Partial Differential Equations (PDE) for Engineers:
Solution by Separation of Variables
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Lecture – 10
Solution of 4 Dimensional Parabolic Problem

Welcome to this session. So, in this class what we are going to do is that in the, we will be taking to of whatever wherever we have left in the last class, that in the last class we have solved a parabolic partial differential equation which is the well posed problem in three dimension, one dimension in time and two dimension in space.

Now in this class, we will be looking into the problem which is basically a four dimensional parabolic partial differential equation which is three dimensional in space and one dimensional in time. It is a transient three dimensional problem, and please remembers that whenever you will be working with a parabolic operator then you are basically looking in to transient problem. Whenever you will be operating with an elliptical operator then you are looking into a steady state problem.

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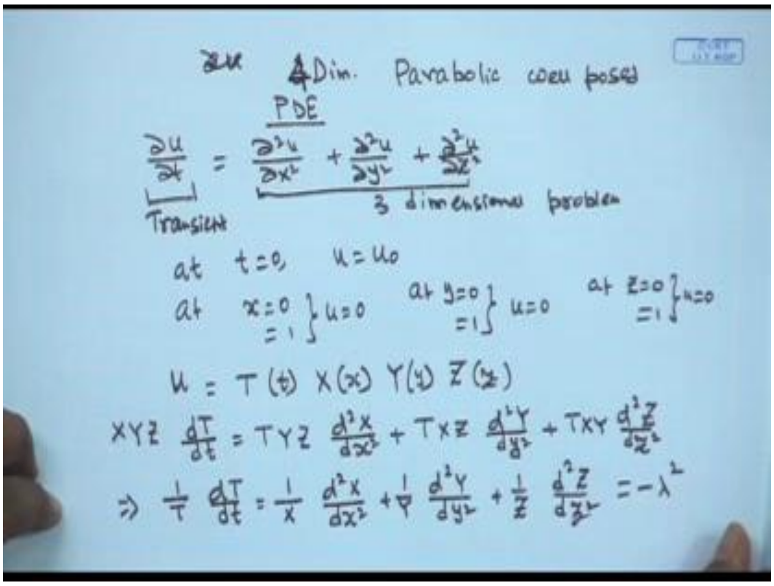

So, that has to be kept in to mind. So, we are still solving a transient problem four dimensional, three dimensional in space and one dimensional in time.

First let us formulate the problem. So, at formulate the three dimensional parabolic well posed partial differential equation. So, it will be del u del t is equal to del square u del x square plus del square u del y square plus del square u del z square. So, we have one dimensional in space, one dimensional in time and three dimensional in space. So, basically it is a four dimensional problem. So, it is a four dimensional problem transient. So, x y z and t these are the four independent variables.

Now let us formulate the boundary conditions at t is equal to 0, we have it is a well posed problem. So, the condition, initial condition at t equal to 0 must be having a non-zero condition, the non-homogeneous condition. Therefore, we assume that at t is equal to 0 u is equal to u naught and we assume all the 6, so there will be two boundaries existing on x, two boundaries existing on y and two boundaries existing on z and all the six boundaries are having the duration boundary condition; that means, they are their values are specified at those boundaries which are basically homogeneous in order to have a well posed problem

So, at x is equal to 0 and 1, we have u is equal to 0 at y is equal to 0 and 1, we have u is equal to 0 at z is equal to 1 and 1, we have u is equal to 0. So, all the six boundaries they are having the homogeneous conditions, there is basically a dirichlet boundary condition prevailing on the six boundaries. So, what we will be expecting? We will be expecting an independent eigenvalue problem in x direction, y direction and z direction as well. So, if you really do that and go ahead with a separation of variable type of solution that mean that then you will be considered to be a product of four independent variable will be a function of time alone. It will be a function of x alone, function of y alone, function of z alone

Now, if you substitute this equation into the governing equation. Let us see what we get. So, if we really do that you will be getting XYZ dT dt is equal to TYZ d square x dx square plus TXZ d square Y dy square plus TXY d square Z dz square. Now if we divide both side by XYZ t then let us see what we get, one over T dT dt plus is equal to 1 over

X d square X dx square plus 1 over Y d square Y dy square plus 1 over Z d square Z dz square. So, the left hand side is entirely a function of time, the right hand side is totally function of space. They will be equal and they will be equal to some constant and these constant has to be a negative constant. If we have a positive constant or zero then will be landing up with a trivial solution.

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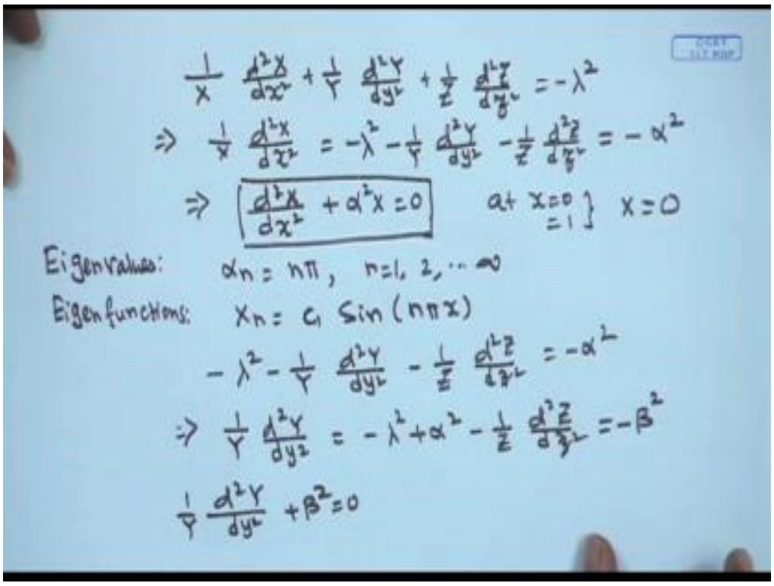

So, these has to be is equal to minus lambda square where a lambda square is a positive constant.

Now let us look into the special varying part. So, if we look in to the special varying part. So, we will be getting 1 over X d square X dx square plus 1 over Y d square Y dy square plus 1 over Z d square Z dz square is equal to minus lambda square. Now as we have seen earlier that as far as the boundary conditions are concerned, we are having homogeneous boundary conditions, both homogeneous boundary conditions in the x direction as well as the y direction and z direction. So, therefore, we will be formulating a standard eigenvalue problem in these three directions independently.

So, let us look in to the formulating the standard eigenvalue problem in the X varying part. So, one over X d square X dx square is equal to minus lambda square minus 1 over

Y d square Y dy square minus 1 over Z d square Z dz square and the left hand side a function of X alone, the right hand side is a function of Y and Z alone. So, therefore, they will be equal and this is equal, they will be equal to some constant and this constant has to be a negative constant in order to have a and standard eigenvalue problem in the X direction.

So, let us formulate that d square X dx square plus alpha square X will be equal to 0. This will be the first eigenvalue problem in the X direction and the boundary conditions of the original problem would have been satisfied by the boundary condition of the X varying part. So, at x is equal to 0 and 1 we have capital X is equal to 0. So, this has been done and we have already seen the solution of this. The solutions are alpha n is equal to n pi at the eigenvalues where n runs from 1 to infinity. So, these are the Eigenvalues and Eigen functions are sin functions.

X n is equal to constant let us say c1 sin n pi x. So, let us formulate the y varying part. So, if you look into governing equation minus lambda square minus 1 over Y d square Y dy square minus 1 over Z d square Z dz square is equal to minus alpha square. So, I take y varying put the on the other side and bring minus alpha square there. So, 1 over Y d square Y dy square should be is equal to minus lambda square plus alpha square minus 1 upon Z d square Z dz square. So, the left hand side is function of y alone, the right hand side is a function of z alone and they are equal and they will be equal to some constant and again these constant has to be a negative constant, otherwise you will be landing up into a trivial solution.

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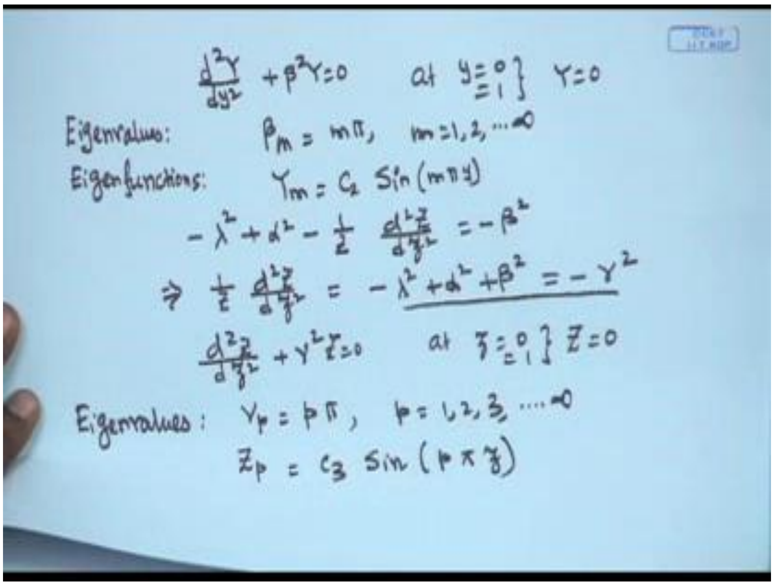

So, therefore, d will be is equal to minus beta square. So, it will be having the standard eigenvalue problem in the Y direction d square Y dy square 1 over Y plus beta square is equal to 0. And the standard eigenvalue problem in the Y direction becomes plus beta square Y is equal to 0 at y is equal to 0 and 1 Y is equal to 0 because the boundary conditions of the Y varying part should be satisfying the boundary condition of the original problem in the Y direction.

So, in the original problem the boundary condition in the Y direction where dirichlet and they are equal to 0 at y is equal to 0 and 1. Similarly therefore, the y varying for must also satisfy that. So, you know the solution of this, since it is an independent eigenvalue problem, the index will be m. So, beta m is equal to m pi where m is equal to 1, 2 up to infinity and these are the Eigenvalues and Eigen functions are Y m is equal to C2 sin m pI y. Now let us look into the z varying part. So, if you looking into the, you know governing equation minus lambda square plus alpha square minus 1 over Z d square Z dz square is equal to minus beta square. So, I take z varying part to the other side and bring beta to the left hand side.

So, if you do that we will be getting one by Z d square Z dz square is equal to minus lambda square plus alpha square plus beta square. So, these they will be a combination of constant and again these constant must be a negative constant, otherwise you will be landing up with a trivial solution. So, this constant must be equal to minus gamma square. So, we will be having d square Z by d z square plus gamma square Z will be equal to 0 subject to the boundary condition at z equal to 0 and 1. We have Z equal to 0. And we know the solution of this. Solutions will be a p pi where index p transforms 1 to infinity and the Eigen functions at the sin functions. So, gamma p will be is equal to p pi at the Eigenvalues where p runs form 1, 2, 3 up to infinity. And Zp will be nothing, but c3 sin p pi z.

Now let us look into what is the value of lambda. So, if you so, this is the governing equation of lambda. So, if you take on to the lambda to the other side and gamma to this side what we will be getting is that lambda square is equal to alpha square plus beta square plus gamma square. So, we have seen alpha is equal to n pi beta is equal to m pi and gamma is equal to p pi.

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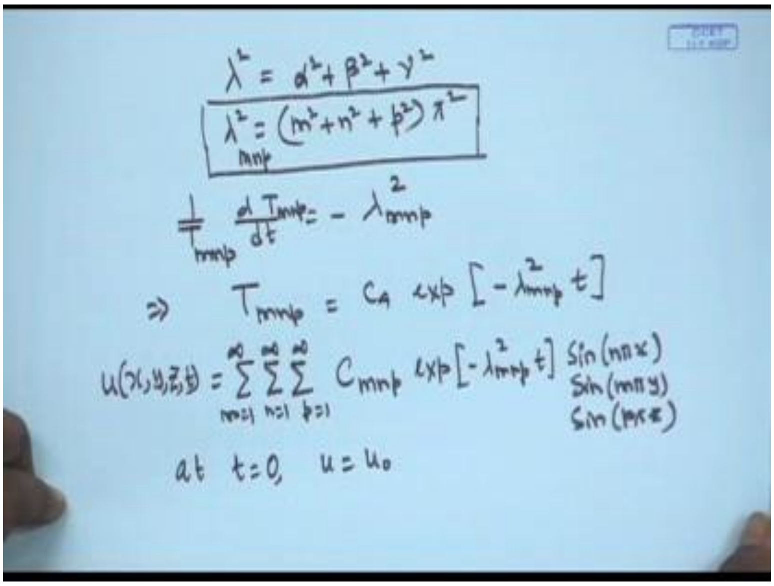

So, therefore, lambda square will be nothing, but m square plus n square plus p square pi square and lambda should be with the subscript written as a subscript m n and p.

So now, we look into the time varying part. So, if you look into the time varying part, we write is as 1 by T dT dt is equal to minus lambda mnp square and since we are using the subscript mnp, the corresponding solution is also T mnp. So, therefore, we have the solution T mnp as equal to C4 exponential minus lambda mnp square of that time t where lambda mnp is given by this expression.

Now we are in a position to construct the complete solution of u either function of xyz and t. Now if we really do that so there will be, it will be appearing three summations because where having three independent eigenvalue problems. So, u x y z and t will be nothing, but summation triple summation 1 over m 1 to infinity, another over n 1 to infinity, another over p 1 to infinity. There will be four constants to be multiplied; they will be giving rise to a new constant. So, we write down their constant as C mnp C mnp exponential minus lambda mnp square of that times t sin n pi x sin m pi y sin p pi z. Now we will be now the problem is almost over only one constant C mnp has to be evaluated. These constant has to be evaluated from the unused initial condition of the original problem that at t is equal to zero u is equal to u naught.

So, utilizing that initial conditions, the non-homogeneous initial condition at t equal to 0 u is equal to u naught. You will be able to evaluate the constant C mnp.

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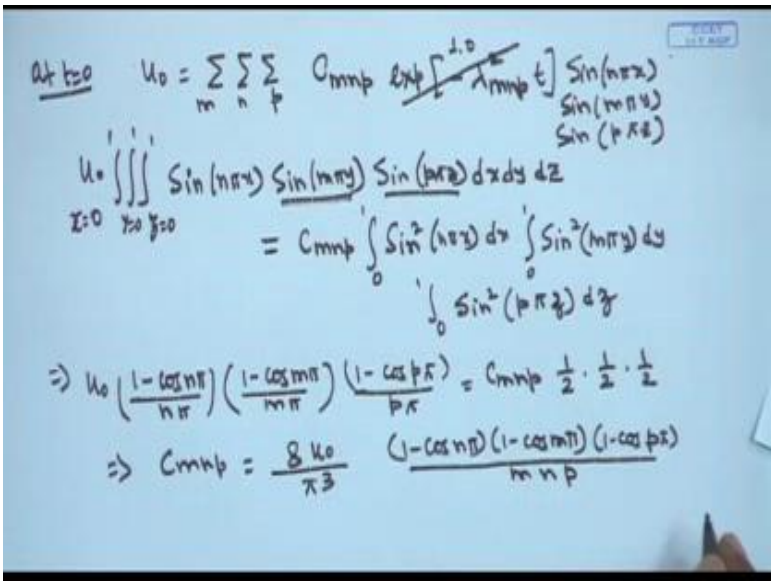

So, if you do that let us see, how we will be obtaining the constant C mnp as u naught is equal to summation triple summation 1 over m, another over n, another over p. C mnp exponential minus lambda mnp square of that t sin n pi x sin m pI y sin p pi z.

Now, since we will be having the three standard eigenvalue problems. In this particular, you know partial differential equation, we will be we will be utilizing the orthogonal property of the Eigen functions; that means, we will be multiplying both side by the sin m pi x sin m pi y sin you know q pi z and indicate over the domain of x y z. And after doing that if you open up this, summation series only one term will survive on the right hand side where m is equal to n and n is equal to m and p is equal to q.

So, therefore, what we will be getting is u naught triple integral 1 over x from 0 to 1, y 0 to 1, z 0 to 1. So, sin n pi x sin m pi y sin p pi z dx dy dz. And on the right hand side, only one term will survive that will be Cmn. So, these will be equal to 1. At t is equal to zero so, at t equal to 0, this will be equal to 1. So, it will be on the right hand side, you will be getting Cmnp integral sin square n pi x dx 0 to 1, 0 to 1 sin square. So, this is at t equal to 0 sin square m pi y dy multiplied 0 to 1 sin square p pi z dz.

So, u naught the first one integral sin n pi x dx it will be nothing, but 1 minus cosine n pi over n pi, the second one integral sin n pi y dy will be nothing, but cosine m pi over m pi, the third one integral sin p pi z dz will be nothing, but 1 minus cosine p pi divided by p pi is equal to Cmnp and sin square n pi x dx, we have already seen this is half, this will be half, this will be half. So, you will be getting 1 by 2 into 1 by 2 into 1 by 2. So, C mnp will be nothing, but 8 u naught pi cube 1 minus cosine n pi one minus cosine m pi 1 minus cosine p pi divided by m n p.

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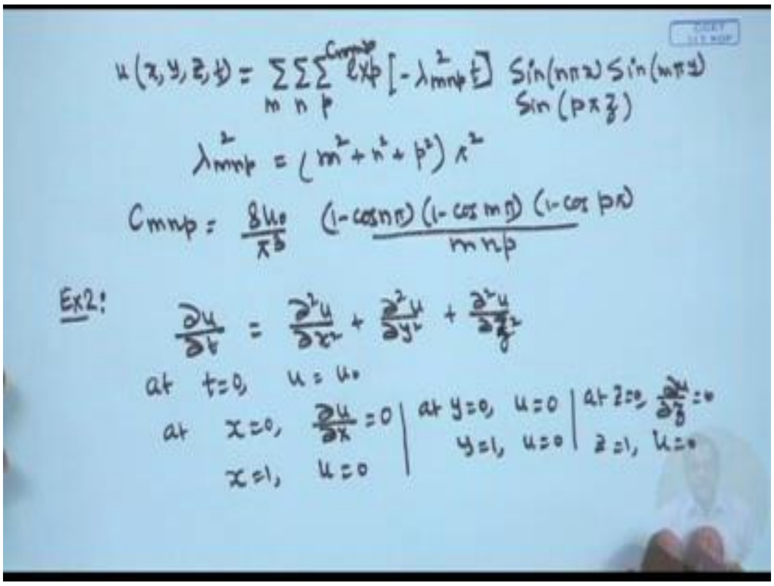

So, that will be the expression of Cmnp and we are coming closer to our complete solution and the complete solution is that u x y z and t will be nothing, but triple integral 1 over m n p exponential minus lambda m n p square t sin n pi x sin m pi y sin p pi z. This is a constant Cmnp where lambda m n p square is nothing, but m square plus n square plus p square time pi square and the constant and constant Cmnp is equal to 8 u zero over pi cube 1 minus cosine n pi 1 minus cosine m pi 1 minus cosine p pi divided by m n p, that gives the complete solution of a well posed parabolic four dimensional partial differential equation, and if we and this can be, if we looking into the some of the boundary conditions are altered.

Now in this particular problem all the boundary conditions are dirichlet boundary condition. Now, if we have you know normal boundary condition in one of these directions; that means, let us say at x equal to zero and y equal to zero and let us say dirichlet boundary conditions are prevailing in the z direction. Now I the cosine functions will be Eigen functions in some of the cases corresponding cases in the eigenvalues will be different.

For example, if we take up another example of this sort that del u del t is equal to del square u del x square plus del square u del y square plus del square u del z square and at t is equal to zero u is equal to u naught at x is equal to 0. We have let us say del u del x is equal to 0. Let us say the boundary is insulated there and x is equal to 1 u is equal to 0. This is a boundary at x is equal to 0, on the x direction. At y direction at y is equal to 0 u is equal to, let us say 0 and y is equal to 1 u is equal to 0 and at z equal to 0. We have let us say del u del z equal to 0 and at z equal to 1, we have u is equal to 0. So, this will be the small z. So, let us look into the solution of this problem. In this problem, in state of dirichlet boundary condition in all the boundaries, we have a normal boundary condition prevailing at x is equal to zero and z equal to zero.

Now let us see how this problem can be solved. Now we will be again looking for a separation of variable type of solution and we considered that u is a function of product of four variables which will be a sole function of time which will be a sole multiplied by sole function of x, multiplied by sole function of y and multiplied by sole function of z.

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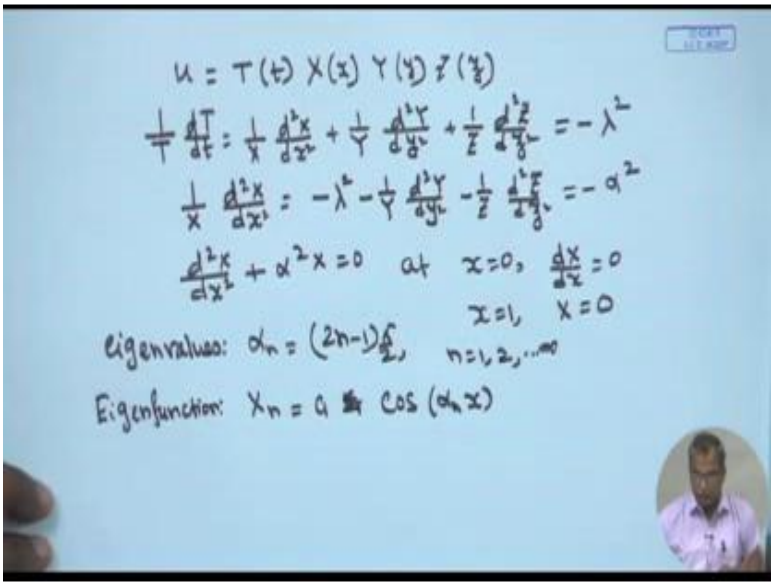

If we do that and put these equation in the governing equation and then separating the variables the final expression will be getting is one over T dT d t is equal to 1 over X d square X dx square plus 1 over Y d square Y dy square plus 1 over Z d square Z dz square. And again as we have seen earlier, the left hand side is entirely a function of time and right hand side is a entirely function of space and they will be equal, they will be equal to negative constant minus lambda square.

Now let us look into the special varying part. The special varying part will be 1 over X d square X dx square will be nothing, but minus lambda square minus 1 over Y d square Y dy square minus 1 over Z d square Z dz square and left hand side is entirely function of x right hand side is a function of y and z. They will be equal to some constant this constant will be minus alpha square.

So, we will be having a standard eigenvalue problem in the x direction as d square X dx square plus alpha square X is equal to 0, subject to the boundary condition of the original problem in the x direction at x is equal to zero dX dx is equal to 0. Because the gradient del u del x was zero in the original problem. At x equal to 0 and at X equal to 1, we had x is equal to 0. So, you know the solution of this problem, the eigenvalues are 2 n minus 1 pi by 2 alpha n is equal to 2n minus 1 pi by 2 where n runs from 1,2 infinity. And Eigen functions are Xn C1 sin. There will be cosine functions, cosine alpha mx. Now, that completes the x varying part.

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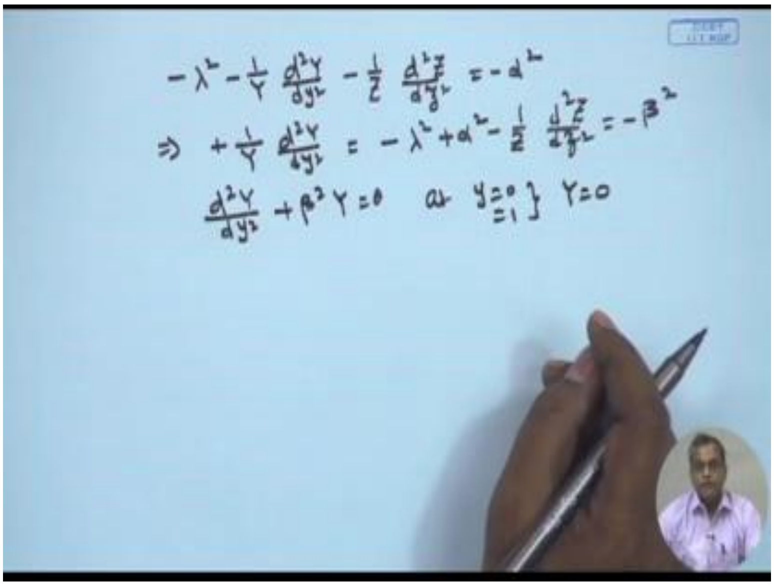

Now let us look into the y varying part, minus lambda square minus 1 over Y d square Y dy square minus 1 over Z d square Z dz square is equal to minus alpha square. So therefore, minus 1 over so, we take this 1 to the other side. So, 1 over Y d square Y dy square is equal to minus lambda square plus alpha square minus 1 over Z d square Z dz square.

So, again the left hand side is the function of y the right hand side is the function of z only that will be equal to some negative constant. In order to get a non-trivial solution, the governing equation of Y will be simply d square Y dy square plus beta square Y will be equal to 0. And in the original problem, you have seen that the dirichlet boundary conditions are prevailing in the y direction. So, therefore, at y is equal to 0 and 1, we have Y is equal to 0.

So, I will stop in this class, in the next class I will be completing this problem and then will be moving ahead with the solution of elliptical partial differential equations which are quite common in all engineering applications which are occurring under steady state.

Thank you very much.