Partial Differential Equations (PDE) for Engineers:
Solution by Separation of Variables
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Lecture – 16
Spherical Polar Coordinate System

Welcome to the session, we will continue whatever we left in the last class. In the last class we are looking into the two dimensional transient heat conduction problems in a cylindrical polar coordinate system. And we used a separation of variable type of a solution and we have found out that the theta varying part is constitutes a standard independent eigenvalue problem. And then we have formulated the radial varying part and we will be looking into the rest of the solution.

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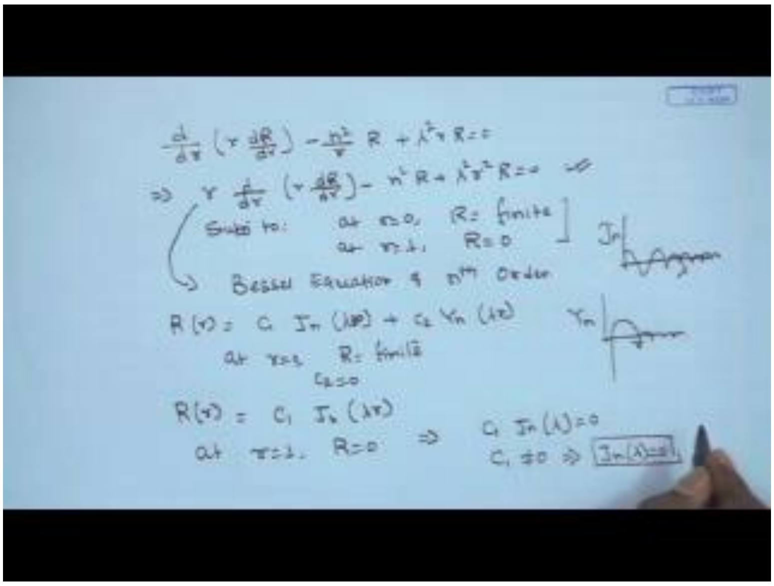

So, if you look into the radial varying part, this is the governing equation, along with, two-boundary condition. And if you now remember that these are nothing this is nothing, but a Bessel equation. Equation of nth order and the solution is composed of Jn and Yn, C1 Jn lambda n lambda r plus C2 Yn lambda r. So, these are the solution and now if we look into the, if we put in the boundary condition. If we know that define definition of Yn, if you Yn is infinite at r is equal to 0.

So, therefore, in order to have this boundary condition that at r is equal to 0, R is finite. So, this is in opposite to the definition of Yn. So, in order to do that, so in order to comply that one, the associated constant has to be equal to 0, that I will be getting a bounded solution of r. So, C1 R r is nothing, but C1 Jn lambda r. Now we put the other boundary condition at r is equal to 1, R is equal to 0. So, therefore, C1 Jn lambda is equal to 0. For a non-trivial solution C1 is not equal to 0.

Therefore, Jn lambda is equal to 0 and if you look into the variation of Bessel function as in terms of lambda. So, they will be assuming a finite value and then they will be oscillating along the x axis or lambda axis at infinite cutting the, you know they are oscillating about the lambda axis with a diminishing magnitude and they will be cutting the lambda axis at infinite number of times. And each of this intersection point is the eigenvalue to a system and the Eigen functions are the nth order Bessel function

So, therefore, we will be having the, we will be writing J1 lambda m n is equal to 0.

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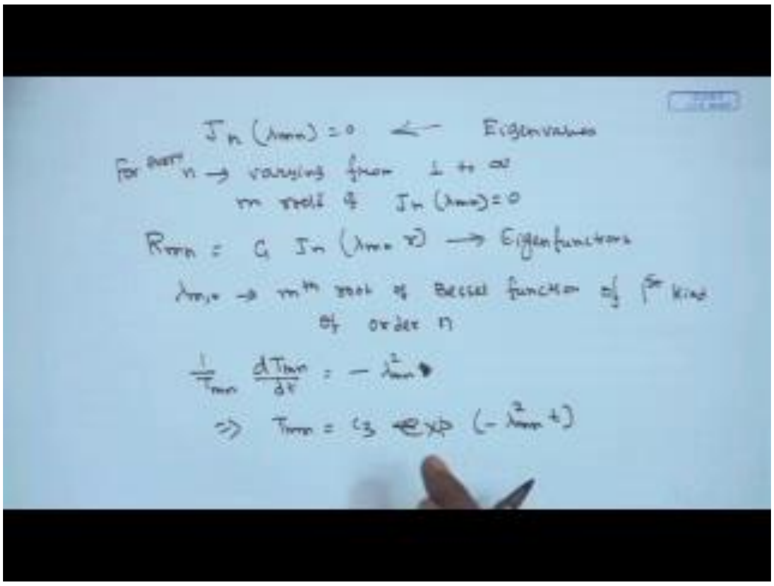

So, there will be infinite number of loop. For every n varying from for every n varying from 1 to infinity will be having m roots of Jn. So, therefore, the Eigen functions are written as Rmn as C1 Jn lambda mnr. So, these are the Eigen functions. Roots of these equations are the Eigen values and lambda mn is nothing, but m’th root of bessel function of first kind of order n. So, now we have seen that both theta varying part and r varying part, they constitute an independent Eigen value problem. So, therefore, we can constitute the complete solution by doing a double summation of individual part of the Eigen value problem. And if you look into the time varying part time, the varying part will be nothing, but Tmn d Tmn dt will be nothing but minus lambda mn square times t lambda mn square. So, Tmn will be nothing, but another constant, let us say C3 exponential minus lambda mn square times t.

Now we can superpose all the three solution and construct the complete solution. So, the complete solution will be a double summation.

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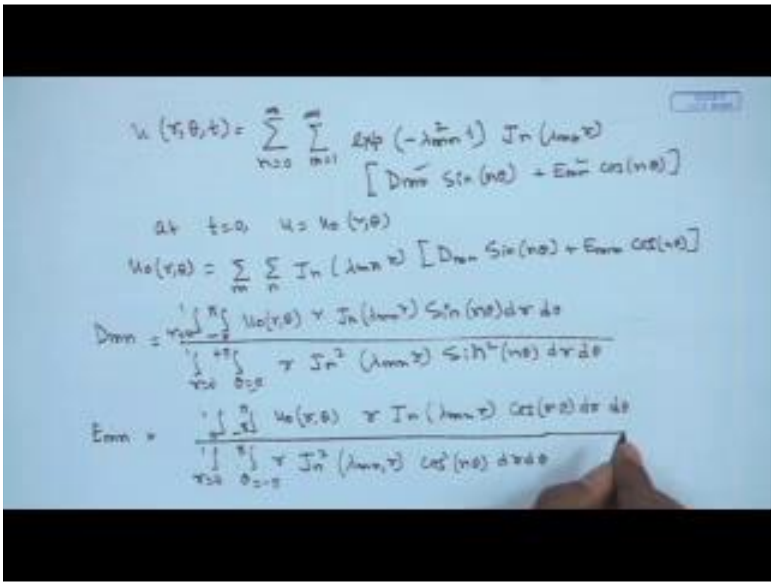

So, u r theta t will be nothing, but summation n is equal to 0 to infinity, m is equal to 1 to infinity exponential minus lambda mn square times t Jn lambda mnr multiplied by Dmn sine n theta plus Emn cosine n theta. So, at t is equal to 0, now this two constants Dmn and Emn will be evaluated from the initial condition at t is equal to 0 u is equal to u zero r theta. So, therefore, u zero r theta will be nothing, but double summation 1 over m another over n Jn lambda mn r Dmn sine n theta plus Emn cosine n theta. And by using the orthogonal property of the sine functions, one can evaluate the Dmn and Emn. So, therefore, Dmn will be nothing, but we have done several times r equal to 0 to 1 theta is equal to minus pi to plus pi u zero r theta r Jn lambda mn r sine n theta dr d theta and r is equal to 0 to 1 theta is equal to pi to minus pi minus pi to plus pi r Jn square lambda mn r sine square n theta dr d theta.

And similarly we can get an expression of Emn as r equal to 0 to 1 theta is equal to minus pi to plus pi r jn square lambda mn r cos square n theta dr d theta and 0 to 1 minus pi to plus pi u zero r theta r Jn lambda mn r cosine n theta dr d theta. So, that gives the complete solution of the cylindrical coordinate system. The partial differential equation for the cylindrical coordinate system and we can have seen that how the Eigen value problem can be formulated in the theta direction, and in the r direction they will be constitute in the independent Eigen value problem and the overall solution can be obtained by using a leap principle of linear superposition by doing a double summation. Since there are two independent Eigen value problem. We are having a double summation in the solution and the associated constants will be evaluated from the orthogonal property of the Eigen functions.

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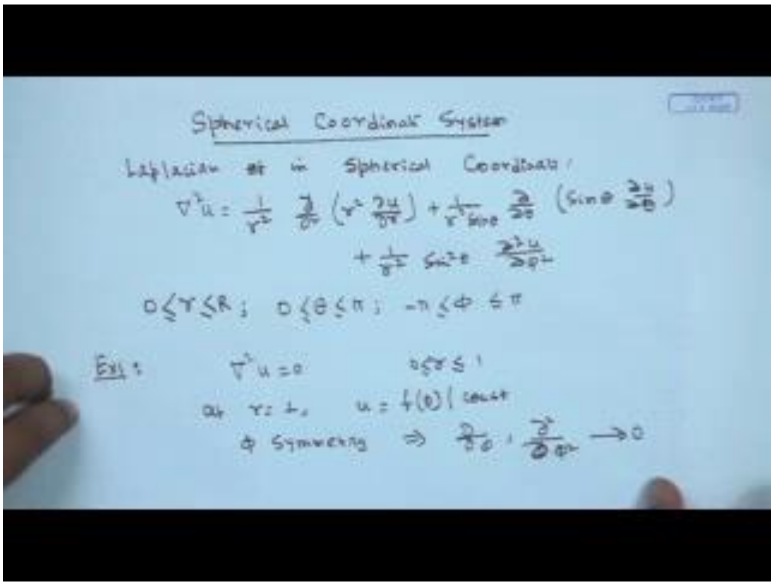

Next we will be moving over to the spherical polar coordinate systems which will quite popular whenever we will be looking into the you know transport properties occurring across the, you know spherical you know system spherical coordinate system spherical vessels or things like that. Then let us look into the Laplacian in spherical coordinate whereas, Laplacian in the spherical coordinate it is del square u is equal 1 over r square del del r square del u del r plus 1 over r square sine theta del del theta sine theta del u del theta plus 1 over r square sine square theta del square u del phi square. So, that is a Laplacian in a spherical coordinate system. And the domain of r is varying from 0 to R or 0 to 1 theta is equal to from 0 to pi and phi is varying from minus pi to plus pi.

Now we will be looking into the first example. Example one is Laplacian of u is equal to 0 and where the domain is r basically varying from 0 to 1 at r is equal to 1, we have u is equal to f of theta or it may be a constant as well and there is a phi symmetry. So, any derivative del del phi del square del phi square all are becoming 0. So, since there is phi symmetry so, this term will be off. So, it becomes a two dimensional problem.

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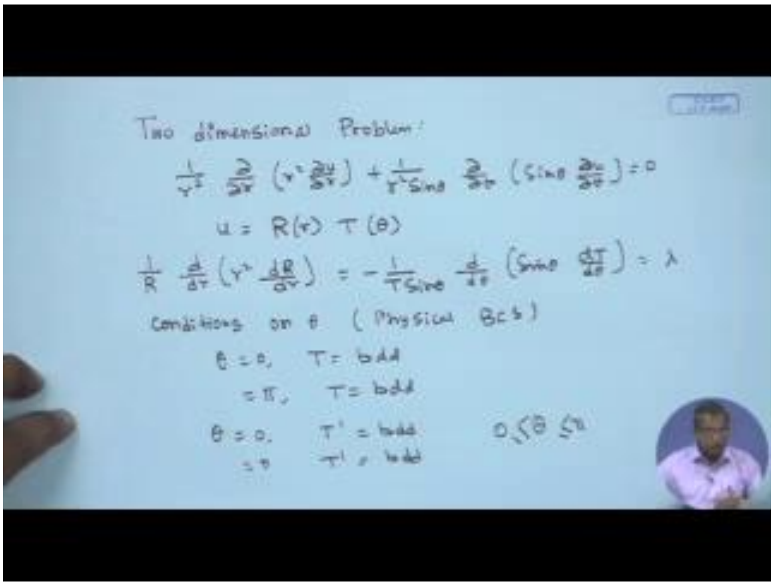

So, let us look into the two dimensional problem. One of our r square del del r r square del u del r plus 1 over r square sine theta del del theta sine theta del u del theta will be equal to 0. Now we assume that u is a, we go ahead with a separation of variable type of solution R of r multiplied by T of theta. So, if you put these in the governing equation and divide by R and T separating the variables, we will be getting 1 over R d d r r square dR dr is equal to minus 1 over T sine theta d d theta sine theta dT d theta.

And again this will be equal to some constant and that constant is equal to lambda. And conditions on theta are coming from the boundary physical boundary condition. Condition on theta will be coming from the physical boundary conditions. Theta is equal to 0 T is bounded. Theta is equal to pi T is bounded. Theta is equal to 0 T prime is bounded and theta is equal to pi T prime is bounded. So, these boundary conditions have to be satisfied and theta is basically varying from 0 to pi.

So, next what we will do we will be substituting small t is equal to cosine theta and see what we get.

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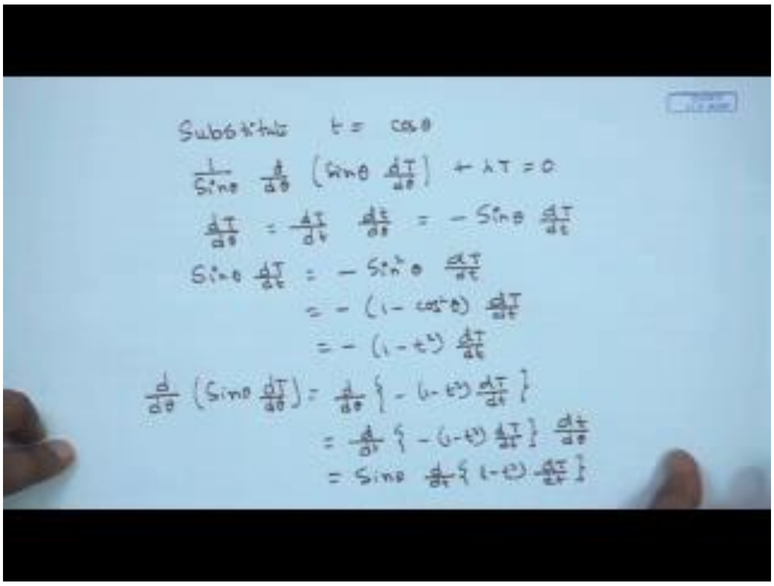

If you substitute small t is equal to cosine theta. Let us see what we get 1 over sine theta d d theta sine theta dT d theta plus lambda t will be equal to 0. So, dT d theta will be nothing, but dT dt times dt d theta. So, this becomes minus sine theta dT dt. And sine theta dT d theta will be nothing, but minus sine square theta dT dt. And this will be nothing, but minus 1 minus cos square theta dT dt is equal to minus 1 minus t square. So, I can I express sine theta dT dt in terms of t and T.

Similarly we can we do for the other variable dt theta sine theta dT d theta is nothing, but d d theta of minus 1 minus t square dT dt and this will become dT d dt of minus 1 minus t square dT dt is nothing, but dt d theta and dt d theta is nothing, but minus cosine minus sine theta. So, it becomes minus minus plus. So, it becomes sine theta d d t of 1 minus t square dT dt. So, 1 by sine theta d d theta of sine theta dT d theta is equal to 1 over sine theta dT d theta sine theta dT d theta is nothing, but d d t of 1 minus t square dT dt.

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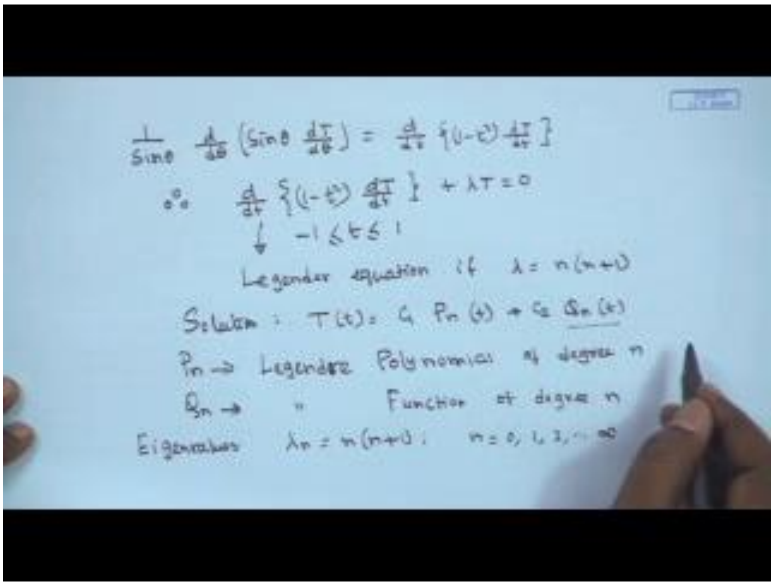

So, I can put in a neat form as d d t of 1 minus t square dT dt plus lambda T is equal to 0 where the dummy variable t is nothing, but it varies from minus 1 to plus 1. Now what is this? This equation is nothing, but a Legendre equation if lambda is represented as n into n plus 1. So, solution of this equation will be constituted of Legendre polynomial and Legendre function. So, T of t will be nothing, but C1 Pn t plus C2 Qn t. So, Pn is nothing, but Legendre polynomial of degree n and Qn is nothing, but the Legendre function of degree n.

Now Eigen values are lambda n is equal to n into n plus 1 where n runs from 0, 1, 2 up to infinity. And we know the property of Qn that Qn is not bounded it is unbounded at plus 1 and (Refer Time: 19:15) t is equal to plus 1 and t is equal to minus 1. But we have a finite value of the solution. So, therefore, the associated constant is equal to 0. So, the Qn is unbounded at t is equal to plus minus 1.

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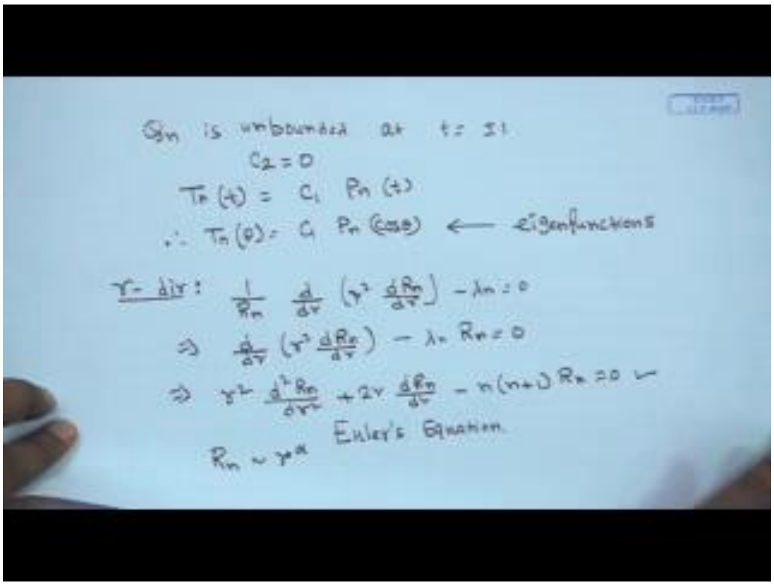

Therefore the associated constant C2 is equal to 0. So, the solution of theta varying part will be nothing, but C1 Pn t. So, therefore, Tn of theta will be C1 Pn cosine theta. So, these are the Eigen functions.

Now, let us look into to the radial direction. In r direction, we have one over Rn d d r r square d Rn d r minus lambda n is equal to 0. So, therefore, we have d d r of r square dRn d r minus lambda n Rn is equal to 0. And lambda n will be basically n into n plus 1. So, just open this up. Open up this differential r square d square Rn d r square plus 2 r dRn dr minus n into n plus 1 Rn is equal to 0. Now, if we remember the special ODs that we have already discussed earlier, this particular equation falls under the category of Euler's equation. So, this is nothing, but the Euler's equation. And we have already discussed that the solution will be in the form of r to power of alpha. The Rn will be in the form of r to the power of alpha. So, if we put r to the power alpha there then what will be getting is that we will be getting a characteristic equation in alpha.

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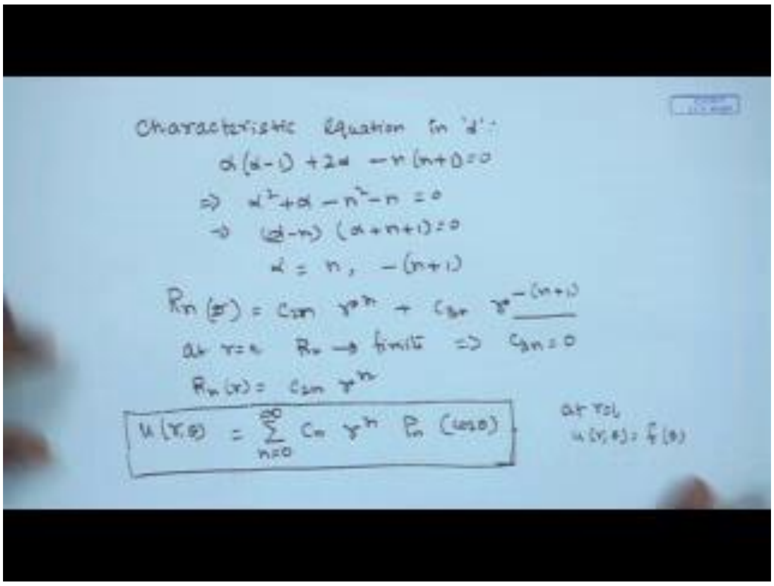

Characteristic equation in alpha and that becomes alpha into alpha minus 1 plus 2 alpha minus n into n plus 1 will be equal to 0. So, it becomes alpha square minus alpha plus alpha plus 2 alpha. So, it will be plus alpha minus n square minus n is equal to 0. This can be factorized alpha minus n into alpha plus n plus 1 is equal to 0. So, therefore, it will be having two roots alpha will be n and minus n plus 1. So, once we get these then will be getting the solution of Rn as Rn which will be function of r is C2 r C2 n r to the power n plus C3 n r to the power minus n plus 1. Now you put the boundary condition at r is equal to 0, Rn is finite. So, if Rn is finite then this term will blow off and it becomes infinite. So, therefore, the associated constant must be equal to 0. So, C3 n must be equal to 0 and Rn r is equal to C2 n r to the power n. So, u r theta is nothing, but summation of n is equal to 0 to infinity Cn r to the power n Pn cos theta. So, these become the solution of u varying part of the problem, the in two dimensional problems.

Now only one constant, now needs to be determined that will be the constant Cn and that constant will be determined from the, you know boundary condition located at r is equal to 1. So, we have the unutilized boundary condition at r is equal to 1 u r theta is given as f of theta. So, from this boundary condition, we will be evaluating r is equal to 1 u of r theta is nothing, but f of theta.

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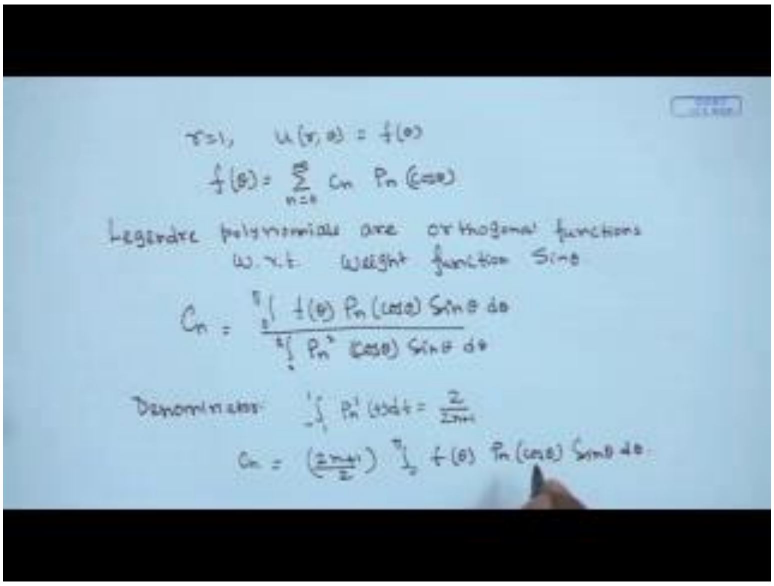

And f of theta will be, therefore summation of n is equal to 0 to infinity Cn Pn cos theta. Now we the Legendre polynomial like the Bessel functions, the Legendre polynomials are orthogonal. If you remember, we have iterated this in the while you have discussed the properties of the Legendre polynomials, their orthogonal functions, but with respect to weight function sine theta.

So, we utilize the orthogonal property of the Legendre polynomial and we can evaluate Cn as integral 0 to pi f of theta Pn cosine theta sine theta d theta and 0 to pi Pn square cosine theta sine theta d theta. And it can be shown that denominator is nothing, but minus 1 to plus 1 Pn square t dt and this will be 2 divided by 2n plus 1. So, Cn is nothing, but 2n plus 1 divided by 2, 0 to pi f of theta Pn cosine theta sine theta d theta. In fact, the numerator can also be simplified, you know in terms of you know the there are recursive formula of the Bessel of the Legendre polynomial. So, that completes this problem in the spherical coordinate system.

Now, in the next class what we will be doing? What we will be taking up one more problem on spherical coordinate system which will be a higher dimensional three dimensional problem. And we will be solving that equation and then that will be computing the problems, the partial differential equations in rectangular coordinate system, in cylindrical coordinate system and spherical polar coordinate system. After that what will be doing? We will be looking into two more realistic problems which will be quite, you know actually occurring in a heat conduction problem and which will be giving rise to a general formulation of an actual problem that will be having the various types of non-homogeneous boundary conditions in different boundaries. Then we will be at the higher dimensional problem how to tackle the various types of non-homogeneities in the boundary conditions and we will be breaking down the problem into sub problem. And at converting them into the well posed problem and we will be going ahead with the complete solution.

So I stopping in this class, we will be taking up this one more problem in this spherical polar coordinate system in the next class.

Thank you very much.