Partial Differential Equations (PDE) for Engineers:
Solution by Separation of Variables
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Lecture - 12
Solution of Elliptical PDE

Welcome to this session. So, we were looking into the solution of elliptical partial differential equations which will be representing the steady state of any process. So, let us look into a two dimensional problem Elliptical Partial Differential Equation.

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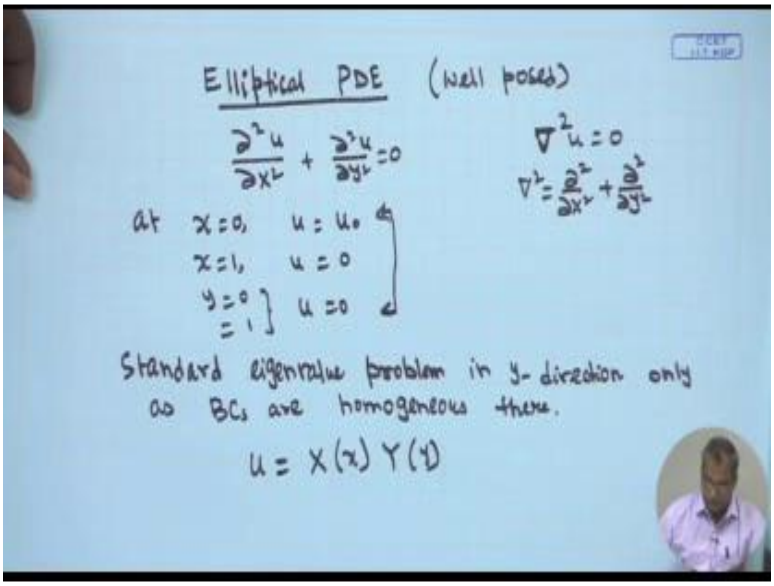

And we will be looking into the well posed problem because in an actual problem it will be an ill posed problem because all the boundary conditions may not be homogeneous at proper places proper boundaries. So, therefore, you have already seen how to break down the problem depending on the number of non-homogeneity is present into the system how to break down the problem into sub problems considering 1 non-homogeneity at a time and thereby you know forcing a converting an ill posed problem into an well posed problem.

So, therefore, now we will be looking into the well posed problem if we know the solution of the well posed problem we can always convert an ill posed problem into a well posed problem and go ahead with the solution. So, the first problem will be dealing with is del square u del x square plus del square u del y square is equal to 0. So, it is basically Laplacian, Laplacian Equation is given by this del square u in two dimensional the graph square operation operator or the Laplacian Operator is represented as del square del x square plus del square del y square. So, let us fix up the boundary conditions at x is equal to 0 you have u is equal to u naught at x is equal 1 we have u is equal to 0 and y equal 0 and 1 we have u equal to 0. Let us consider a Dirichlet boundary condition at x is equal to 0 that is u is equal to u naught that is specified in all the three boundaries; that means, at x is equal 1 and y is equal to 0 and 1 we have the homogeneous boundary conditions.

So, if you look into this system we will be having a standard eigenvalue problem in the y direction only not in x direction because the boundary conditions in the y direction are homogeneous. So, we will be having a standard eigenvalue problem in y direction only as b c's are homogeneous there. So, again it is a linear operator. So, we go ahead with a separation of variable type of solution consider u is a product of sole function of x and sole function of y. So, we are then going to substitute this into the governing equation. So, if we do that you will be getting y d square x d x square plus x d square y d y square is equal to 0 we multiply divide both sides by x y.

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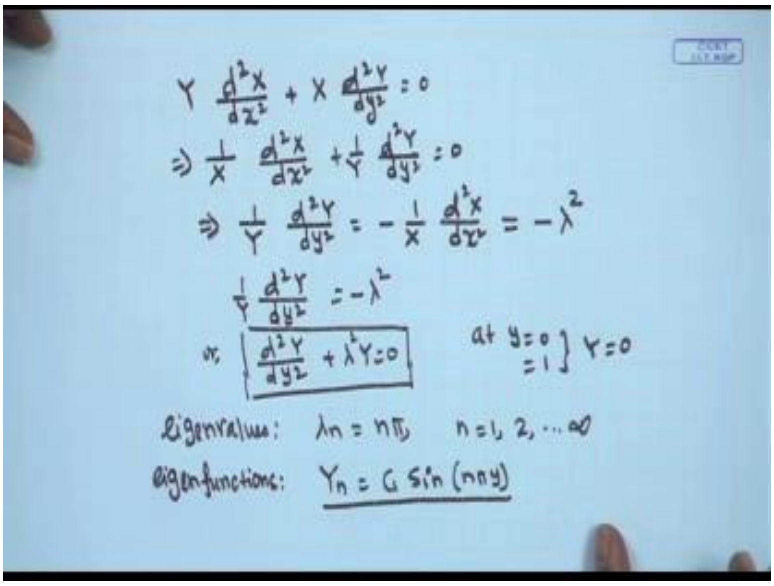

So, we will be getting 1 over x d square x d x square plus 1 over y d square y d y square is equal to 0. So, we will be formulating a standard eigenvalue value problem in the y direction because the boundary conditions are homogeneous in that direction. So, write 1 over y d square y d y square is equal to minus 1 over x d square x d x square, the left hand side is a function of y the right hand side is a function of x alone they equal they will be equal to some constant and this constant has to be a negative constant in order to have a Non-Trivial Solution in the y direction.

So, you formulate the eigenvalue problem in the y direction d square y d y square 1 over y is equal to minus lambda square or d square y d y square plus lambda square y is equal to zero. So, this is the eigenvalue problem that we formulate in the y direction and the boundary conditions are also satisfied by the original boundary condition in the y direction at y is equal to 0 and y is equal to 1 capital Y is equal to 0 and we have already seen the solution of this the eigenvalues are n pi at the index n runs from 1 to infinity and eigen functions or the sine functions it is y n is equal to c 1 sine n pi y. So, these will be giving the solution of y varying part. Now let us look how the solution of x varying part will be obtained. So, if you look into the x varying part that is very very important 1 minus 1 over x; d square x d x square is equal to minus lambda n square.

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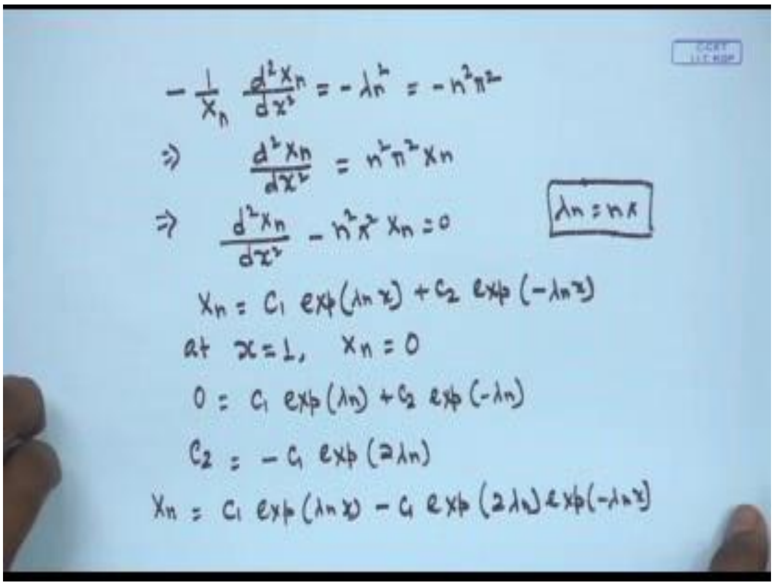

So, that is since lambda n is n pi. So, that will be n square pi square. So, write a subscript in to denote the solution corresponding n'th eigenvalue. So, d square x n d x square is equal to n square pi square x n. So, if we take it to the other side d square x n d x square should be minus n square pi square x n will be equal to 0. So, we know the solution of these the solution will be consisting of e to the power n pi x and e to the power minus n pi x. So, if we do that. So, x n will be nothing, but c 1 exponential lambda n x plus c 2 exponential minus lambda n x where lambda n is nothing, but n pi. So, this is the solution of x varying part

Now let us look into the governing equation the boundary condition, the boundary condition of the x varying part must satisfy the boundary condition of the original problem; that means, at x is equal to 1 we had the homogeneous boundary condition that is x is equal to 1 we had capital X n is equal to 0. So, if you put that boundary condition let us see what we get 0 is equal to c 1 exponential lambda n plus c two exponential minus lambda n. Therefore, we can get c 2 is equal to minus c 1 exponential 2 lambda n. So, once we do that then we can write down the governing equation of X n, so c 1.

Exponential lambda n x c 2 is minus c 1 exponential 2 lambda n exponential minus lambda n x so.

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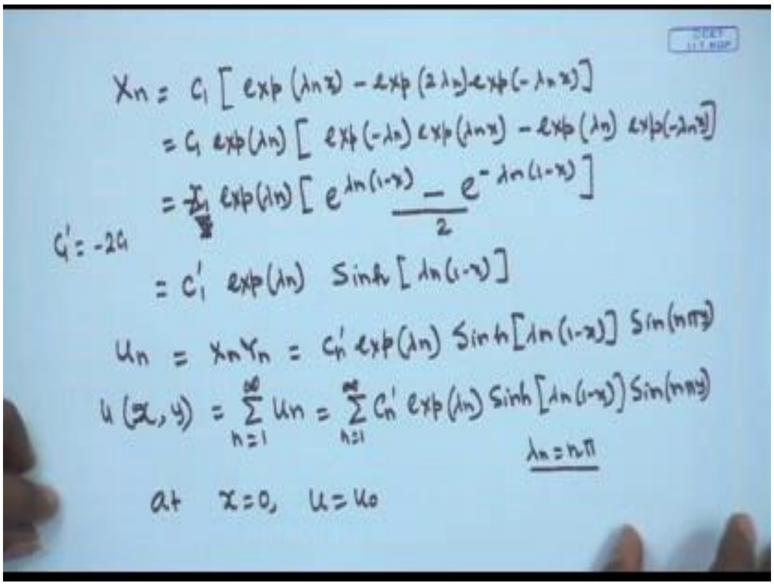

We take c 1 common, so that it will give you exponential lambda n x minus exponential 2 lambda n exponential minus lambda n x and then we take exponential lambda n x exponential lambda n common outside. So, it becomes exponential minus lambda n exponential lambda n x minus exponential lambda n exponential minus lambda n x bracket end then you take minus common. So, it will be a c 1 exponential lambda n these becomes this will be in the front so it will be e to the power lambda n 1 minus x minus e to the power minus lambda n 1 minus x then we multiply and divide by 2 if we do that this becomes sine hyperbolic function.

So, it minus c 1 by 2 will be a new constant let us say c 1 prime exponential lambda n and these become this will be 2 this will be multiplied by 2, multiplied by 2 and divide by 2. So, minus 2 C 1 will be C 1 prime C 1 prime will be a new constant that will be minus 2 C 1. So, c 1 prime exponential lambda n and these will become sine hyperbolic lambda n 1 minus x. So, that constitutes the solution of x n.

So, we will be able to write down the complete solution u n is nothing, but x n y n. So, u n y this is c 1 prime exponential lambda n sine hyperbolic lambda n 1 minus x times sin n pi y and u will be as a function of x and y will be summation of all these solution by putting them up by using the principle of linear superposition. So, we construct the complete solution by summation of u n where the index n runs from 1 to infinity and this will be c 1 prime multiplied by c 1 it will be a new constant may be. So, it will be new constant C n prime. Summation of C n prime n is equal 1 to infinity exponential lambda n sine hyperbolic lambda n 1 minus x sine n pi y while lambda n is nothing, but n pi.

Now we have the complete solution in front of us except the evaluation of the constant c n prime and c n prime now will be evaluated from the unused boundary Non-Homogeneous Boundary Condition that is at x is equal 0 u is equal to u naught. So, utilized the condition at x is equal 0 u is equal to u naught if we put that boundary condition we will be getting u is equal to u naught summation n is equal to 1 to infinity C n prime exponential lambda n sine hyperbolic lambda n 1 minus 0 that is x is equal 0 sine n pi y.

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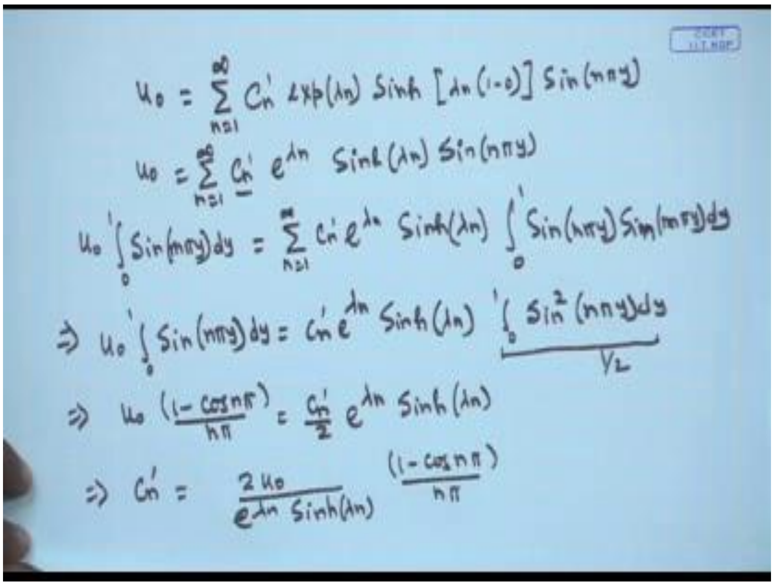

So it will becomes u naught is equal to n is equal 1 to infinity c n prime e to the lambda n sine hyperbolic lambda n sine n pi y. Now this C n prime now evaluated by utilizing the Orthogonal Property of the Eigen functions that is the sine n pi y sine function. So, I multiply both sides by sine n pi y d u y and integrate over the domain of y from 0 to 1 see what you get 0 to 1 sine n pi y d y and this will be sine n pi y d y and this will be n equal to 1 to infinity C n prime e to the power lambda n sine hyperbolic lambda n this will be sine n pi y sine n pi y d y from 0 to 1 and using the Orthogonal Property.

Once you open up the summation series all the terms will vanish except one term where m is equal to n and that term will be having on the right hand side the summation side by opening up the summation we will be having one term in the right hand side that will survive it will be integral sine square n pi y d y. Then changing the running index from m to n we will be having u 0 0 to 1 sine n pi y d y will be C n prime e to the power lambda n sine hyperbolic lambda n 0 to 1 sine square n pi y d y and you know these will be returning a value of half.

So, it becomes u 0 1 minus cosine n pi over n pi C n prime by 2 e to the power lambda n sine hyperbolic lambda n and we will be able to get the solution of C n prime as 2 u 0 e to the power lambda n sine hyperbolic lambda n 1 minus cosine n pi divided by n pi. Now we will be substituting C n prime into the complete solution and see what we get. So, if you look into that the complete solution is now.

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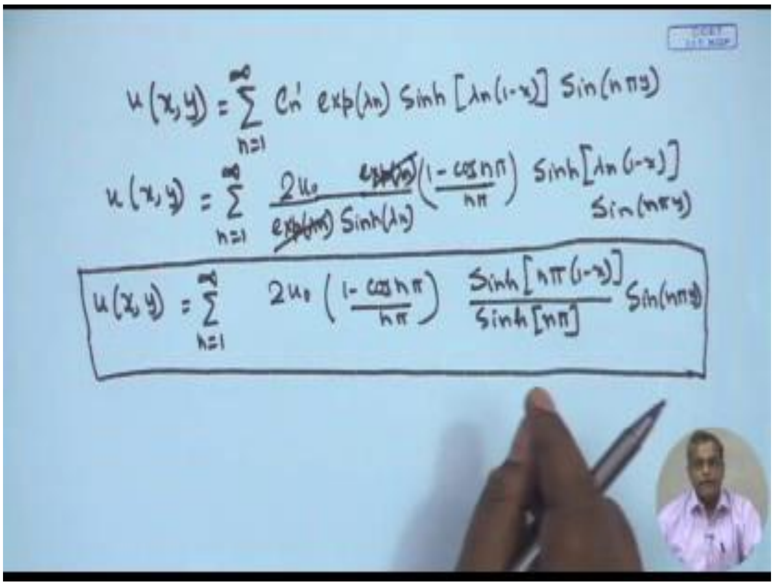

If you look into the solution as in terms of C n prime it will it was n is equal to 1 to infinity C n prime exponential lambda n sine hyperbolic lambda n 1 minus x sine n pi y.

So, therefore, now I am going to substitute c n prime expression of c n prime. So, this will be n is equal to 1 to infinity C n prime is 2 u 0 exponential lambda n sine hyperbolic lambda n 1 minus cosine n pi over n pi sine hyperbolic lambda n 1 minus x sine n pi y multiplied by exponential lambda n. So, there was exponential lambda n here. So, you can write exponential lambda n there.

So, this exponential lambda n and exponential lambda will be canceled out and you will be getting the complete solution as; n is equal 1 to infinity 2 u 0 1 minus cosine n pi over n pi sine hyperbolic lambda n is n pi, so it will be n pi 1 minus x divided by sine hyperbolic lambda n that is n pi sine n by y. So, that gives the complete solution of an elliptical partial differential equation in two dimension where the boundary condition at x is equal to 0 is u naught and all the three boundary conditions are homogeneous and Dirichlet. So, you will be getting the complete solution like this.

So, in the next example I will be considering the boundary condition instead of boundary condition at x is equal 1 x is equal 0 I will be considering the boundary condition at x is equal 1 is non-zero all the other boundary conditions are homogeneous and zero. So, let us look into that. So, we will be solving the partial differential equation del square u del x square plus del square u del y square is equal to 0.

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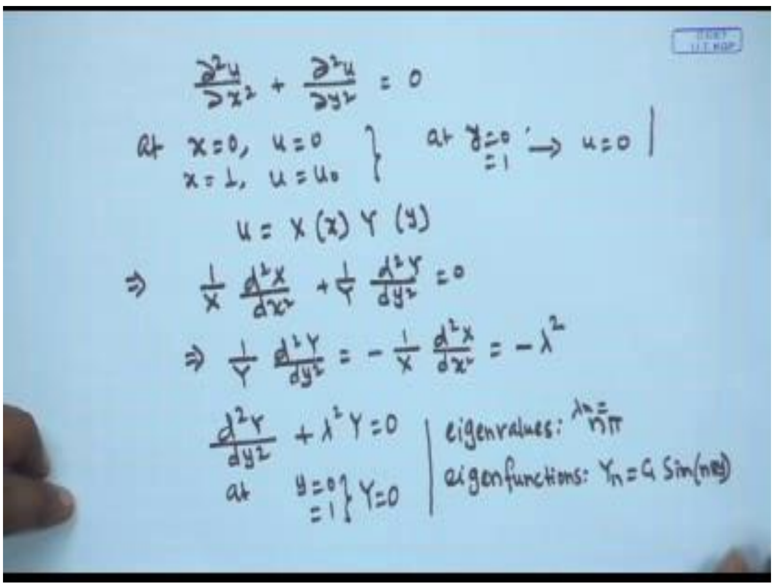

And at x is equal to 0 we have u is equal to 0 at x is equal 1 we have u is equal to u naught and at x is equal at y is equal to 0 and 1 we have u is equal 0. So, now, in this particular problem the boundary condition located at x equal to 1 will be assuming a non-zero boundary condition Non-Homogeneous Boundary Condition, but everywhere else it is a Dirichlet boundary condition let us say how the solution looks like we will be getting a separation of we will be attempting a separation of variable type of solution after separating the variables you will be having 1 over x d square x d x square plus 1 over y d square y d y square is equal to 0 we will be formulating a standard eigenvalue problem in the x in the y direction because the boundary conditions are homogeneous there.

So, therefore, 1 over y d square y d y square will be is equal minus 1 over x d square x d x square is equal to minus lambda square in order to have a Non-Trivial Solution. So, if we now formulate the governing equation of y it will be nothing, but lambda square y is equal to 0 at y is equal 0 and y is equal 1 your capital y is equal to 0 in order to confirm the boundary condition of the original problem in the y direction we know the solution of this problem the eigenvalues are n pi Eigen functions are sine n pi y.

So, you have the eigenvalues as n pi lambda n is equal n pi and Eigen functions as y n is equal to c 1 sine n pi y now let us look into the uh x varying part. So, if you look into the x varying part we are going to get is d square x d x square minus lambda square x is equal 0.

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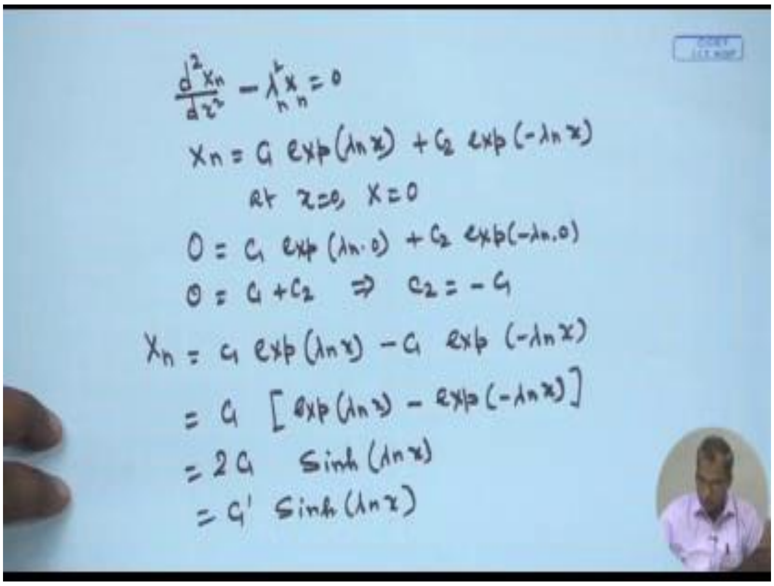

So, you can put n corresponding to n'th eigenvalue. So, x n will be c 1 exponential lambda n x plus c 2 exponential minus lambda n x. Now, the x varying part we had the in the boundary condition in the original problem at x is equal 0 capital x must be equal to 0 in the original problem we had at x is equal to 0 u is equal to zero. So, that part and at that boundary capital x must satisfy that; that means, at x is equal to 0 capital x must to be equal to zero. So, if you do that you get 0 is equal to c 1 exponential lambda n into 0 plus c 2 exponential minus lambda n into zero. So, it will be 1. So, we will be getting 0 is equal to c 1 plus c two from there will be getting c 2 is equal to minus c 1 if you do that then we will be getting x n is equal to c 1 exponential lambda n x minus c 1 exponential minus lambda n x. So, you will be having c 1 exponential lambda n x minus exponential minus lambda n x you multiply divide by 2. So, 2 C 1 will be having this will be sine hyperbolic function sine hyperbolic lambda n x.

So, this will be new constant let us say C 1 prime sine hyperbolic lambda n x. Now we will be able to write down the complete solution.

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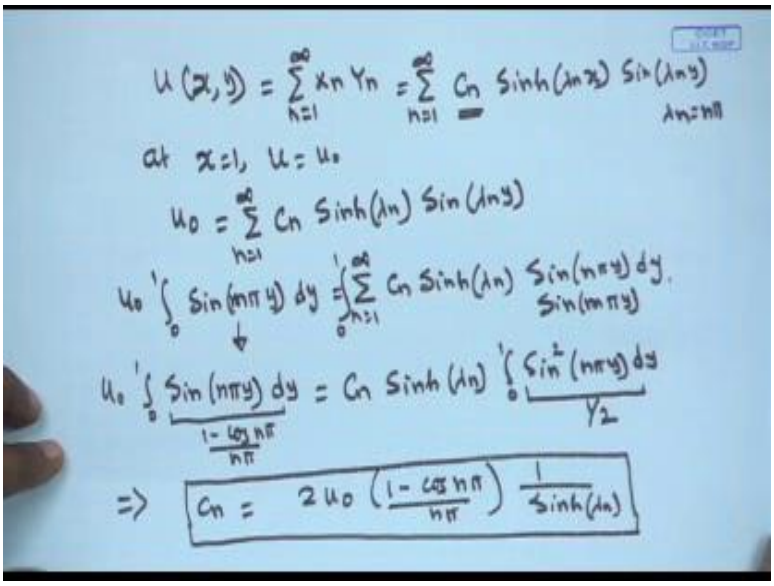

So, u will be u as a function of x and y nothing, but summation of x n y n where index n runs from 1 to infinity and this will be c 1 prime and c two it will be a new constant C n prime or may be C n sine hyperbolic lambda n x cosine sine lambda n y lambda n is basically n pi. Now we will be utilizing the boundary condition at the original problem at x is equal 1 u is equal to u naught and then to evaluate the undermined constant the left over coefficient c n and it will be u naught is equal to summation C n sine hyperbolic lambda n sine lambda n y.

Then you will be evaluating c n by utilizing the Orthogonal Property of sine function. So, if you do that you be getting u 0 integral 0 to 1 sine n pi y d y is equal to summation n is equal to 1 to infinity c n sine hyperbolic lambda n integral 0 to 1 sine n pi y d y and if you open up the summation series all the terms except 1 term all the other terms will vanish only 1 term will survive when lambda m is equal to lambda n. So, it will be sine we multiply both sides by sine m pi y and all the terms will vanish except m is equal to n and then we change the running coefficient m into n.

So, we will be getting 0 to 1 sine n pi y d y is equal to c n sine hyperbolic lambda n integral sine square n pi y d y 0 to 1 and you know this will be nothing, but half and these will be 1 minus cosine n pi over n pi. So, I will be in a position to evaluate c n now

I will be writing directly the expression of c n 2 u 0 1 minus cosine n pi over n pi divide by sine hyperbolic lambda n or n pi. So, that will give the complete solution of c n. Now you will be able to write down the entire solution in terms of u and x y u x y will be nothing, but summation of c n.

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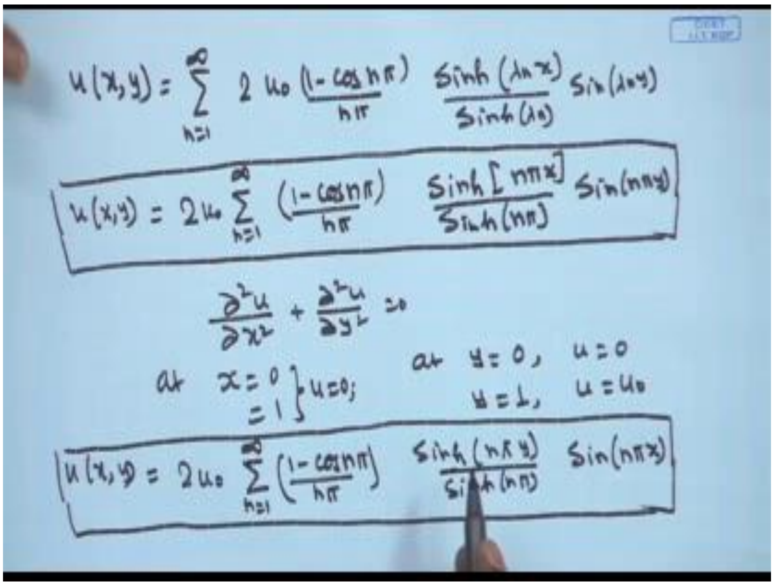

So, this will be two u naught 1 minus cosine n pi over n pi sine h hyperbolic lambda n x divide by sine hyperbolic lambda n sine lambda n y. So, lambda n is basically n pi so, it will be 2 u naught n is equal to 1 to infinity 1 minus cosine n pi divided by n pi sine hyperbolic n pi x divided by sine hyperbolic n pi sine n pi y that will be giving you the complete solution.

Similarly, we may have a non-homogeneity appearing in the x direction in the y direction and the boundary condition in the x direction may be homogeneous for a well posed elliptical partial differential equation; that means, we may have a problem something like this del square u del x square plus del square u del y square equal to 0 at x is equal to 0 and 1 u is equal to 0 at y is equal to 0 you have u is equal to 0 at y is equal 1 you have u is equal to u naught.

In this case you will be we formulating a standard eigenvalue problem in the x direction because the boundary conditions are homogeneous in the x direction and in the y direction you will be having the hyperbolic functions. So, the solution of this equation will be exactly similarly to this only the thing is that the eigenvalue value problem will be defined in the x direction and two u naught it will be two u naught n is equal to 1 to infinity 1 minus cosine n pi divided by n pi sine hyperbolic n pi y divide by sine hyperbolic n pi and this will be sine n pi x because you will be having a standard eigenvalue problem in the x direction.

Similarly, we can have the non-homogeneity at y is equal to 0 and homogeneity at y is equal to 1 and x in the x direction both the boundary are homogeneous. So, in that case we will be having a standard eigenvalue problem in the x direction and in the other side of y direction will be having the other problem. So, it will be the first elliptical problem we have solved, but in this case the eigenvalue problem will be in the x direction instead of y direction. So, that is how the well posed problems in the elliptical well posed elliptical partial differential equation can be solved.

So, I will stop here. In this class in the next class what will we doing we will be looking into a an actual three dimensional which will be having more than one non-homogeneity and I will break down the problem into other sub problems and we will be then convert the ill posed problems into the well posed problem and we know the solution of the well posed problem. So, after that we will be looking into the solution of hyperbolic function differential equation well posed hyperbolic partial differential equation then we will move ahead with the Cylindrical Coordinate System and Spherical Polar Coordinate System which will be quite often in any process in generic system.

Thank you very much.