Partial Differential Equations (PDE) for Engineers:
Solution by Separation of Variables
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Lecture – 13
Solution of Hyperbolic PDE

Good morning everyone, welcome to this session. So, in the last class we have seen the solution of electrical partial differential equation, and we have seen that we have looked into a well posed problem and we have seen that if there is the if 1 condition is non homogeneous and all other boundary conditions are homogeneous, we have to formulate it, construct a standard eigenvalue problem in the direction where the boundary conditions are homogeneous and then we have looked into some special type of boundary conditions and then we have solved the problem to its completion and we have already also seen that if there is a process which is occurring on the study state that will be represented or modeled by an electrical partial differential equation.

Then what is left is that we have to look in to the hyperbolic partial differential equation in rectangular coordinate system, and will be solving a well posed problem because we already know that if actual problem which is not well posed all the time. It may be an ill posed problem and we have seen the techniques, how to convert the ill posed problem into an well posed problem and then we have to solve now, in today’s class what will be doing is that, will be solving a hyperbolic partial differential equation. Let us look into the hyperbolic partial differential equation.

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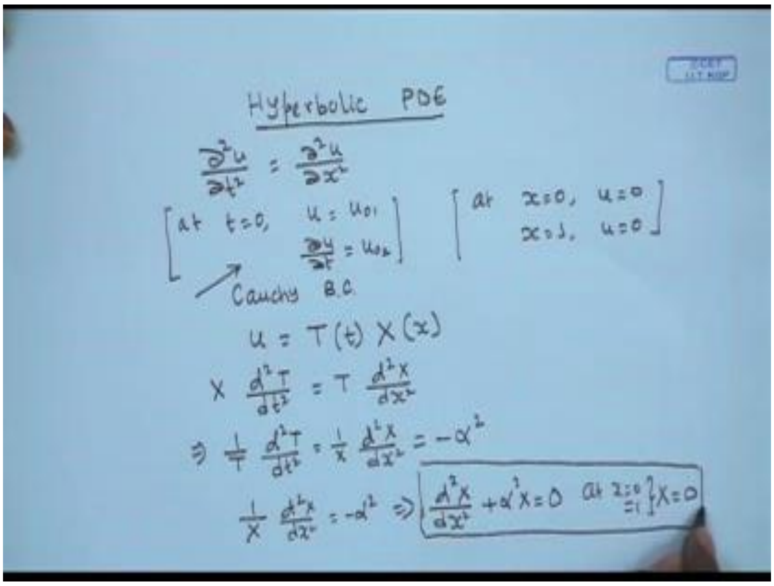

So, will be solving a problem del square u del t square is equal to del square u del x square and at t is equal to 0 u is equal to u naught 1 and at t is equal to 0 del u del t will be is equal to let us say u naught 2 and boundary conditions are at x is equal to 0 we have u is equal to 0 and x is equal to 1 we have u is equal to 0. So, if you remember the in case of hyperbolic partial differential equation the conditions both con 2 conditions are specified at t is equal to 0. So, these are the Cauchy boundary condition when both the boundaries are specified at the same boundary.

Now, what will be doing next is that. So, this is a well posed problem for a hyperbolic partial differential equation, where the initial condition is already set and it is both of conditions are specified. So, it is a well posed problem because the boundary conditions are homogeneous. Now, what will do next we will be looking for a separation of variable type of problem and since the boundary conditions in the x direction they are homogeneous we have to construct a standard eigenvalue problem in the x direction.

So, we assume that u is a product of 2 functions 1 is function of time alone and I is the function of space alone. So, going to substitute this in the governing equation, if you do that will be getting x d square t d t square is equal to t d square x d x square and we divide by x t. So, it will be 1 over t d square t d t square is equal to 1 over x d square x d x square the left hand side is a function of time alone the right hand side is function of space alone they will be equal and they will be equal to some constant minus alpha square because if this constant is 0 or positive will be landing into a trivial solution.

Now, let us formulate the standard eigenvalue problem 1 over x d square x d x square is equal to minus alpha square. So, therefore, d square x d x square plus alpha square x is equal to 0 subject to the boundary condition of the original problem in the x direction that means, at x is equal to 0 and 1 capital x is equal to 0. So, this formulates the standard eigenvalue problem in the x direction and we have already seen the solution of this equation the solution with these is a standard eigenvalue problem with eigenvalues n pi where the index n runs from 1 to infinity and the eigenfunctions are sin functions.

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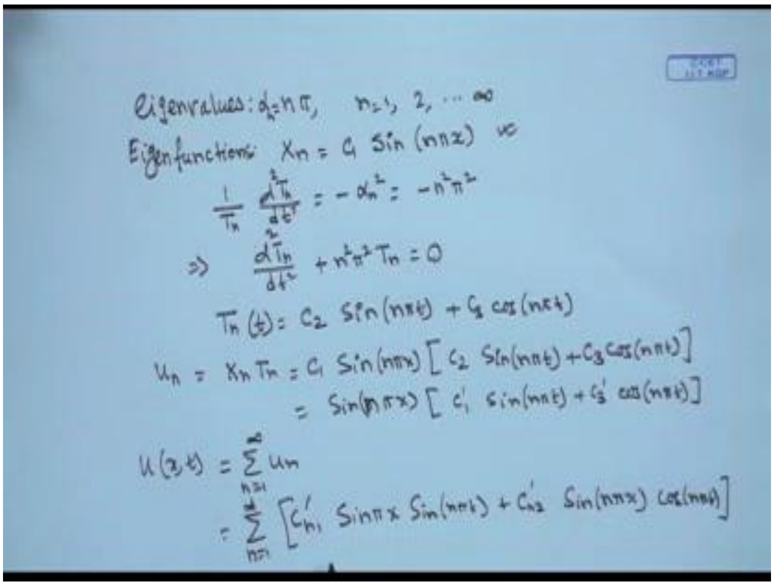

So, let us look into that, the eigenvalues are n pi alpha n is equal to n pi n runs from 1, 2 up to infinity and Eigen functions are x n C 1 sin n pi x, now let us so that computes the solution of x varying a part or special varying part, now will be looking into the time varying part. So, let us look in to the time varying part 1 by t n d t n d t square is equal to minus alpha n square or minus n square pi square.

So, if we do that, it becomes d square t n d t square plus n square pi square t n will be is equal to 0, will be looking into the solution of this. So, if this be d, square to n t n d t square plus n square pi n pi square t n is equal to 0. So, as you have seen earlier the solution of this equation will be the combination of sin function then cosine function. So, t n of t will be, let us say C 2 sin n pi t plus C 3 cosine n pi t.

So, let us look into the solution corresponding to nth eigenvalue. So, this will be nothing, but x n t n. So, it will be C 1 sin n pi x multiplied by C 2 sin n pi t plus C 3 cosine n pi t. So, I take this inside, this will be sin n pi x C 1 C 2 it will be new constants, let us say C 1 prime sin n pi t plus C 3 prime cosine n pi t. So, complete solution becomes now summation superposition of all the solution n is equal to 1 to infinity. So, this will be summation of n is equal 1 to infinity C 1 prime C n C n 1 prime sin n pi x sin n pi t plus C n 2 prime sin n pi x cosine n pi t.

So, that gives the complete solution now there are 2 constants in this in this final expression C n 1 prime and C n 2 prime and we have we have 2 initial conditions which have remained unutilized. So, will be utilizing these 2 conditions and see what we get. So, you utilize these 2 initial conditions in order to evaluate the constants C n 1 prime C n 2 prime.

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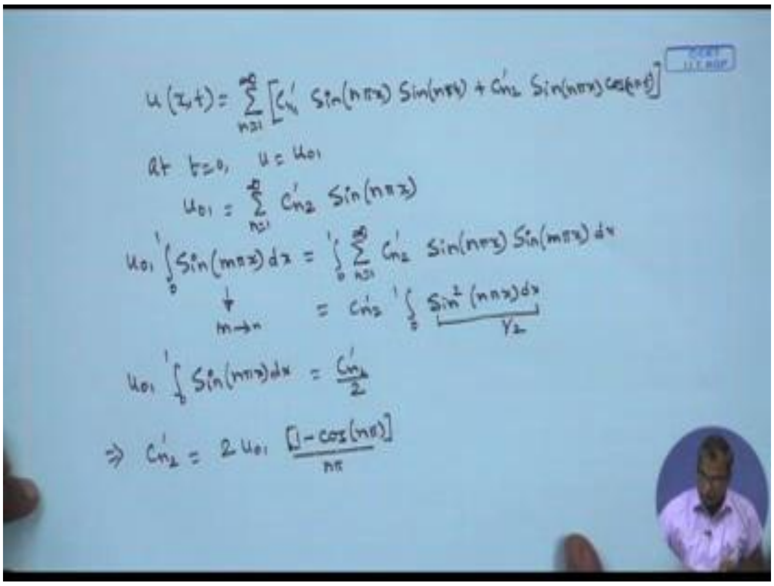

So, u of x t is equal to summation of n is equal to 1 to infinity C n 1 prime sin n pi x sin n pi t plus C n 2 prime sin n pi x cosine n pi t. So, that have that is the complete solution may utilize the 2 initial condition at t is equal to 0 u is equal to u 0, 1. So, therefore, you write u 0 1 is equal to summation n is equal to 1 to infinity. So, at t is equal to 0 sin 0, 0 is 0 and cos 0 is 1. So, only 1 thermal survive out of these two. So, this will be C n 2 prime sin n pi x. Now, will be evaluating the constant C n 2 prime by exploiting the orthogonal property of the Eigen functions if you do that I will be multiplying both side by sin n pi x d x and u 0, 1 is constant. So, it will be out. So, 0, 2, 1 sin m pi x d x is equal to 0 to 1 summation n is equal to 1 to infinity C n 2 prime sin n pi x sin m pi x d x.

So, if we open up the summation series all the terms will vanish only 1 term will survive that will be 1 n is equal to m. So, it will be 0 2 1 sin square n pi x d x next I will be changing the running variable on the left hand side to n m from n to m. So, it becomes u 0 1 0 to 1 sin n pi x d x is C n 2 prime and we have already seen that this integral is half. So, it will be C n 2 prime by 2. So, therefore, we can have C n 2 prime is equal to 2 u 0 1 one minus cosine n pi divided by n pi. So, that will be the expression of C n 2 prime. So, next what will be doing will be utilizing the other initial condition and to get the other constant C n 2 prime.

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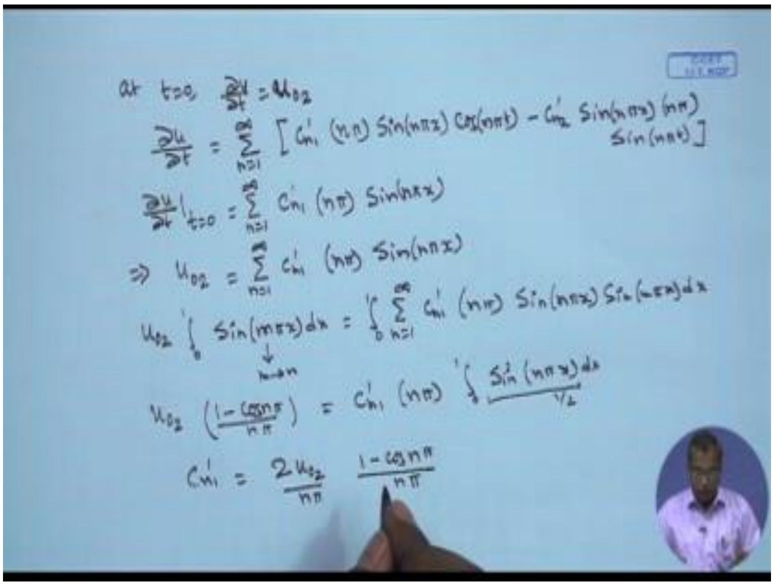

So, the other condition is at t is equal to 0 del u del t will be is equal to 0. So, it has evaluate what is del u del t del u del t is summation n is equal to 1 to infinity C n 1 prime n pi sin n pi x cosine n pi t and minus C n 2 prime sin n pi x into n pi sin n pi t. So, there is del del u del t we evaluate del u del t at t is equal to 0, if you do that when this term will vanish because sin 0 is 0 and cos 0 is 1. So, that will survive. So, I will be having C n 1 prime multiplied by n pi sin n pi x and del u del t at t is equal to 0 is u 0, 2.

So, if you remember the original boundary condition or initial condition at t is equal to 0 del u del t was u 0 2. So, you write u 0 2 is equal to summation n is equal to 1 to infinity C n 1 prime n pi sin n pi x. So, I multiply again I will be exploiting the orthogonal property of the Eigen functions and evaluate sin n 1 prime. So, I multiplied both side by sin m pi x d x and then integrate over the domain of x from 0 to 1. So, u 0 2 is constant. So, this will be 0 to 1 sin m pi x d x is equal to 0 to 1 summation n is equal to 1 to infinity C n 1 prime n pi sin n pi x sin m pi x d x and will be 1 term will survive all the other terms will vanish because of the orthogonal property of the sin function then I change the running index m into n.

So, this will be giving you 1 minus cosine n pi divided by n pi and on the right hand side I will be getting C n 1 prime n pi integral sin square n pi x d x that will be nothing, but half. So, C n 1 prime is nothing, but 2 u 0 2 by n pi 1 minus cosine n pi divided by n pi.

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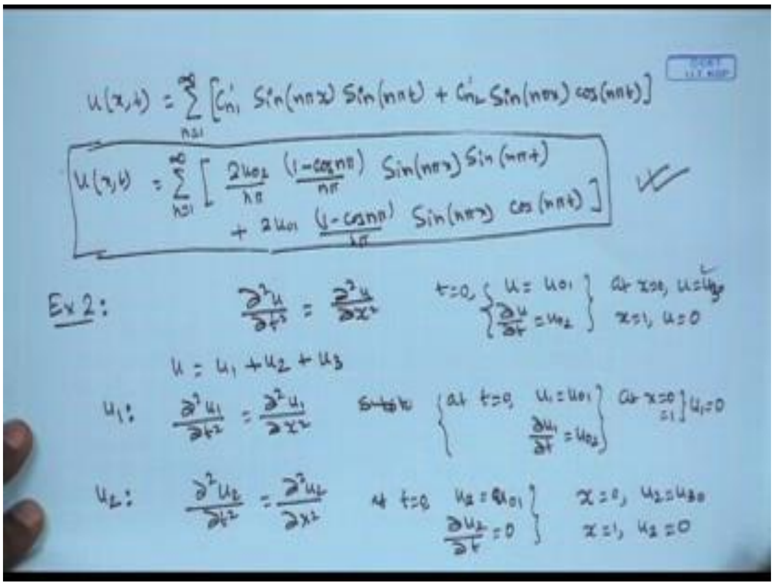

Now, will be able to get the complete solution, if you look into the complete solution the complete solution will be u x t is equal to summation n is equal to 1 to infinity C n 1 prime sin n pi x sin n pi t plus C n 2 prime sin n pi x cosine n pi t. Now, will be substituting the value of C n 1 prime and C n 2 prime, this will be n is equal to 1 to infinity bracket 2 u 0 2 by n pi 1 minus cosine n pi over n pi sin n pi x sin n pi 2 n pi t plus C n 2 prime is plus 2 u 0 1, 1 minus cosine n pi divided by n pi sin n pi x cosine n pi t.

So, that gives the complete solution of u as a function of x and t for the hyperbolic problem we are dealing with next what I will be doing I will be taking 1 more example of hyperbolic problem where 1 of the boundary condition is having a non homogeneous value then will be breaking down this problem into 2 sub problems considering 1 non homogeneity at a time and then will see how to get the solution of the sub problem and constant the complete solution.

So, next example that will be taking up is del square u del t square is equal to del square u del x square where at t is equal to 0 I of u is equal to u 0 1 and del u del t is equal to u 0 2 and at x is equal to 0 I have u is equal to u 1 naught and u 1 naught is already there. So, let us say u 3 naught and at x is equal to 1 u is equal to 0. So, I have a non homogeneity in the in all the boundary condition. So, what I do as usual I will be breaking down these 2 pro this problem into 2 sub problems considering 1 non homogeneity at a time.

So, I will be substituting I will be breaking down this problem into 2 sub problems says that u is equal to u 1 plus u 2 and the formulation of u 1 is del square u 1 del t square is equal to del square u 1 del x square subject to at t is equal to 0 u 1 is equal to u 0 1 and del u 1 del t is equal to u 0 2. So, there is the conditions at t equal to 0 at x is equal to 0 and 1 u 1 is equal to 0. So, this is a very well posed problem of the hyperbolic partial differential equation and we have already seen the solution of this problem the solution of this problem is given by these expression.

So, there is there next will be looking into the other part u 2 u 2 is del square u 2 del t square is equal to del square u 2 del x square, where at t is equal to 0 my u 2 is equal to 0 del u 2 del t is equal to 0 subject to x is equal to 0 I have u 2 is equal u 3 naught and at x is equal to 1 I have u 2 is equal to 0. So, I will be considering, 1 non homogeneity at a time. So, therefore, we can say that will be u 2 is equal to 0 and del u 2 del t equal to 0.

In fact, there are there sources on non homogeneity, will be considering non-homogeneity at a time. So, I can keep this non homogeneity intact u 2 is equal to u 0 1 and in the second case I will be considering the other non homogeneity. So, it will divided into 2 sub 3 sub problems because there are 3 sources of non homogeneity 1 2 and 3.

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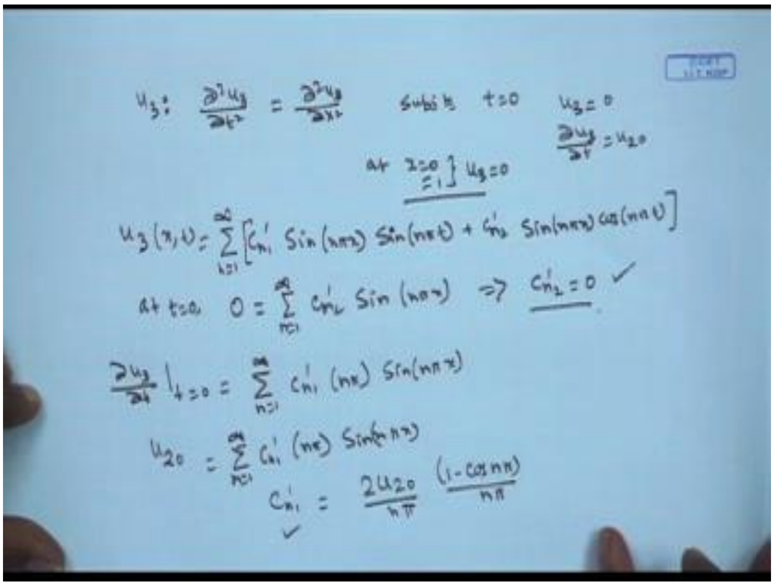

So, u 3 will be del square u 3 del t square is equal to del square u 3 del x square subject to t is equal to 0 u 3 is equal to 0 and del u 3 del t is equal to u 2 naught and at x is equal to 0 and 1 my u 3 is equal to 0. So, I have to solve just 1 equal 1 one 1 sub problem u 2 or u 3 u 1 is completely solve earlier, and let us see how because if you solve either u 2 and u 3 and will be no how to solve individually in a each sub problem, otherwise look in to this u 3. So, it will be depending on, will be having an initial condition which is 0. So, I will be getting this sub problem. So, the first 1 is, this we how to how this solve this equation. So, these will be a standard eigenvalue problem in the x direction and we have already seen how to how to solve this equation.

Therefore, if you look in to the standard solution the solution will be u x u 3 x t will be nothing, but summation of n is equal to 1 to infinity C n 1 prime sin n pi x sin n pi t plus C n 2 prime sin n pi x cosine n pi t. So, at t is equal to 0 at t is equal to 0 u 3 is equal to 0. So, these will be equal to 0 and n is equal to 1 to infinity sin 0 0 is 0. So, C n 2 prime sin n pi x.

So, now since this coefficient is 0 the associated constant will be equal to 0 then we took the other 1 del u 3 del t is equal to 0. So, del u 3 del t at t is equal to 0 will be will be giving you the value n is equal to 1 to infinity C n 1 prime n pi sin n pi x and del u 3 del t is equal to u 2 naught. So, I will be having u 2 naught summation n is equal to 1 to infinity C n 1 prime n pi sin n pi x and I can evaluate C n 1 prime as u 2 naught 1 minus cosine n pi divided by n pi and there will be 2 over here n pi. So, this way we can evaluate C n 2 prime and C 1 n 1 prime this will be evaluated by the using orthogonal property and we can constant the complete solution of u 3 by substituting these 2 coefficients.

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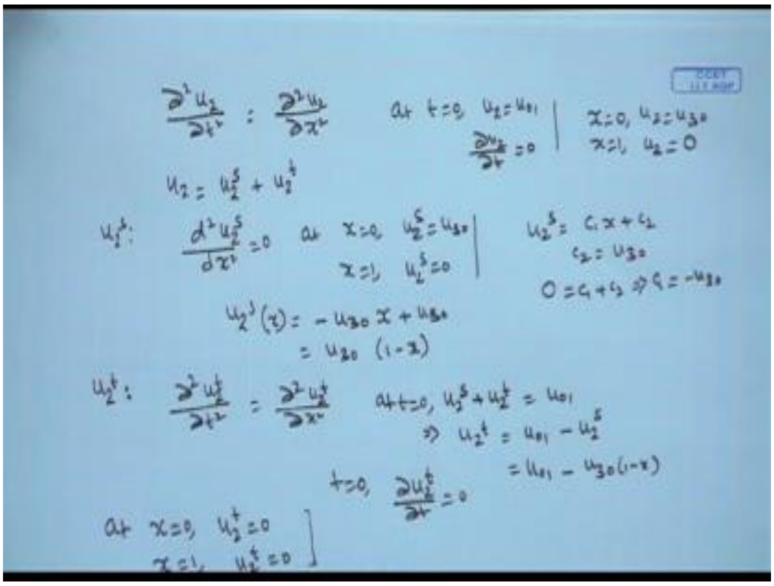

Now, which will be very, very interesting to solve the you know problem of u 2 now let us look in to the problem of u 2 del square u 2 del t square will be is equal to del square u 2 del x square at t is equal to 0 u 2 is equal to u 0 1 and t is equal to 0 del u 2 del t will be equal to 0 and we have at x is equal to 0 u 2 is equal to u 3 naught and at x is equal to 1 we have u 2 is equal to 0. So, will be having a non homogeneous boundary condition here and therefore, we can we can we can we can. So, we can break down this problem into 2 sub problem u 2 is equal to u 2 s plus u 2 t

So, the governing equation of u 2 s will be d square u 2 s d x square that we have done earlier is equal to 0 at x is equal to 0 u 2 s is equal to u 3 naught and at x is equal to 1 u 2 s is equal to 0 we know the solution of this solution of this u 2 s is equal to C 1 x plus C two. So, at x is equal to 0 u 2 s is equal u 3 naught. So, C 2 is equal to u 3 naught. So, and at x is equal to x equal to 1 u 2 s is equal to 0. So, it will be C 1 plus C 2. So, C 1 is equal to minus u 3 naught.

So, the solution of u 2 s usually be a function of x is minus u 3 naught x plus u 3 naught. So, u 3 naught will be nothing, but 1 minus x. So, therefore, we can look into the solution of governing equation of u 2 t u 2 t is be will be del square u 2 t del t square is equal to del square u 2 t del x square at t is equal to 0 u 2 is equal to u 2 u 0 1. So, it will be u 2 s plus u 2 t is equal to u 0 1 and u 2 t will be nothing, but u u 0 1 minus u 2 s and it will be u 0 1 minus u 3 naught into 1 minus x that is u 2 t at t is equal to 0.

Similarly, at t is equal to 0 del u 2 t del del t equal to 0. So, at t is equal to 0 and in the del u 2 t del t will be equal to 0 because the special varying part the time with the time derivative of that will be equal to 0 and the boundary condition at x equal to 0 I have my u 2 t is equal to 0 and at x is equal to 1 u 2 t is equal to 0. So, will be having a standard eigenvalue problem in the x direction and we will be having a non 0 initial condition at t equal to 0 and del u 2 del t will be equal to 0 at t equal to 0. So, will be will be getting the almost same similar type of solution if you look in to the solution you can see that the solution will be will be something like this.

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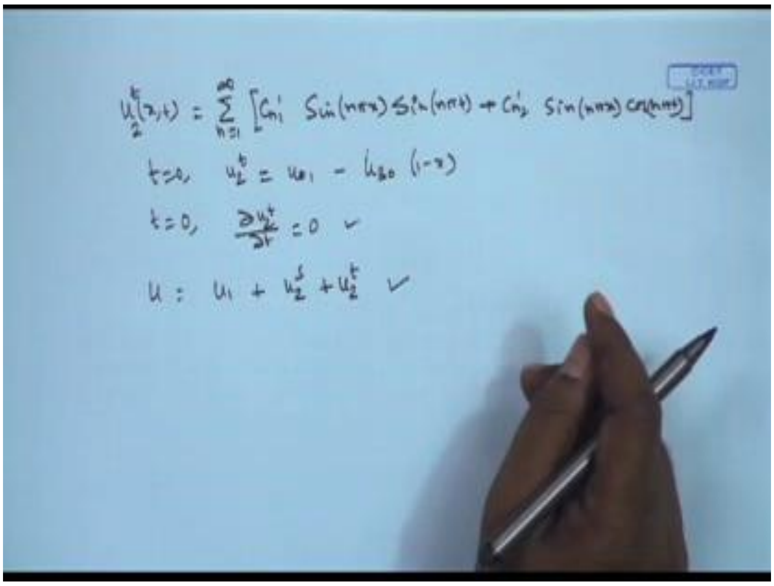

U x t will be summation n is equal to 1 to infinity C n 1 prime sin n pi x sin n pi t plus C n 2 prime sin n pi x cosine n pi t and we have 2 initial condition at t is equal to 0 u 2 u u 2 u 2 is equal to u 2 t.

So, this will be u 2 t will be equal to u 0 1 minus u 3 naught 1 minus x and at t is equal to 0 we have del u 2 t del t will be equal to 0. So, because of this boundary initial condition 1 of the constants will vanish and the other constant will be surviving. So, this will be the solution of u 2 t then will be adding up all individual solution and getting the complete solution u is equal to u 1 plus u 2 s plus u 2 t. So, that will be that will be constructing the complete solution. So, that computes the solution of hyperbolic partial differential equation I stopping in this class the next class I will be moving over to the radial polar coordinate cylindrical polar coordinate system.

Thank you very much.