Partial Differential Equations (PDE) for Engineers:
Solution by Separation of Variables
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Lecture 15
Cylindrical Co-ordinate System – 3 Dimensional Problem

Welcome to the session of a course. So, we were in the last class we are looking into the solution of partial differential equation 2 dimensional problem in cylindrical polar coordinate system. It was a transient 1 dimensional problem. One-dimension in time other dimension in radial direction, where we had a theta symmetry. So, we looked into the in the formulation of the governing equation, as well as the boundary condition. Then we are formulating the eigenvalue problem in the radial direction. And we have found out the eigenvalue problem in the radial direction the corresponding constant can be 0 or it can be positive. If it is the constant at 0 or positive then we will be landing up to a trivial solution.

Now, in this class we will be completing this problem and then, we will be moving forward to solve the 3 dimensional problems in cylindrical polar coordinate system, which is quite common in various engineering applications.

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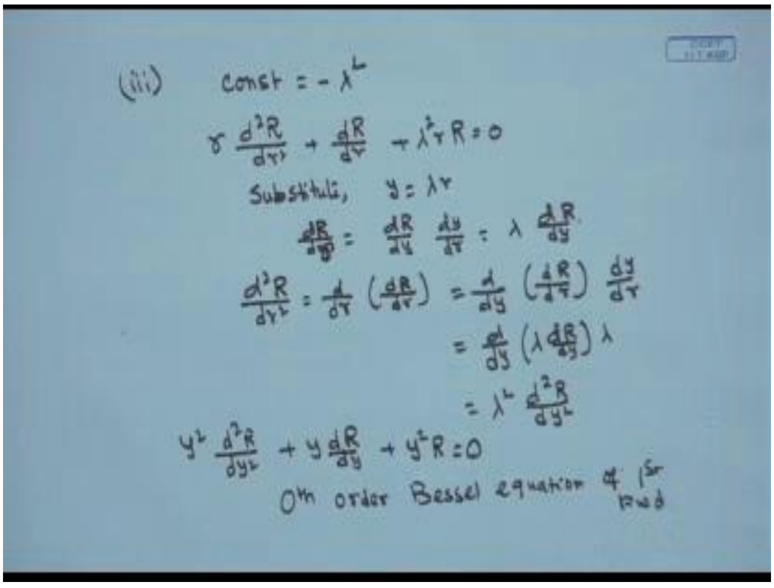

So, we will be looking into the case number 3, where this constant is a negative constant. So, this will be equal to minus lambda square. So, the governing equation of r varying part this is nothing, but d square r d r square plus d r d r plus lambda square r r is equal to 0. Now we substitute y is equal to lambda r. Then after then we can we can express the independent variable d r with respect to y.

So, what is d r d y. d r d y is nothing, but d r d y d r d r will be nothing, but d r d y d y d r. So, it will becomes, it will become d d y d r is nothing, but lambda, lambda d r d y. So, we will be having d r d y as d r d r as lambda d r d y. Similarly one can have d square r d r square is 1 more differentiation with respect to d r d r. So, it becomes d d y of d r d r d y d r and d r d r is already expressed in terms of y. So, d d y of lambda d r d y d y d r is lambda. So, it becomes lambda square d square r d y square. So, you will be, so, we get r. So, after substituting all these it will be getting y square d square r d y square plus y d r d y plus y square r is equal to 0. So, this will be this is nothing, but zeroth order Bessel equation of first kind the solution. We have already looked into this problem earlier and the solution of this of this equation is nothing, but composing of zeroth order Bessel function j 0 and y 0.

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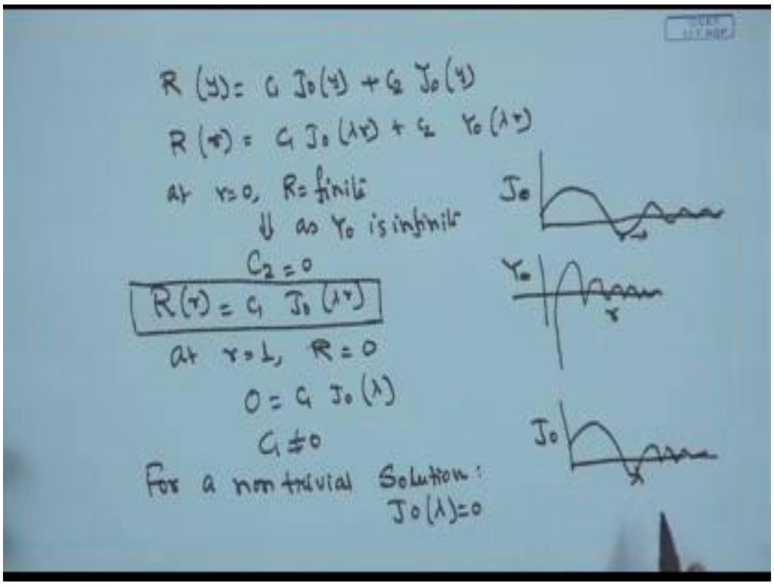

So, if you really look into the solution r of y becomes c 1 j 0 y plus c 2 y 0 y. So, put it back to lambda r. So, r of r is nothing, but c 1 j 0 lambda r plus c 2 y 0 lambda r. And at r is equal to 0 your r is finite capital r is finite. So, if you look into the variation of j 0. It becomes it becomes a positive value and then it will be oscillating about the x axis with r axis with diminishing magnitude on the other hand if you look into the y zero, y 0 will be basically infinity at r equal to zero, and then it will be oscillating about the r axis with diminishing magnitude. So, therefore, in order to have a finite value of the solution, the associated constant must be vanished otherwise at that particular boundary the solution becomes unbounded.

In order to avoid that, the associated constant because as y 0 is infinite making the solution to be infinite, the corresponding constant will be equal to 0. So, therefore, r of r is equal c 1 j 0 lambda r. now we use the so, this is the solution or the Eigen function of the particular problem. And we put the other boundary condition that is at r is equal to one, r is equal to 0 so; that means, 0 is equal to c 1 j 0 lambda.

Now, in order to have a non trivial solution to this particular problem we must be having c 1 not equal to 0, in order to have c 1 not equal to zero, for non trivial solution. Trivial solution only option is j 0 lambda is equal to 0 and if you plot plot plot j 0 lambda then it will be as we have seen earlier, that will be crossing the it will be oscillating the lambda axis and it will be intersecting the axis at infinite point number of points. And all these intersection points are the roots of this transcendental equation one can take request to the you know any numerical sub routine for example, Forton or C++ we have the there will be a standard sub routines are available, one can solve the first 5 roots or first 7 roots of this transcendental equation and one will be getting the you know solution and eigenvalues.

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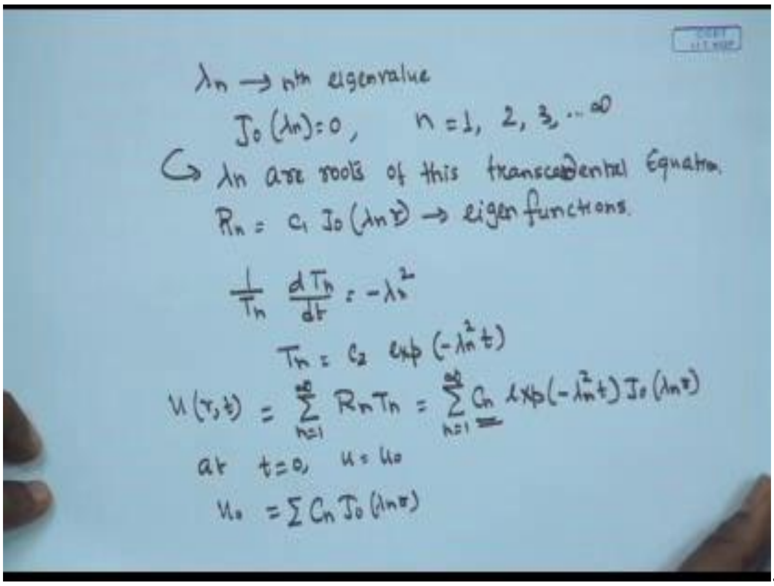

So, there are infinite number of eigenvalues back to this particular problem and we will be denoting them as lambda n lambda n corresponds to nth eigenvalue and we will be writing j 0 lambda n is equal to 0 where the index n runs from 1, 2, 3 up to infinity. So, the lambda n is the eigenvalues which are roots of this transcendental equation. And the corresponding Eigen functions are Bessel function of zeroth order Bessel function j 0 lambda n r are the Eigen functions.

Then next what we will do we will look into the time varying part, and we will try to construct the complete solution. If we look into the time varying part, this is becomes t t 1 by t n d t n d t, is equal to minus lambda n square, and t n will be is equal to c 2 exponential minus lambda n square t, therefore, u as a function of r t. We can construct the complete solution, by superposing all the individual solution corresponding to every eigenvalue and Eigen function. So, this will be c 1 into c 2 it will be multiplied with c n n is equal to 1 to infinity, exponential minus lambda n square t j 0 lambda n r.

Now the undetermined coefficient c n will be evaluated from the un utilized initial condition that are t is equal to 0 u is equal to u naught. If you write that t, t is equal to u, is equal to u naught, we will be getting u naught, is equal to summation of c n j 0 lambda n r. And we have already proved, that the Bessel functions are orthogonal to zeroth order

Bessel function or any order Bessel function will be orthogonal to each other, with respect to weight function r. That is already proved using the property orthogonal property of the Bessel functions. We will be utilizing we will be evaluating the constant c n.

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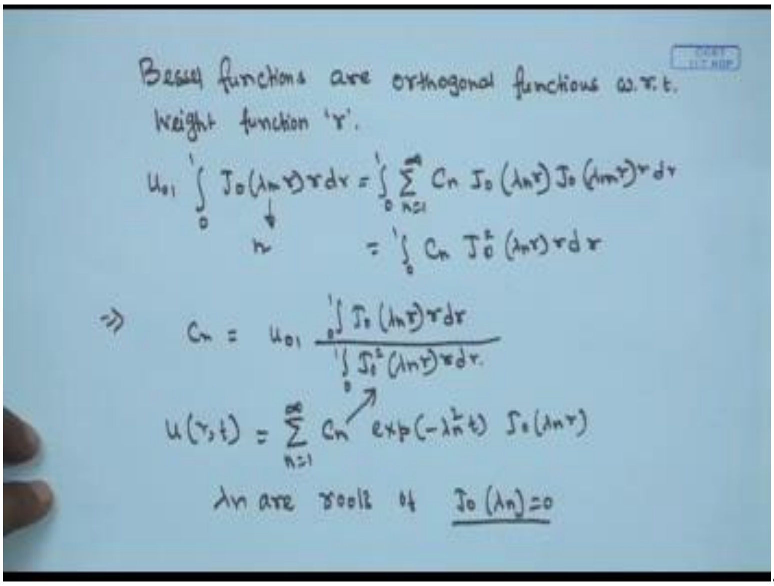

So, Bessel functions are orthogonal functions, with respect to weight function r. So, if you do that I multiply both side by j 0 lambda m r r d, r n is equal to 1 to infinity that is in right hand side c n j 0 lambda n r j 0 lambda m r r d r. Now if you open up this summation series all the terms will vanish only the 1 term that will be when m is equal to n that will survive. So, c n j 0 square lambda n r r d r. Now we change the running variable m into n which will be corresponding to our solution. So, we can get the expression of c n as u 0 1 0 to 1 j 0 lambda n r r d r, and 0 to 1 j 0 square lambda n r r d r.

So, once we get that we get the complete solution of u as r as a function of r and t, as summation of n is equal to 1 to infinity c n exponential minus lambda n square t j 0 lambda n r. Where c 1 is obtained from this expression and by evaluating these integrals either by using Simpsons rule or trapezoidal rule and then lambda 0 are the roots, lambda n are roots of the transcendental equation j 0 lambda n is equal to 0. So, that gives the complete solution of 2 dimensional that is 1 dimensional transient heat conduction problem in a spherical in a cylindrical polar coordinate system. Next we will move ahead we will go forward and we will be solving a 3 dimensional problem in cylindrical coordinate system.

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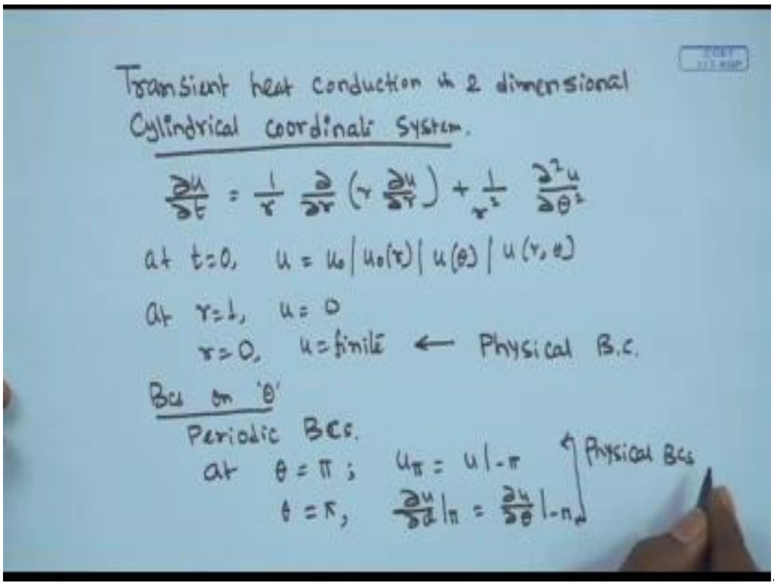

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So, let us look into a it is a, so, let us look into a it is a it is equivalent to a transient heat conduction problem, in 2 dimensional cylindrical coordinate system. So, let us look into a governing equation. The governing equation will be del u del t is equal to 1 over r del del r, r del u del r plus 1 over r square del square u del theta square. Now we will be having 1 condition on t, that will be the initial conditions 2 conditions on r because it is order 2 with respect to r, and 1 is already known at r is equal to 1, theta u, u is equal to zero; that means, we are maintaining the constant temperature at the outer boundary and at r is equal to 0 u is finite or del u del r is equal to zero, that is basically coming from the physics of the system, and it is a physical boundary condition, but we need to have 2 boundary conditions on theta, because this is order 2 with respect to theta.

So, let us write down the initial condition at t is equal to zero, u is equal to u naught it may be a constant, it may be a function of r, it may be a function of theta only, it may be a function of r and theta both so.

At r is equal to 1 we have u is equal to zero, at r is equal to 0 u is finite. So, this is an example of a physical boundary condition. Now let us define the boundary conditions on theta and these are known as the periodic boundary conditions. There are known as the periodic boundary condition again they will be coming from the physics of the problem at theta is equal to pi; u at pi is equal to u at minus pi. And at theta is equal to pi del u del theta evaluated at pi should be is equal to del u del theta evaluated minus pi. And both of these conditions on theta are known as the physical boundary condition, because they are formulated from the physics of the problem; when we will be going ahead with the separation of variable type of solution, to attempt to solve this equation. So, next what we will do is that we will be looking into the solution of the transient 2 dimensional heat conduction problem in the cylindrical.

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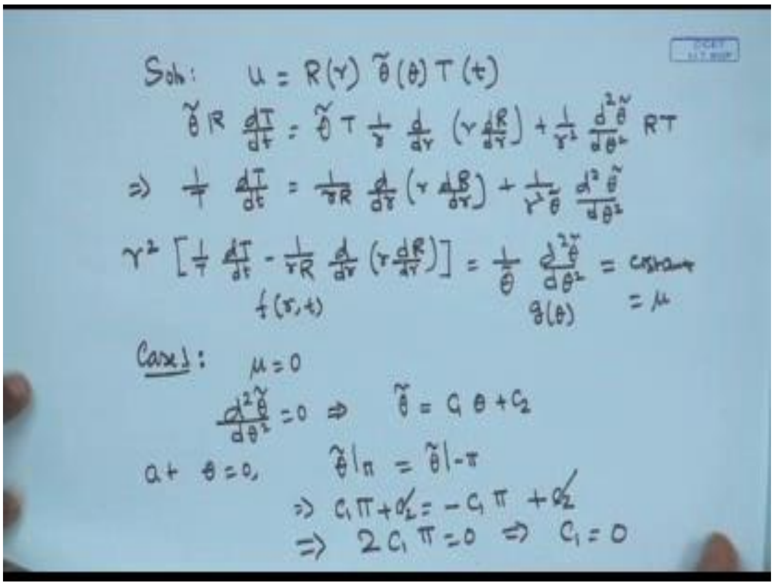

Coordinate system the solution goes like this. We will be we will be assuming that u is a function it is a product of 3 function 1 is function of r alone. Another is theta hat it will be the function of theta alone. Another will be t capital t that will be the function of time alone. Now if we substitute this into the governing equation then what will be getting is, theta hat r d t d t should be is equal to theta hat t 1 over r d d r of r d r d r, plus 1 over r square d square theta hat d theta square r t. So, therefore, if we divide both sides by theta hat r t, we can separate out the variables and then see what we get we will be getting 1 over t d t d t, and then 1 over r r d d r of r d r d r plus 1 over r square theta hat d square theta hat d theta square.

Next we will be multiplying r square on both side and try to prepare formulate standard eigenvalue problem in the theta direction let us see what we get. We will be bringing r radial r varying part on the left hand side and see what we get. So, what we will be getting is multiplied by r square 1 over t, d t d t minus 1 over r r, d d r r d r d r is equal to 1 over theta hat, d square theta hat d theta square. The left hand side entirely functions of r and t. The right hand is side entirely a function of theta. They are equal they will be equal to some constant. So, this will be a function of r and t. This will be some function of theta alone. They will be equal to some constant and this constant can be a positive constant can be 0 can be positive can be negative.

Let us investigate this separately and let us try to formulate the eigenvalue problem in the theta direction. If you do that then the case number 1 will be mu is equal to zero, when this constant is equal to 0. So, the governing equation become d square theta hat d theta square is equal to 0. So, these will be having a solution which is nothing, but a linear equation c 1 theta plus c 2. So, at theta is equal to zero, we have theta hat evaluated at pi is equal to theta hat evaluated at minus pi. So, theta hat at pi is nothing but c 1 pi plus c 2 and this will be minus c 1 pi plus c 2. So, c 2 c 2 will be canceling out. So, what will be having is 2 c 1 pi is equal to 0. Then 2 is not equal to 0 pi is also a constant not equal to 0. So, therefore, the only option is c 1 is equal to 0. So, the solution becomes solution. So, c 1 is equal to 0 the solution, becomes theta hat is equal to constant, theta is equal to constant.

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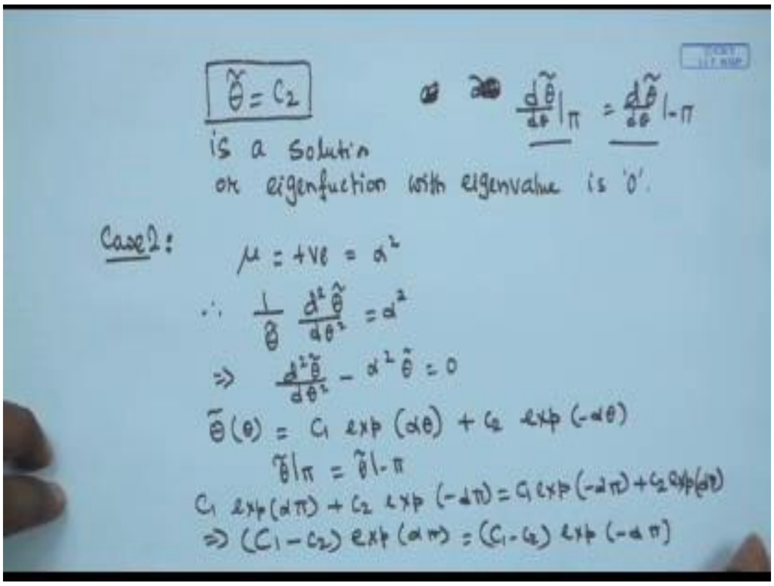

Let us try to see the other boundary condition and see how what is the solution implies in terms of other boundary condition. The other boundary condition is at del u d theta hat d theta at pi is equal to d theta hat d theta at minus pi. Since theta hat is a constant and it is the solution to this particular problem therefore, it is derivative with respect to theta is 0. So, that will be always satisfied by this boundary. So, therefore, theta hat is a constant is a solution. It is not a trivial solution, is a solution; that means, the constant mu is equal to 0 is eh will not give raise to a trivial solution. It will be giving a solution and the Eigen function is nothing, but a constant.

So, eigenvalue mu is equal to 0 is an eigenvalue to this problem. And the corresponding Eigen function is constant. Is a solution or Eigen function with eigenvalue is 0. Now next we look into case number 2. In case number 2 we consider mu is positive, that is equal to alpha square. So, 1 over theta hat d square theta hat d theta square is equal to alpha square. So, d square theta hat d theta square minus, alpha square theta hat will be equal to 0. So, we get the solution theta hat as a function of theta is nothing, but c 1 exponential alpha theta plus c 2 exponential minus alpha theta. Then we evaluate the constant c 1 and c 2 from the 2 boundary condition, periodic boundary condition. That is theta hat at pi is equal to theta hat at minus pi.

So, c 1 exponential alpha pi plus c 2 exponential minus alpha pi, is equal to c 1 exponential minus alpha pi, plus c 2 exponential minus minus plus alpha pi. And we can have c 1 minus c 2 exponential alpha pi, and c 1 minus c 2 exponential minus alpha pi.

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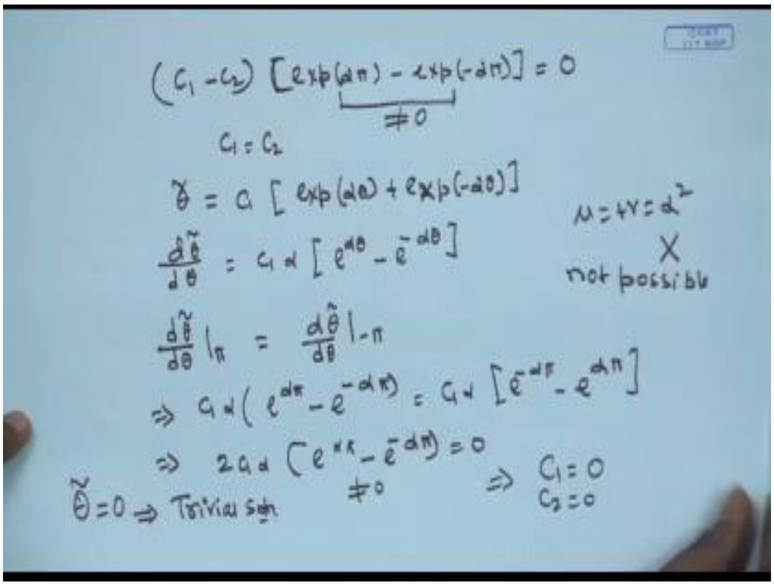

So, if we take c 1 minus c 2 to the other side we having c one, minus c 2 exponential alpha pi minus exponential minus alpha pi now that should be equal to 0. And exponential alpha pi you will be, you will be assuming a positive value this will be a positive value and this cannot be equal to 0. So, this can never be equation to 0. So, therefore, the only option here is c 1 is equal to c 2.

So, therefore, in this case number 2. We have the solution as theta hat is equal to c 1 exponential alpha theta, plus plus plus exponential minus alpha theta, and now if we try to satisfy the other boundary condition. So, you will be getting d theta hat before that we have to evaluate d theta hat d theta is equal to c 1 alpha e to the power alpha theta minus e to the power minus alpha theta. So, d theta what is the other boundary condition, the other boundary condition is d theta hat d theta at pi is equal to nothing, but d theta hat d theta evaluated at minus pi.

So, if you do that we will be getting c 1 alpha e to the power alpha pi, minus e to the power minus alpha pi is equal to c 1 alpha, e to the power minus alpha pi minus e to the power minus minus plus alpha pi and therefore, we can have 2 c 1 alpha into e to the power alpha pi minus e to the power minus alpha pi. Again this cannot be equal to 0 and alpha cannot be equal to 0. So, therefore, c 1 has to be equal to 0 and since c 1 is equal to c 2 c 2 has to be equal to 0. So, what is the solution the solution is theta hat is equal to, 0 and it is a trivial solution. So, therefore, mu is positive is equal to alpha square that is ruled out and not possible. Because that will be leading to that will be leading to a trivial solution, and we are really looking for a non trivial solution. So, this constant has to be a negative constant. So, let us consider the case number 3.

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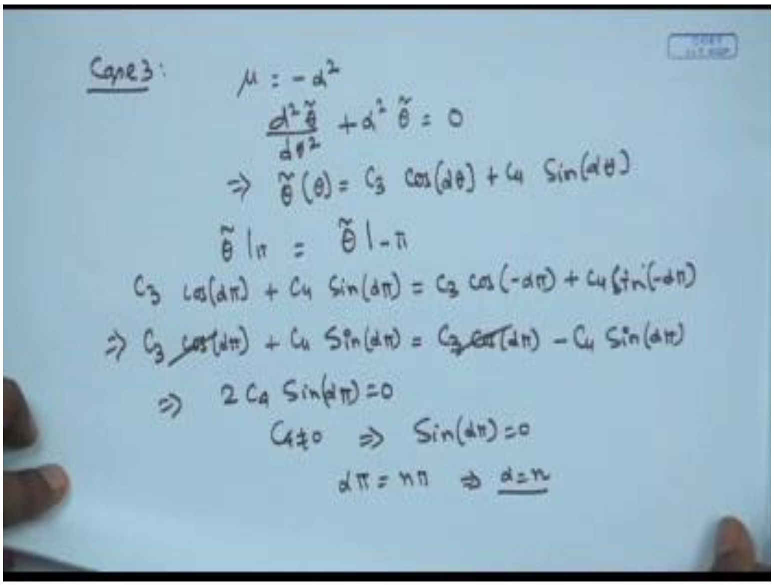

So, if you look into the case number 3 mu is equal to minus alpha square. So, now, we formulate the standard eigenvalue problem with negative constant. So, if you take it bring it to the other side, d square theta hat d theta square plus alpha square theta hat will be is equal to 0. We know the solution the solution will be constituted by the sin function combination of sin function and cosine function.

So, if you really do that. So, you will be getting the solution theta hat as the function of theta is equal to c 3, cosine alpha theta plus c 4 sin alpha theta. Now we put theta hat at pi is equal to theta hat at minus pi. So, if you do that we will be getting c 3 cosine alpha pi plus c 4 sin alpha pi is equal to c 3 cosine minus alpha pi plus c 4 minus sin minus alpha pi. So, you know cos of minus x is equal to cos, sin of minus x is minus sin x. So, this will be c 3 cosine alpha pi, plus c 4 sine alpha pi, is equal to c 3 cosine alpha pi, minus c 4 sine alpha pi.

So, c 3 cosine alpha pi will be canceling from both sides. So, what is left is 2 c 4 sine alpha pi is equal to 0. Now in order to have a non trivial solution c 4 must not be equal to 0. So, c 4 must not be equal to 0. So, what is left is sin alpha pi is equal to 0. And what is the general solution alpha pi alpha pi will be nothing, but n pi or alpha will be is equal to n. now let us try to use the other boundary condition. If you use the other boundary condition that is the d theta hat d theta at pi is equal to d theta hat d theta hat minus pi.

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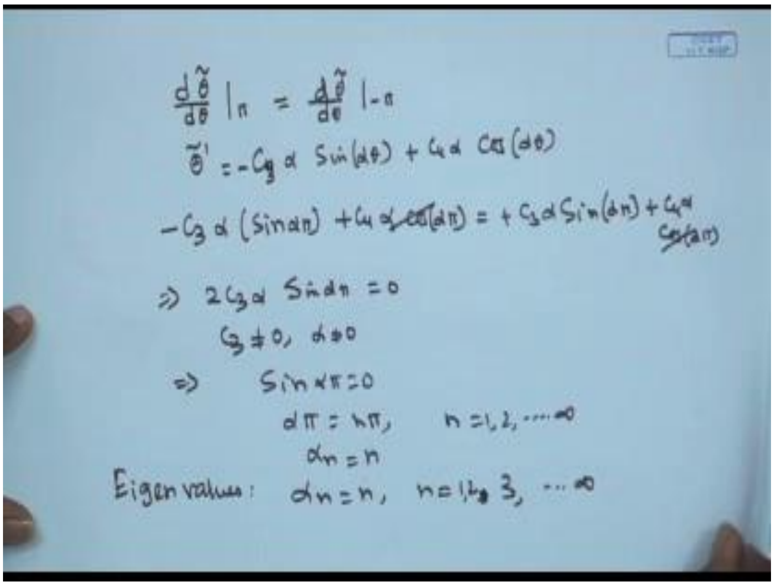

So, let us first evaluate theta hat theta hat at v c 4 alpha c 3. So, this will be c 3 alpha minus of that sin alpha theta plus c 4 alpha cosine alpha theta. And theta hat at pi will be minus c 3 alpha sin alpha pi plus c 4 alpha cosine alpha pi. And these will be evaluated at minus pi. So, minus c 3 alpha minus pi means minus minus plus sin alpha pi plus c 4 alpha cosine of minus means it will be cosine alpha pi. And this c 4 alpha cosine alpha pi will be canceling out. And again we will be getting 2 c 3 alpha sin alpha pi is equal to 0.

So, in order to have a non trivial solution we should have that c 3 is not equal to 0. Because therefore, alpha should also not be equal to 0 because alpha equal to 0 we have already considered.

So, there therefore, these becomes sin alpha pi is equal to 0. In both the cases we have considered that whenever we are using the 2 both the boundary conditions, we will be landing up with the condition, that sin alpha pi is equal to 0 and we have the general solution alpha n alpha pi is equal to n pi, where the index n runs from 1 to infinity. So, therefore, pi will be cancel. So, alpha n is equal to will be equal to n. So, the eigenvalues, of this particular problems, problem is alpha n is equal n where n is equal to 1 2 3 up to infinity and the corresponding Eigen functions are corresponding Eigen functions. Theta hat theta c 3 n cosine n theta plus c 4 n sin n theta; because alpha has now become n.

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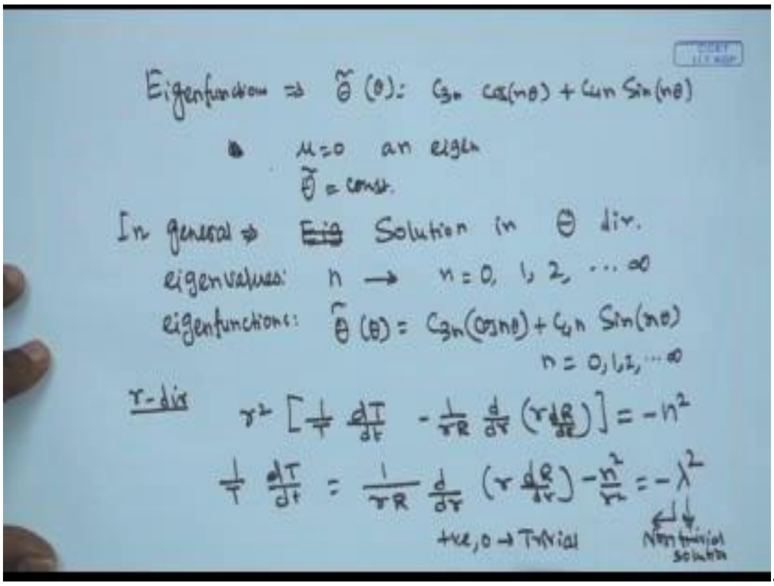

Now, if you remember n is equal to the constant mu is equal to zero, was also an eigenvalue to this particular problem. And the Eigen function is constant. So, therefore, we can in general write the Eigen function in this form, but if n starts from so, in actually in this particular form n starts from 1 to infinity if n starts from 0 will be so, let us let us see what is the value of this Eigen function at n equal to 0. Sin 0 is eh 0 cos 0 is 1, and it will be giving c 3 n that is a constant.

So, in general if I start my running index n from 0, 1, 2 up to infinity, instead of 1, 2, 3 up to infinity, then I will be including the Eigen function the eigenvalue value n equal to zero, and Eigen function equal to constant in this particular form is inside it. So, therefore, the in general, we can write down the Eigen solution of in theta direction, as a eigenvalues are in where n runs from 0, 1, 2, 3 up to infinity.

So, n equal to 0 is a 0 is an eigenvalue, 1 is an eigenvalue, 2 is an eigenvalue, 3 is an eigenvalue, up to infinite terms. Where the corresponding Eigen functions the general form of the Eigen functions are theta hat which will be a function of theta is nothing, but c 3 n cosine n theta plus c 4 n sine n theta, where n runs from 0 1 2 up to infinity, instead of 1 2 3 up to infinity. Now let us write down the governing equation in the r direction that is only left behind, it will be nothing, but r square 1 over t d t d t minus 1 over r r d d r of r d r d r, is equal to minus lambda square, with them alpha square. So, that will be n. So, it will be n square. So, we can separate out the variables d t d t is equal to 1 over r r divided by d r r d r d r minus n square r square. And again that will be a negative constant. This is a function of time this is a function of space this will be equal to some constant, that negative constant in order to if we have a positive constant or zero, then we will be getting a trivial solution.

So, in order to have a non trivial solution, we will be getting the negative value of this constant that will give you giving us a non trivial solution. So, I will stop you here, in this class in the next class I will be taking up this problem and from this point onwards, we will be completing this problem entirely.

Thank you very much.