Partial Differential Equations (PDE) for Engineers:
Solution by Separation of Variables

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Lecture – 14
Orthogonality of Bessel Function and 2 Dimensional Cylindrical Coordinate System

Welcome to the session. Now, as it is discuses in the last class will be starting the Cylindrical Polar Coordinate System. These cylindrical coordinate systems are quite common in various injuring problem for example, the transport through a pipe in a tubular reactor, you know there are tubular cross section equipment where the mass transfer, heat transfer can take place, heat exchanges mass exchanges like dialysers and other things. So, the tube light geometry and cylindrical geometry are quite common in various engineering problems.

Now, in the last few classes we have looked in detail how the governing equation takes the form in rectangular coordinate system and their solution. Now from this class onwards will be looking into the cylindrical polar coordinate system electron will be moving over to the spherical polar coordinate system.

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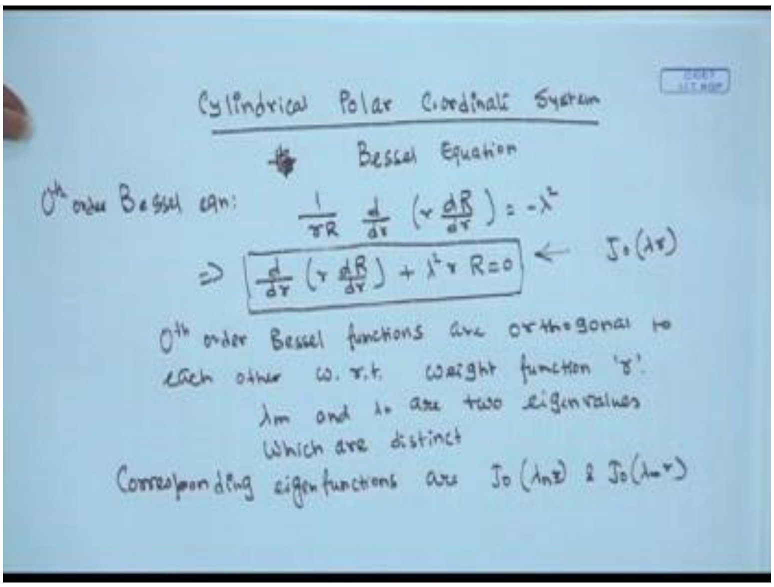

So, first will be taking about the cylindrical polar coordinate system; so as you have discuss that will be encountering with the Bessel equation in case of cylindrical coordinate system. And typically 0th order Bessel equation will be the form of 1 over rR d dr r dR dr is equal to minus lambda square. This will be in a sense d dr of r dR dr plus lambda square rR is equal to 0. So, these are typical Bessel equation and first will be proving in this. So, will start that this is a 0th order Bessel equation; Bessel function they are orthogonal each other in fact that is true for we will be later on generalizing it that any order of Bessel functions they are orthogonal each other.

So, in this class will just proofs at the very beginning that 0th order Bessel functions are orthogonal to each other with respect to weight function ‘r’. Like in the rectangular coordinate system we have seen the Eigen functions like sine functions and cosine functions are orthogonal to each other with respect to the (Refer Time: 03:35) function 1. Now, in case of cylindrical coordinate system will prove that Bessel functions orthogonal to each other with respective at (Refer Time: 03:43) function r. So, let us say lambda m and lambda n is two Eigen values which are distinct. Corresponding Eigen functions 0th order Bessel functions of the Eigen functions to this Bessel equation. Therefore, the corresponding Eigen functions are J0 lambda nr and J0 lambda mr.

So, Eigen functions to this problems are the J0 lambda r and corresponding to lambda m and lambda n the Eigen functions are J0 lambda nr and J0 lambda mr. That means, this equation will be satisfied by the Eigen function J0 lambda mr and this equation will be satisfied also by the Eigen function J0 lambda nr.

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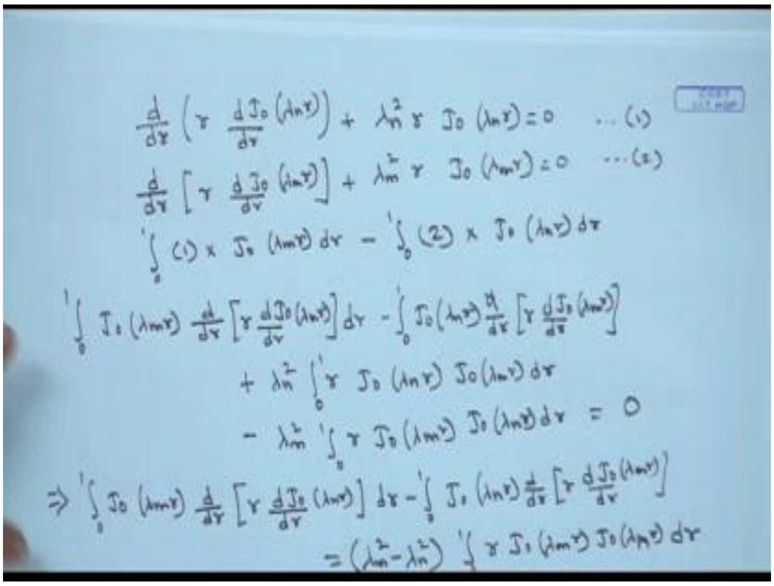

Therefore, we can write that d dr of r dJ 0 lambda nr dr plus lambda n square r J0 lambda nr is equal to 0. And we can also write that d dr of r d J0 dr lambda mr plus lambda m square r J0 lambda mr is equal to 0. We write it as equation 1 and equation 2. So, what next I will do will be multiplying equation 1 with J0 lambda mr and integrate over the domain of r and then will be multiply equation 2 with respective to J0 lambda nr and then integrate over domain of r and then subtract. That means, what I will be doing next is I will be multiplying equation 1 with J0 lambda mr dr then integrate from over the domain of r from 0 to 1 and then minus 0 to 1, equation 2 multiplied by J0 lambda nr dr.

If we do that let us see what we get. It will be 0 to 1 J0 lambda mr d dr r d J0 dr lambda nr dr minus 0 to 1 J0 lambda nr d dr of r d J0 dr lambda mr then plus lambda n square r J0 lambda nr J0 lambda mr dr 0 to 1 minus lambda m square 0 to 1 r J0 lambda mr J0 lambda nr dr is equal to 0. If you look into these two terms the r J0 lambda mr and J0 lambda nr dr will be a common so there will be lambda n square minus lambda m square,

So, we will be breaking into the two equations left hand side and right hand side completely J0 lambda mr d dr r d J0 dr lambda nr dr minus 0 to 1 J0 lambda nr d dr r d J0 lambda mr dr; and right hand side we write lambda m square minus lambda n square integral 0 to 1 r J0 lambda mr J0 lambda nr dr. Now let us evaluate the left hand side completely then will be turning to the right hand side.

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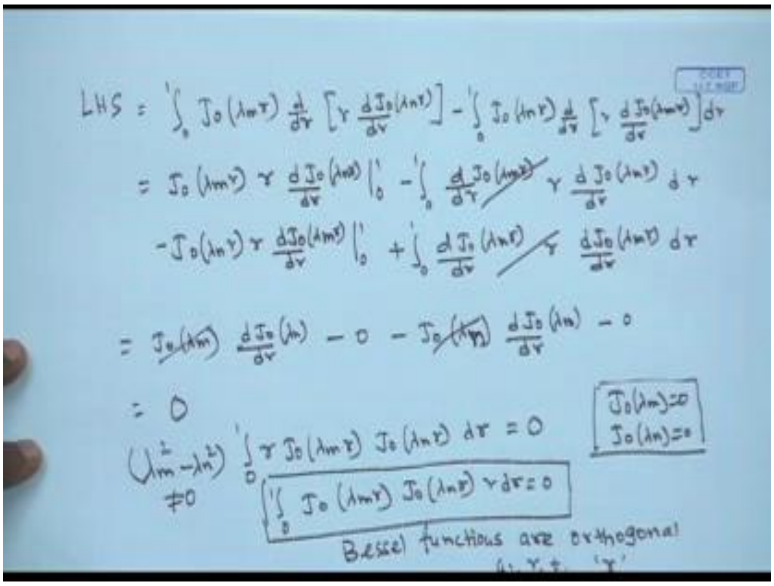

So, let us evaluate left hand side. I write down the left hand side completely 0 to 1 J0 lambda mr d dr of r d J0 lambda nr dr minus 0 to 1 J0 lambda nr d dr r d J0 lambda mr dr dr. We integrate this by parts, so first function J0 lambda mr; first function integral of the second function so r d J0 lambda nr dr 0 to 1 minus 0 to 1 differential of the first function so d dr of J0 lambda mr; integral of the second function so r d J0 lambda nr dr dr minus first function integral of the second functions. So, J0 lambda nr r d J0 lambda mr dr from 0 to 1 minus minus plus 0 to 1 differential of the first function, so d J0 lambda nr dr r integral second function d J0 lambda mr dr d r.

So, if you look into this integral is r d J0 lambda mr dr d J0 lambda nr dr and this will be equal and identical, but they will be having the opposite sign so this two will cancelling out. Now let us put the boundary condition J0 lambda m this nothing but the binary combined term r at 1. So, d J0 lambda n dr minus when r is equal to 0 this term is equal to 0 so this will be returning a value 0 minus J0 lambda m r is 1 so this will become d J0 lambda m this becomes lambda n d J0 lambda m dr and r equal to 0 this will be 0.

Now, since lambda m and the lambda n are the eigenvalues and we know that the eigenvalues will be the routes of the transiental equation J0 lambda m is equal to 0. Therefore, J0 lambda m is equal to 0 and J0 lambda n is equal to 0, because the eigenvalues are routes of this equation, so this will be is equal to 0 this will be is equal to 0. Therefore, the whole left hand side will be will be is equal to 0.

If we now write down the right hand side it becomes lambda m square minus lambda n square integral 0 to 1 r J0 lambda mr J0 lambda nr is equal to dr dr is equal to 0. Now lambda m and lambda n are two distinct eigenvalue, so therefore lambda m square minus lambda n square cannot be is equal to 0. So, the option that is left behind is this integral will be equal to 0 this will be 0 to 1 J0 lambda mr J0 lambda nrR dr will be is equal to 0. That proves that the Bessel function are orthogonal with respective to weight function r.

So, we will be utilising this property of the Bessel function in order to solve the partial differential equations in cylindrical polar coordinate system. After this once you prove that the Bessel functions are orthogonal then will be looking into the complete solution say of the (Refer Time: 13:34) this cylindrical coordinate system.

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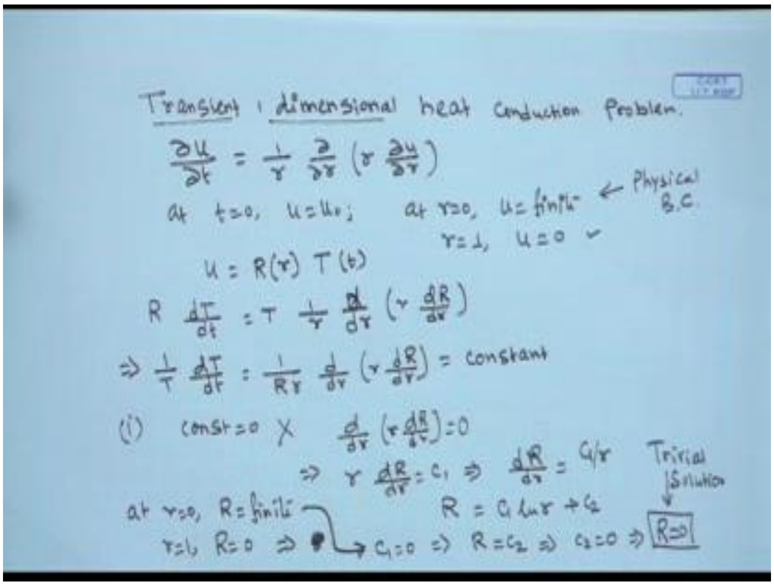

So, let us first take a 2 dimensional problem. This is transient 1 dimensional heat conduction problem, so it is basically 2 dimensional problems; 1 dimensional in space 1 dimensional is time so governing equations we will take in this form del u del t is equal to 1 over r del del r r del u del r. At t is equal to 0, we have u is equal to u naught, at r is equal to 0, we have u is equal to finite and at r is equal to 1, we have u is equal to 0. Now this boundary conditions we need to discuss about this so at time t equal to 0 u is equal to u naught means it is maintaining a uniform temperature at time t is equal to 0. The boundary condition at r is equal to 1, u is equal 0 means at the wall, so this r and u coordinates system let us consider they are non-dimensional.

So, r is equal to 1 means at the r by r is equal to 1, so therefore at the edge of the tube or the pipe we are maintaining a constant temperature. After doing a non-dimensionalization we can make the boundary condition to be homogeneous. Suppose t is equal to t naught and r is equal to capital R, now if you defined r star is equal to r by r then r star is equal to 1 and we define theta or u as non-dimensional temperature like t minus t naught divided by t naught then u will be equal to 0 and r is equal capital 1. This means we are maintaining a constant temperature at the wall. And at r is equal to 0 u is finite means we do not have any value; we cannot measure the experimental value of temperature at middle point of the tube. But we know that there will be existing temperature at the centre point, so we put u is equal to finite there.

This is a example of physical boundary condition. This will be dictated by the physics of the problem, so this is an example of physical boundary condition. Once we do that then we go ahead with separation variable type of solution. So, u is supposed to be a product of two functions; one is completely with the function of r and another is entirely a function of time. If you do that then we will be getting R dT dt is equal to T 1 over r del del r (Refer Time: 16:43) d dr of r dr dr.

So, then we separate out the variables on over T dT dt is equal to 1 over Rr d dr of r dR dr. The left hand side is entirely a function of time, the right hand side is a entirely function of space and they will be equal to some constant. Now if you remember in this case of rectangular coordinate system whether the very first instants and we solve the first problem we check the constant whether it is 0 positive and negative, and then ultimately came to conclusion that for a particular case this will be giving a non trivial solution other two cases will be become a trivial solution. Similarly, in this case also will be will be checking three values of this constant; 0, positive and negative and then will be looking into the solution and looking for a non trivial solution.

So, case one will be let us say, this constant equal to 0. If that is the case then d dr of r dR dr will be equal to 0. This means r dR dr will be equal to constant c 1. So, dR dr is equal to c 1 by r, and then we have one more integration c 1 l nr plus c 2. Now at r is equal to 1 u is equal to 0 that means, at r is equal to 0 here capital R is finite so it will be obeying the original problem or original boundary condition in the r direction and r is equal to 1 your capital r is equal to 0. So therefore, if you put for r is equal to 0 r is finite means l n 0 is not defined.

Therefore, the associated constant will be equal to 0. Therefore, this imposition of this boundary condition means c 1 is equal to 0, so therefore is equal to c 2 and r is equal to 1, capital R equal to 0, so therefore c 2 equal to 0, so your capital R is equal to 0. If you have this constant to be a 0 constant value to be assume a value 0 then will be getting in a trivial solution, so we are landing with a trivial if this constant is 0. So therefore, this constant cannot be 0 because we are not looking for a trivial solution we are looking for a non trivial solution.

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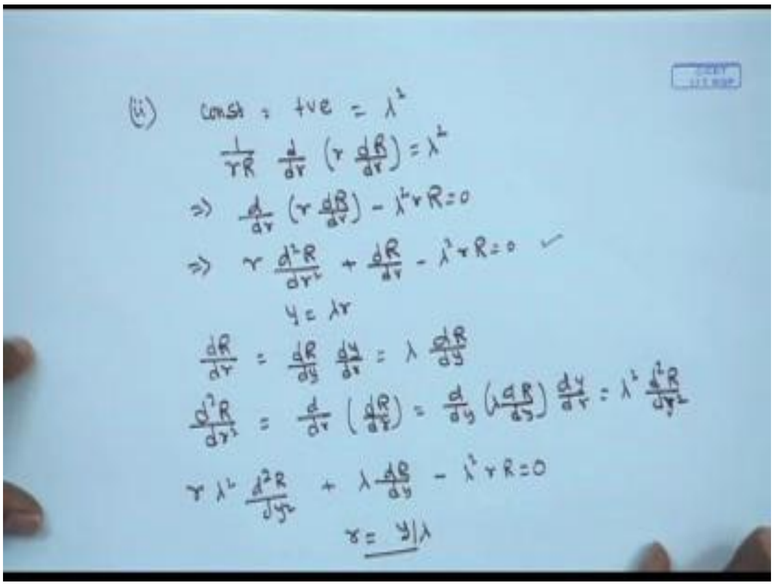

And let us check the next possibility. The next possibility is this constant is positive lambda square is positive and let us say this is becomes 1 over rR d dr of r dR dr is equal to lambda square. So, will be having d dr of r dR dr minus lambda square rR is equal to 0. If you open up this it becomes r d square R dr square plus dR dr minus lambda square rR is equal to 0. Now we substitute y is equal to lambda r and see what we get if you substitute y is equal to lambda r then the derivatives dR dr will be nothing but dr dy dy dr, so this becomes lambda dr dy.

And similarly we can get d square R dr square in terms of y as d dr of d dR of dr. So therefore, this becomes d dy dR dr is lambda dr dy or divided by dy dr and this will be lambda so it will be lambda square d square R dr square. Then we substitute this values this derivative in this equation in terms of y if you really do that will be having r lambda square d square R dy square, this will be dy square plus lambda dr dy minus lambda square rR is equal to 0.

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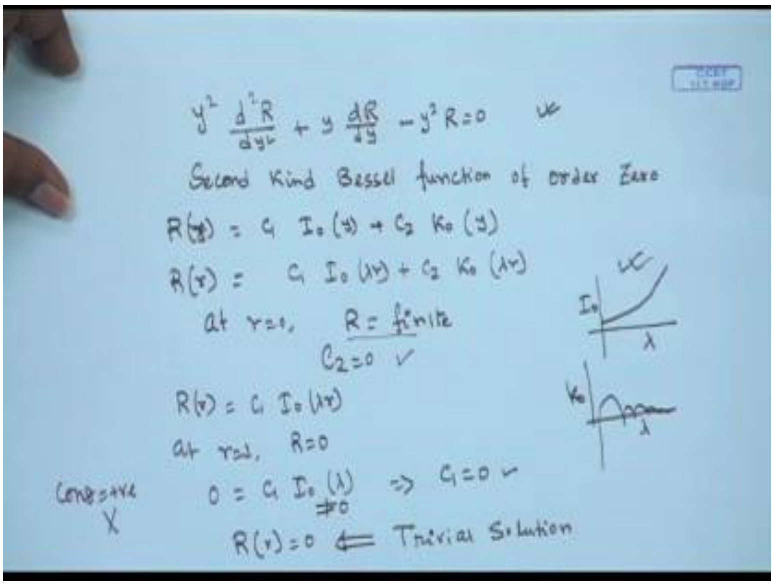

Then we substitute r by y by lambda in this equation and after substituting that the neat form of the equation becomes y square d square R dy square plus y dR dy minus y square r is equal to 0. So, this equation is nothing but if we can identify that it is a second kind Bessel function of order 0.

As we have discussed about the special (Refer Time: 23:00) or special ordinary differential equation that will have discussed earlier we can identify that this equation is nothing but the 0th order of the Bessel function of second kind. So, the solution is composed of I 0 and the super position of I 0 and k 0. If you now write the solution, the solution will be nothing but R of y is c 1 I naught y plus c 2 k naught y and y is lambda r. So therefore, R which is a function of small r is nothing but c 1 I 0 lambda r plus c 2 k 0 lambda r.

And now we can put the values boundary condition of the original problem at r is equal to 0, capital R is finite. That means, if you remember the property of I naught, the property of I naught as a function of lambda it will be ever positive. On the other hand if you look into the property of k naught by lambda it will be infinity at r is equal to 0, so therefore the associated constant c 2 is equal to 0. Therefore, Rr is equal to c 1 I 0 lambda r right. Since k 0 is infinite at small r is equal to 0, but a now capital R should be finite there so there will be a conflict of concept. Therefore, the associated constant must vanished at the particular boundary.

So, what is left is Rr is equal to c 1 I 0 r. Now let us put the other boundary condition at r is equal to 1. We have capital R is equal to 0, so this will be is equal to 0 c 1 I 0 lambda. Now in this equation as you have seen the property of I 0 is that it will be ever positive for every value of finite lambda I 0 will be assuming a positive value. So therefore, it can never be equal to 0. Therefore, in order to satisfy this equation c 1 has to be equal to 0, this is not equal to 0 therefore this means c 1 is equal to 0. We have already seen that c 2 is equal to 0 and c 1 is equal to 0, so therefore solution is r is equal to 0. This gives me a trivial solution. Therefore, the case that the constant is a positive constant is ruled out. So, constant is positive that cannot happen it is ruled out; therefore what is left is that the constant has to be a negative constant.

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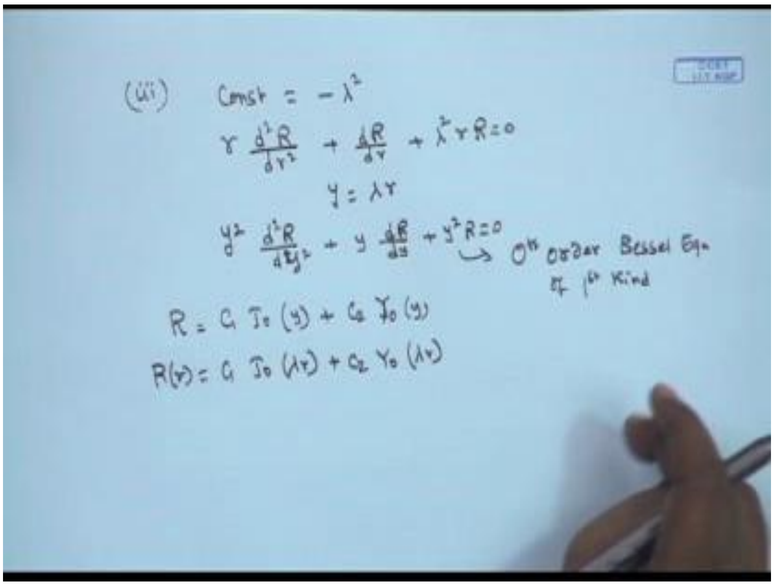

So, let us look into that option the constant is case number three. The constant is negative constant and that will be equal to minus lambda square. So, that is the case we write down the governing equation of redial varying part is r d square R dr square plus dR dr plus lambda square eR is equal to 0.

Now, next again will be doing y is equal to transformation y is equal to lambda r, so this gives you y square d square R dy square plus y dR dy plus y square R is equal to 0. This is the 0th order of the Bessel equation of first kind. So, the solution is composed of J0 and Y0; c 1 J0 y plus c 2 Y0 y. In terms of R this is becomes c 1 J0 lambda r plus c 2 Y0 lambda r.

So, I will stop here in this class. In the next class I will be taking up this problem and moving ahead and I will be completing this solution by using the separation of variable.

Thank you very much.