to C1 lambda cosine minus lambda pi is same as last lambda pi and this will be minus C2 lambda sin minus lambda pi. So, it will be minus minus plus lambda pi. So, these two will be canceled out. So, you will be having 2 C2 lambda sin lambda pi will be equal to 0, therefore, for a for a non-trivial solution sin lambda pi equal to 0. So, lambda pi is equal to m pi, there is general solution infinity. And therefore, you will be having lambda m is equal to m which are the natural number.

So, from both the boundary conditions, we are going to get lambda pi will be equal to m where the index m transform 1, 2 infinity. Now if we use a value of lambda to be equal to 0, then we can see that what will happen to the constant, you know our eigenvalues. So, d square phi d phi square will be equal to 0 for constant lambda equal to 0, so for the case one lambda equal to 0. So, you will be having phi is equal to C1 phi plus C2.

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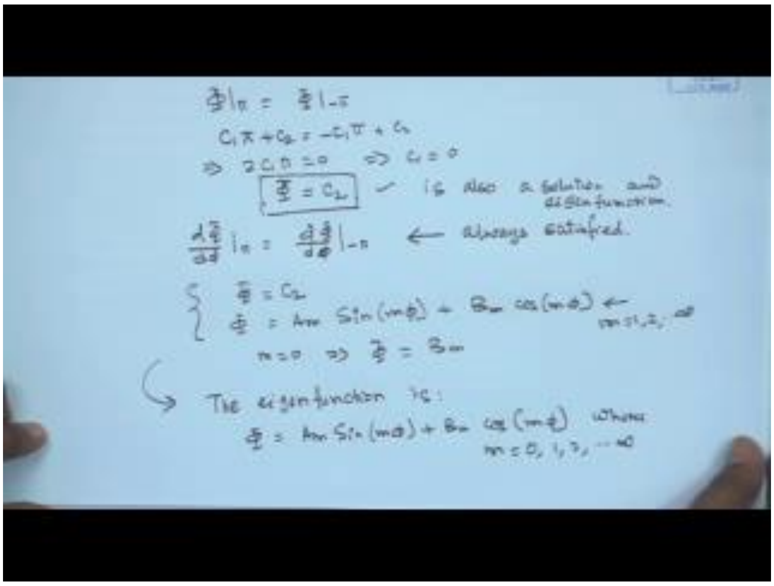

Now if you use the boundary condition that phi at pi is equal to phi at minus pi, then this becomes C1 pi plus C2 is equal to minus C1 pi plus C2. So, C2 will be comes in out. So, you will be have been 2 C1 pi equal to 0 so, C1 will be equal to 0. So, your phi will be nothing by, but C2. Now for, if you look into the other boundary condition, other boundary condition is d phi d phi at pi is equal to d phi d phi at minus pi and if phi is equal to c2 is the solution, and then this will be always satisfied. It is, if phi is a constant then these will be always satisfied. So, therefore, phi is equal to constant is also an Eigen function is also a solution and Eigen function.

So, therefore, will be having, now if you combine so, now what are the Eigen functions we are getting? One is the phi is equal to constant that is a constant and then another will be phi is equal to, let say Am sin m phi plus Bm cosine m phi where the in this equation m runs from 1, 2 up to infinity. Now we have a true solution or two eigenvalues functional form. One is phi is constant, another is phi is equal to Am sin m phi plus Bm is a cosine m phi. Now, m is equal to 1 to infinity where there m are basically the natural numbers. Now we can combine these two equations, these two solutions Eigen functions into one Eigen function, if we run our running index m starts from 0.

let us look into (Refer Time: 17:04) if we start counting our aim from 0 instead of 1. Now let us see what will get at phi at m equal to 0? At m is equal to 0 if I evaluated this equation, I will be getting sin 0 is 0 and Bm will be a constant. So, therefore, if I take m count, if I count m from 0 to infinity instead of 1 to infinity, I will be taking care of the Eigen function phi is equal to constant. So in general I can, the Eigen function for phi varying part is, in general phi is equal to Am sin m phi plus Bm cosine in phi where m starts from 0, 1, 2 up to infinity. So, that completes the eigenvalue problem in the phi direction.

Now, we can always show that if lambda so, we have considered the constant lambda is 0. The constant negative and if it is positive and bring it to the other side, then you will be getting a non trivial solution. So, you can just have a look at do that one.

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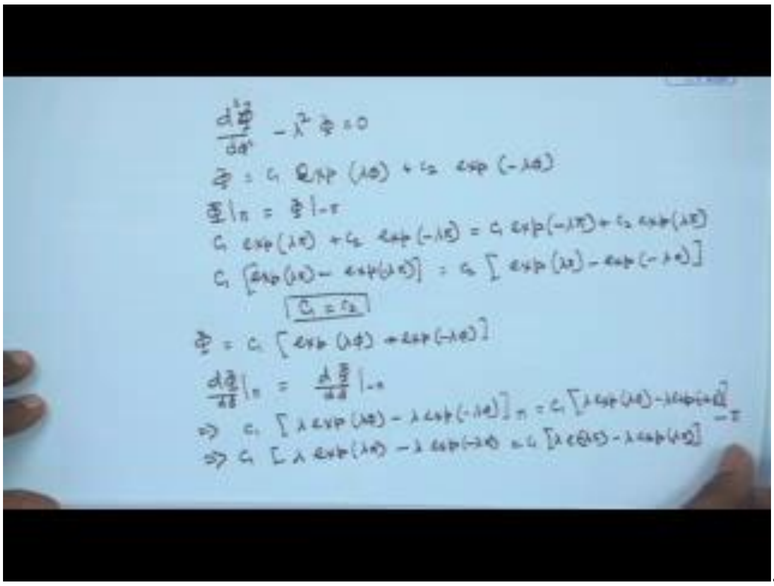
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So, d square phi d phi square minus lambda square phi is equal to 0 and you know the solution of this. Solution of these will be C1 exponential lambda phi plus C2 exponential minus lambda phi, then we write down the boundary condition phi at pi is equal to phi at minus pi. So, C1 exponential lambda phi plus C2 exponential minus lambda pi is equal to C1 exponential minus lambda pi plus C2 exponential plus lambda pi. So, this if we take it to the other side c1 exponential lambda pi minus exponential minus lambda pi. And here it will be having C2 common exponential lambda pi minus exponential minus lambda pi. So, these will say that C1 will be is equal to C2. These two are equal and opposite. So, you will be having phi is equal to C1 common exponential lambda phi plus exponential minus lambda phi. Then we will be utilizing other two-boundary condition. Other boundary condition d phi d phi at pi is equal to d phi d phi at minus pi.

So, if you utilize this, this will become C1 lambda exponential lambda phi minus lambda exponential minus lambda phi valuated at pi and here it will be 1 lambda exponential lambda phi minus lambda exponential minus lambda phi evaluated at minus pi. And if you really put this C1 lambda exponential lambda pi minus lambda exponential minus lambda pi and this will be C1 lambda exponential lambda pi minus lambda exponential. This will minus lambda, this is minus minus plus. So, it will be lambda pi. So, if you take minus common. So, it will be the identical and same and they will not be equal to zero.

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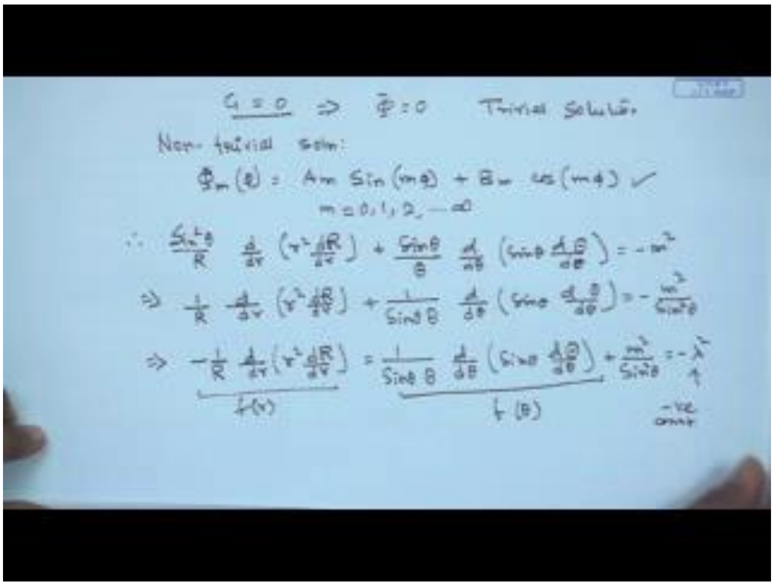

So, you will be getting C1 will be equal to 0. So, one will be getting a trivial solution. So, you will be getting ultimate landing with phi is equal to zero.

So, one will be getting a trivial solution for the non-trivial solution. We are really looking for the non trivial solution. Therefore, phi m of phi will be nothing, but Am sin m phi plus Bm cosine m phi where the index m runs from 0, 1, 2 up to infinity. So, this will take the case of constant eigenvalue as well you know constant Eigen function as well. So, you will be having the Eigen function for m is equal to 0. It is constant for m not equal to 0, for any other value it will be combination of sin function and cosine function .

Now, let us look into the r varying part and other parts. So, sin square theta divided by R d dr of r square dR dr plus sin theta divided by theta d d theta of sin theta d theta d theta is equal to minus m square where m is from 0 to infinity. Now we take it to the, we constitute the eigenvalue problem in the theta direction may be. So, it will become minus. So, one more step I divided by sin square theta. So, 1 by r d dr of r square dR dr plus 1 over sin theta theta d d theta sin theta d theta d theta is equal to minus m square sin square theta. So, I take it on the other side and m square sin square theta on these side it will be 1 over R d dr of r square dR dr is equal to 1 over sin theta theta d d theta sin theta d theta d theta plus m square sin square theta is equal to minus lambda square.

So, these will be equal to this is a function of r only, this will be a function of theta only. They are equal and there will be equal to some constant and that constant may be new constant may be lambda minus lambda square and it has to be a negative constant because in order to have a non trivial solution. Now you will be able to formulate the eigenvalue problem of theta direction in the theta.

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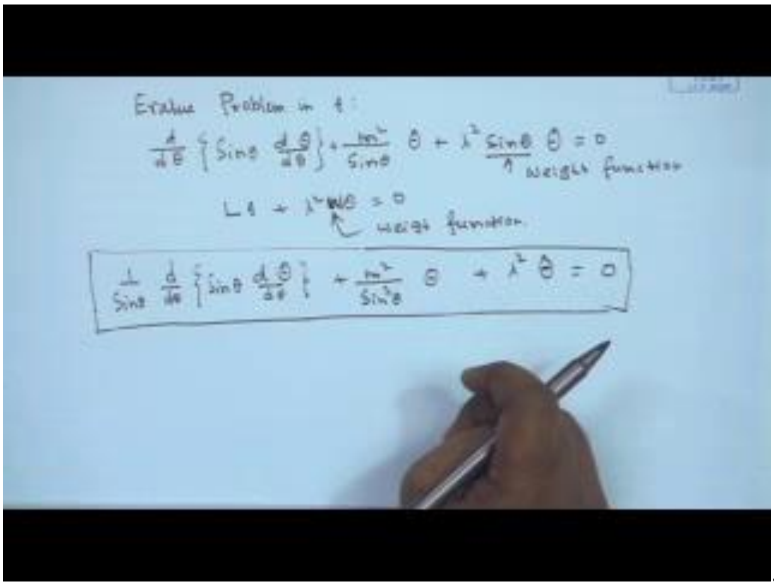

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If you did at the eigenvalue problem in theta direction becomes, in theta direction becomes d d theta sin theta d theta d theta plus m square divided by sin theta capital theta plus lambda square sin theta capital theta is equal to 0.

Now if you remember, if you look into this equation, you can find out that this sin theta is nothing, but the weight function. If you look into the standard eigenvalue problem in the form of L theta plus lambda R theta is equal to 0. So, this will be in the in the form of L theta plus lambda square r theta is equal to 0. It is basically w theta W is nothing, but the weight function. So, here the weight function in this spherical polar coordinate system is sin theta. And now once we get that, I divide both sides by sin theta and let us see what we get. 1 by sin theta d d theta sin theta d theta d theta plus m square sin square theta theta plus lambda square theta will be is equal to 0. So, these will be the form of the eigenvalue problem in the theta direction and you will be looking into the solution of this. And I will be taking this up in the next class.

I will stop in this class, we will complete this problem in the next class and then we will be looking into other problems as well.

Thank you very much.