Partial Differential Equations (PDE) for Engineers:
Solution by Separation of Variables
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Lecture - 11
Solution of 4 Dimensional Parabolic Problem (Contd.)

Welcome to this session. So, in the last class were looking into the solution of 4 dimensional problem, 3 dimension in space and 1 dimension in time and we are looking into the boundary conditions; Neumann boundary condition at x equal to 0 and at z equal to 0, rest 4 boundaries they are remaining at the Dirichlet boundary conditions. So, now, we solve the problem to the half way, we have completely solved the x varying part and we have seen the Eigen values, where basically the 2 n minus 1 pi by 2 and Eigen functions were the cosine functions.

Now, we are we came up to the y varying part and we have formulated the governing equation of the y varying part with the boundary conditions.

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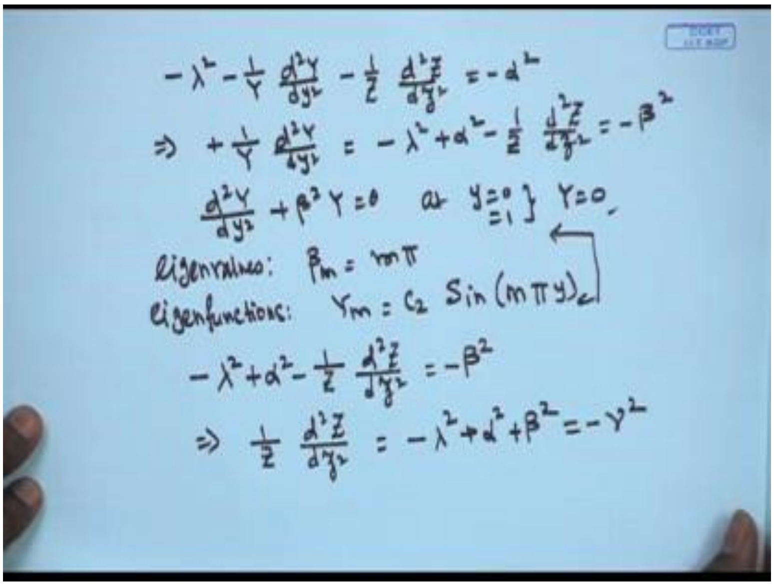

So, we know the solution of this Eigen values of this problem are beta m is equal to m pi because we have all homogeneous boundary conditions in all the 3 directions. We can construct the standard Eigen value problems independent which are independent to each other in all reactions. Therefore, will be writing 3 subscripts corresponding to 3 directions x y and z, corresponding Eigen functions are sin function. So, y m is equal to c 2 sin m pi y.

Now, let us look into the so that gives the complete solution of the y varying part and let us look into the z varying part, so that will be coming from there. So, this is the governing equation of z varying part minus lambda square plus alpha square minus 1 over z d square z d z square is equal to minus beta square. So, therefore, 1 over z d square z d z square will be nothing, but minus lambda square plus alpha square plus beta square alpha square. We have seen 2 n minus 1 pi by 2 and beta square, we have seen m square pi m square pi square. So, therefore, this constant has to be a negative constant, minus gamma square in order to have a non trivial solution of z. Now, let us formulate the Eigen value the governing equation of z varying part.

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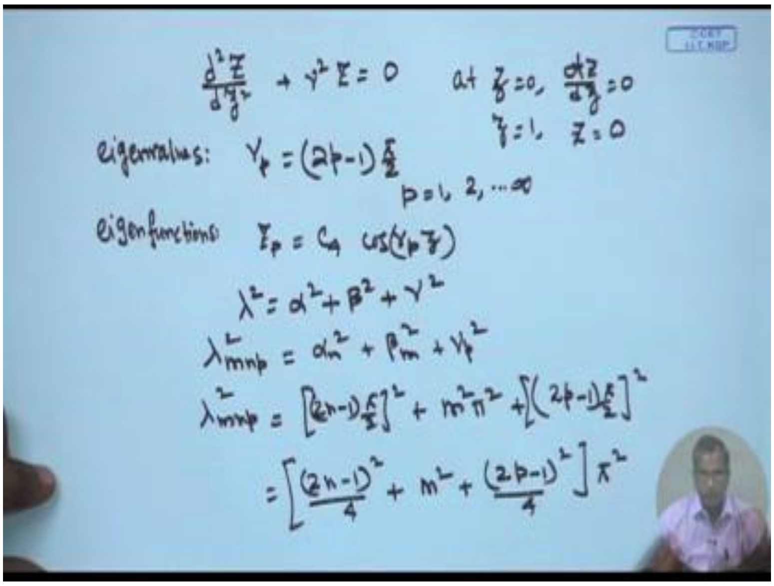

So, it will be d square z d z square plus gamma square z will be equal to 0 and the boundary conditions of the z varying part must satisfying the boundary condition of the original problem at z equal to 0. We have d z d z equal to 0 because in original problem we had the Neumann boundary condition prevailing at z equal to 0 and at z equal to 1, we had z equal to 0. So, again we know the solution of this problem the solution is that Eigen values are 2 n minus 1 pi by 2. In fact, this will be 2 p minus 1 pi by 2 Eigen values basically gamma p is equal to 2 p minus 1 pi by 2 where index p runs from 1 2 up to infinity and Eigen functions will be z p is equal to c 4 cosine gamma p z.

So, the complete solution we are now able to get the complete solution before that, we have to get the time varying part. So, let us look into the time varying part lam, what is lambda square? Lambda square will be nothing, but now alpha square plus beta square plus gamma square now let us put the subscript. So, this will be lambda m n p square is equal to alpha n square plus beta m square plus gamma p square. So, lambda m n p square alpha n was 2 n minus 1 pi by 2 square of that beta m beta m is m pi. So, m square pi square this will be 2 p minus 1 pi by 2 square of that. So, this will be 2 n minus 1 square by 4 plus m square plus 2 p minus 1 square by 4, all thing multiplied by pi square. So, this is the complete solution of lambda m n pi and once we get this lambda m n pi into the time varying part.

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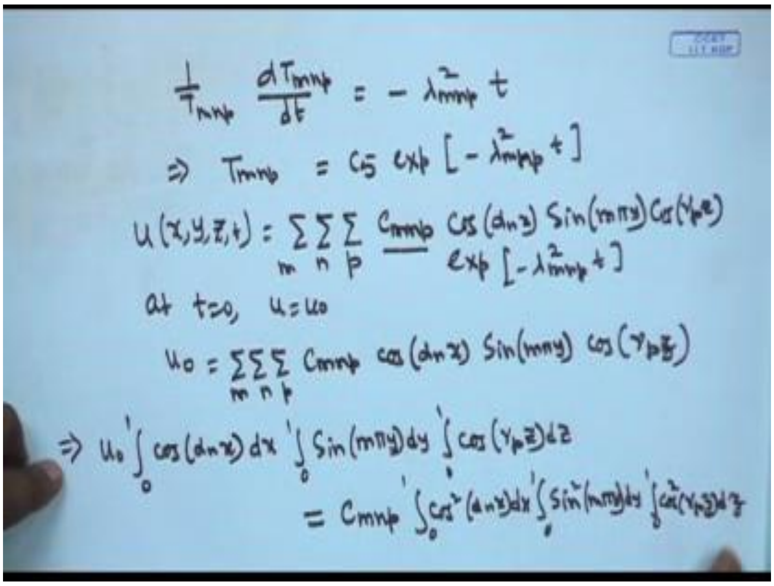

Then will be will be getting the solution of the time varying part 1 over t m n p d t m n p d t is equal to minus lambda m n p square times t. So, we will be getting t m n p is equal to lets a c 5 exponential minus lambda m n m n p square of that times t.

So, we will be able to construct the complete solution now and the complete solution is u as a function of x y z and t will be is equal to triple summation 1 over m another over n another over p all the constant will be multiplied giving rise to a new constant c m n p cosine alpha n x sin m pi y and cosine gamma p z where alpha n is nothing, but 2 n minus 1 pi by 2 gamma p is nothing, but 2 p minus 1 pi by 2. Now, will be getting the and multiplied by exponential the time varying part minus lambda m n p square times t we will be evaluating the c m n p by using the unutilized initial condition t is equal to 0 u was equal to u naught. If you put it there, you will be getting triple integral triple summation 1 over m n and another over p c m n p cosine alpha n x sin m pi y and cosine gamma p z.

So, this constant c m n p; now evaluated by exploiting the orthogonal property of the Eigen functions this sin functions and cosine functions and as we have done earlier will be getting the solution of c m n p as u naught integral cosine alpha n x d x from 0 to 1 multiplied by 0 to 1 sin m pi y d y sin cosine gamma p z d z on the left hand side in the right hand side when you open up the summation series all the terms will vanish because of the orthogonal property of the Eigen functions only 1 term will survive will be c m n p cosine square alpha n x d x from 0 to 1 0 to 1 sin square m pi y d y and 0 to 1 cosine square gamma p z d z and we have seen that all of these constants or all of these integrals will be nothing, but assuming the value of half. So, if you do that then will be getting the u naught 1 mi this will be the sin alpha n x by alpha n from 0 to 1 sin only 1 term will survive.

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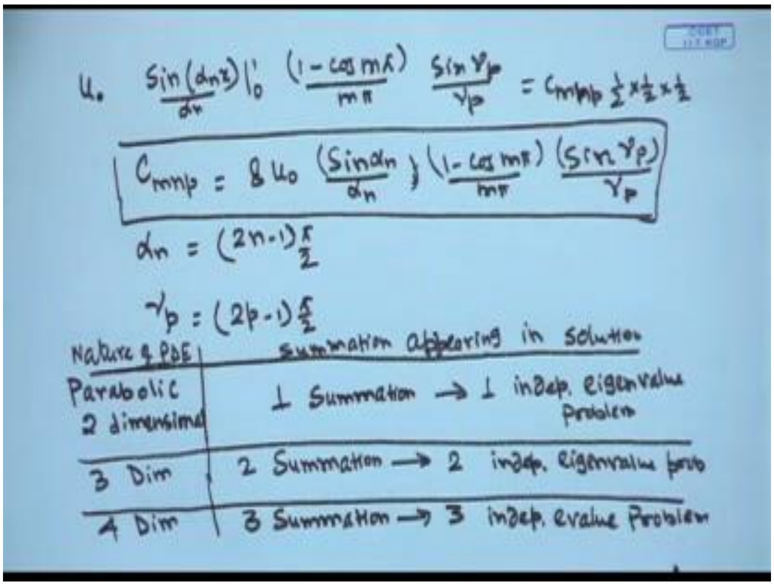

So, and this will be 1 minus after integral we have done several times 1 minus cos m pi over m pi and this will be sin gamma p divided by gamma p and in the right hand side we had half into c m n p half into half into half. So, c m n p will be nothing but 8 u 0 sin alpha n by alpha n multiplied by 1 minus cosine m pi over m pi sin gamma p over gamma p where alpha n is nothing, but 2 n minus 1 pi by 2 and gamma p is nothing, but 2 p minus 1 pi by 2.

So, that will be giving the expression of the constant and determined constants c m n p that has to be inserted into governs into the solution of u into this equation and it will be giving you the complete solution. So, depending on the type of boundary conditions that are prevailing in the governing equation 1 may land into the sin functions cosine functions at the Eigen functions and various types. So, Eigen values that we have come across that whether it will be the sin functions are the if the Dirichlet boundary conditions are prevailing on both the boundaries then the Eigen values will be n pi if the 1 of the condition at x equal to 0 is prevailing is the Neumann is presented by the Neumann boundary condition than 2 n minus 1 pi by 2 at the Eigen values and cosine functions is the Eigen functions and if a robin mixed is presented at x is equal to 1 and at x equal to 0 it is a Dirichlet then it will be a transcendental equation will arise involving the tan functions over the roots of this equations of the Eigen values and sin functions of the Eigen functions.

Now, if we can summarise this they have this problem then we can write down that the nature of p d e p d e and summation appearing in solution it will be very interesting problem that if it is a parabolic 2 dimensional problem then I will be have 1 summation in my solution because I have 1 independent Eigen value problem in my system if we are dealing with a 3 dimensional problem I will be having 2 summation in my solution because I have a 2 independent Eigen value problem if I deal with 4 dimensional problem then there will be 3 summation appearing in my governing equation because I will be dealing with 3 independent Eigen value problem.

So, now I will be considering 1 case before going into the elliptical partial differential equation we know the 3 standard boundary conditions of first kind second kind and third kind that for example, if we if we take the second kind boundary condition at x is equal to 0 del u del x is equal to 0 at x is equal to 1 u is equal to 0 now in this particular problem the boundary at x is equal to 0 is insulated and we know the Eigen values are cosine functions Eigen values are 2 n minus 1 pi by 2 and Eigen functions are the cosine function now if we reverse this boundary condition then; that means, the instead of insulation boundary at x is equal to 0 if it appears in x is equal to 1 the how my solution how will be getting into the solution. So, once that is cleared then if we have in in actual problem if we have the boundary conditions at different boundary or then will be able to tackle that problem as well.

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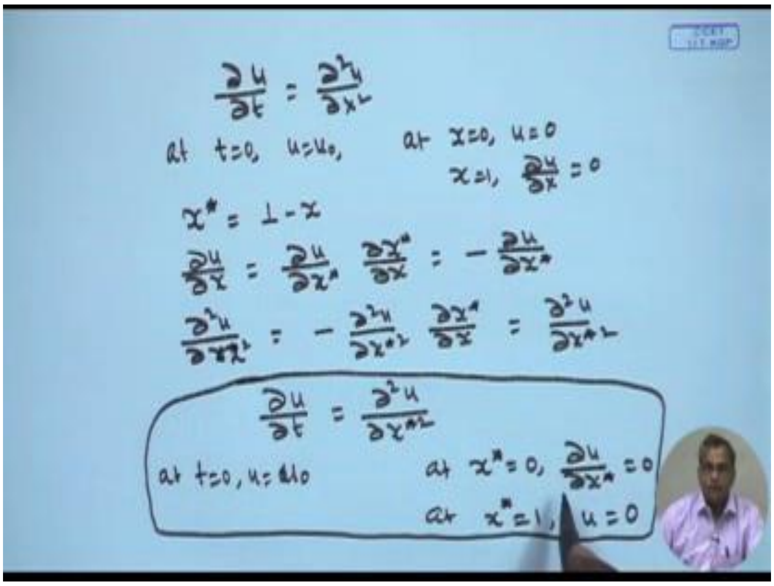

For example I am looking into a 2 dimensional problem 1 dimension in space 1 dimension in time del square u del x square at t is equal to 0 let us say u is equal to u naught and at x is equal to 0 u is equal to 0 and x is equal to 1 let us say the del u del x is equal to 0; that means, a Neumann boundary condition is prevailing at x is equal to 1 instead of x is equal to 0.

Now, what I will do next I will have a transformation in the coordinate system I will define x star is equal to 1 minus x now if I define the x star let us see del u what is del u del x del u del x is equal to nothing,, but del u del x star del x star del x. So, these will be del x star del x will be minus one. So, minus del u del x star and del square u del x star square will be nothing, but, one more derivative. So, it will be minus del square u del x star square del x star del x del x star del x is again minus one. So, it will become del square u. So, del x star square. So, I will be I will be I will be getting my problem as a fixed del u del t is equal to del square u del x star square at t is equal to 0 u is equal to 0 u is equal to u naught at x equal to 1 means at x star is equal to 0 at x star is equal to 0 my del u del x star is equal to 0 and at x is equal x star is equal to one; that means, x is equal to means x star equal to 1 my u is equal to 0.

So, now this is a problem that we have already solved in this class only thing is now the variable has been changed from x to x star where x star is defined as 1 minus x. So, if you remember we have already seen the solution of this problem the solution is basically the Eigen values are cosine 2 n minus 1 pi by 2 the Eigen values are 2 n minus 1 pi by 2 and Eigen functions are cosine functions in terms of x star. So, if you really look into this problem, now let us try to do a justice by completing the solution of this problem.

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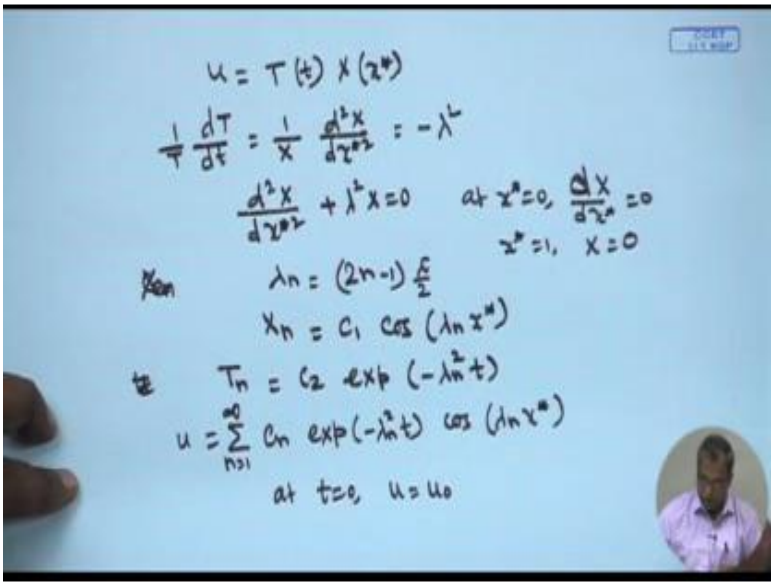

So, u is equal to a function of time and function of space. So, it will be 1 over t d t d t is equal to 1 over x d square x d x star square it will be equal to minus lambda square. So, let us write down the Eigen value problem in the x star direction d x star square plus lambda square x is equal to 0 subject to at x star is equal to 0 d x d x star is equal to 0 and x star is equal to 1 capital x is equal to 0.

So, let us look into the solution the solution will be the Eigen function Eigen values will be let us say lambda n is equal to 2 n minus 1 pi by 2 and Eigen functions x n is c 1 cosine lambda n x star and if you construct the complete solution and at T n will be let us say c 2 exponential minus lambda n square t and the complete solution is u is equal to summation n is equal to 1 to infinity c n exponential minus lambda n square t cosine lambda n x star and we can evaluate the constant c n by using the initial condition at t equal to 0; that means, at t equal to 0 u is equal to u naught let us completely solve this problem.

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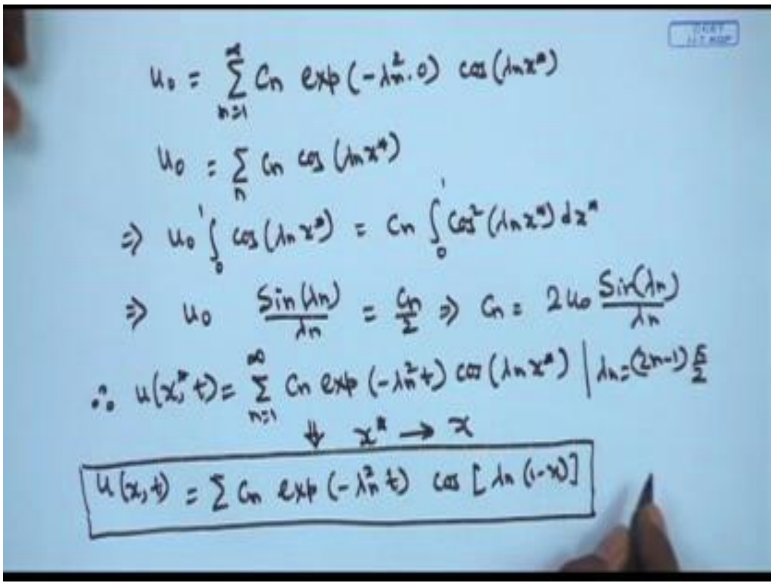

So, u naught is equal to summation c n exponential minus lambda n square times 0 cosine lambda n x star. So, this will be u naught c n cosine lambda n x star and again will be utilizing the orthogonal property of the cosine functions Eigen functions will be u 0 0 to 1 cosine lambda n x star is equal to c n cosine square lambda n x star d x star 0 to 1. So, only 1 term will survive if you open up the summation series utilizing the orthogonal property of the cosine functions or Eigen functions. So, this will be sin lambda n divide by lambda n is equal to c n by 2 and will be having c n is equal to 2 u naught sin lambda n divided by lambda n.

So, the complete solution is u x star t will be summation of c n exponential minus lambda n square t cosine lambda n x star where n is equal to 1 to infinity and lambda n is 2 n minus 1 pi by 2 pi by 2 and c n is 2 u 0 sin lambda n divided by lambda n and now we get back into a original coordinate system convert x star to x. So, therefore, u as a function of x and t will be nothing, but summation c n exponential minus lambda n square t cosine lambda n 1 minus x. So, that will be gives you the complete solution.

Similarly, if you have the standard, the third kind of problem, if you remember that boundary condition at x is equal to 0 is Dirichlet and boundary condition at x is equal to 1 is robin mixed and if the boundary condition is now reversed, if the boundary condition at x equal to 0 is robin mixed and it is Dirichlet at x equal to 1 then what will be doing will be having a transformation of coordinate system by defining x star is equal to 1 minus x star and then it will be recast in a standard form of second kind or third kind and will be it getting the solution in terms of standard solution in terms of the transformed coordinate system and then will be reverting back to or ordinary original coordinate system by writing x star is equal to 1 minus x.

So, that will be giving the solution in those cases of particular problems now will be moving ahead with the solution for an Elipson, we have we have covered more or less all the different kinds of problem in parabolic partial differential equation 1, dimensional 2, dimensional 3, dimensional and 4 dimensional, we have constant the various types of boundary conditions, how they will be appearing and how they will be affecting the in the final form of the solution and also we have seen that if the first second kind or third kind of problems are the boundaries are in the in the reverse direction then how to have a change of transformation of the access and can recast them into the standard form in order to get the solution.

Also we have seen that in an actual problem there maybe with it the solution the problem may not be a well posed problem the problem maybe an ill posed problem with lots of you know non homogeneity in the boundary conditions as well as the initial condition then we have seen how to break down the problem into sub problems considering 1 on homogeneity at a time. So, that in the 1 over out of the suppose there are 3 sources of non homogeneity in a parabolic partial differential equation then what we have done we have broken down the problem into 3 sub problems considering 1 on homogeneity at a time and out of this 3 sub problems 1 sub problem is well posed which will be having a non 0 initial condition and homogeneous boundary conditions.

Out of the 2 sub problems it will be homogeneous initial condition and boundary condition is non homogeneous then we have to break down the problem into 2 sub parts once again 1 one sub part will be time dependent another sub part will be time independent in the time will be formulating the governing equation of the time dependent part and time independent part and judiciously attach the non homogeneous boundary condition with the time independent part. So, that will be forcing the time dependent part to have a homogeneous boundary condition and it will be become well posed problem. So, we have seen how to convert an ill posed problem into well posed problem and we have the db standard solution we already seen. Therefore, we have looked into the well posed problem for 4 dimensional; 3 dimensional as well as 4-dimensional parabolic partial differential equation.

Now, will be moving ahead with the solution of the elliptical partial differential equation which will be nothing, but a steady state solution of a physical system which where there will be no time variation and most of the engineering applications they occurred either in the steady state or in the transient problem, for example, if you are looking for a you know process engineering from point of view if you looking into any process engineering system, whereby in putting the initial parameters, let us say feed conditions and you are prevailing the reaction conditions, etcetera. Then you will be looking for a product stream and then that product stream must be confirming some particular properties if that property is within the desirable property then your product can be marketable in the market otherwise it will be a waste lot.

So, therefore, the 1 one has to have a steady a quality of the product in order to have in order to achieve that 1 one ones to run his plant under a steady state condition. So, therefore, transient is very, very important as well as the steady state is very, very important transient gives you start up start up or shut down situation where the where the system will be achieving a steady state and this transient problems can be modelled by writing the parabolic partial differential equation.

On the other hand, steady state is very, very important because 1 must be having a steady solution in the steady solution that means 1 has to have a steady set of property within a tolerable range in the product. So, therefore, the steady state problems also very important in any process engineering and the steady state problems are modelled by time independent writing down the time independent governing equation and the time independent governing equation in more than 1 variable will lead to the elliptical partial differential equations.

So, that is the importance of elliptical partial differential equations in the process engineering which will be corresponding to the steady state of the system and in the next class will be looking into the solution of elliptical partial differential equation and I will stop here and will be taking up this problem in the next class.

Thank you very much.