Partial Differential Equations (PDE) for Engineers:
Solution by Separation of Variables

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Lecture – 01
Introduction to PDE

Good morning every one welcome to the course of partial differential equations in separation of variables. So, in this course will be looking into the in various problems those will be encountering for any engineer this course will be useful for any engineers. So, whenever we will be doing the actually conducting the experiments and trying to predict the performance of the system then we have to conduct a huge number of experiments if you would like to optimize the operating conditions.

Now, these will incur lots of cost in terms of man power, energy, resources and as well as the materials. So, in order to minimize all the (Refer Time: 01:06) cost for optimizing a particular set of operating conditions in a system in order to get a appropriate performance prediction then one has to undergo if you want someone right now the physical situation in terms of mathematical equations then solve these problems then one can undergo some the one can do have a with the regoutity of (Refer Time: 01:32) conducting experiments by doing virtual experiments on the computer. So, modeling and simulation is one of the integral parts in modern in any modern processes. So, let us look into the difference between modeling and simulation, what is modeling? Modeling is basically to write down the physical system in forms of mathematical expressions.

So, it is basically mathematical interpretation of the physical situation or system that we are dealing with. Once we write down the mathematical expression to describe the process then will be doing will be solving those equations with the set of you know (Refer Time: 02:13) boundary conditions or initial condition. Now this is known as the simulation solution on the computer is basically simulation by doing modeling and simulation one will be getting the design performance or one will be getting the optimum design if your system geometry etcetera everything is known to a known then one can really solve the problem in order to optimize the operating conditions to get the particular. So, you know yield or you know conversion or a (Refer Time: 02:45) any reaction of any process performance of any process in general.

On the other hand if the conversion or the final output or the performance will be desired performance is known to us then we can define what will be the design parameters. For example, what will be the length of the reactor or the volume of the reactor or they or we can design the equipment. So, these are the various advantages of modeling and simulation we can do have with large number of experiments by conducting virtual experiments by you know by simulating the model equations on a computer.

So, therefore, modeling and simulation is very very important in any process engineering operation and in order to do that one has to undergo you know one has to model the situation by writing the appropriate mathematical expression to describe the situation physical situation appropriately. Now, there are various tools to solve this set of equations and in most of the engineering applications will be encountering the partial differential equations and this partial differential equations will basically a multiplied multi-variable problem the number of independent variables are more than one then only one can one will be landing up with the partial differential equations.

Now, there are how to solve the partial differential equations there are various tools, will be looking into some strong analytical tools. So, numerically any equation can be solved that can be already that can be done in any numeric by adopting to any strong numerical technique, but the analytical solutions are also available from for certain case of partial differential equations.

So, will be why analytical solutions are important analytical solutions are important because the variation the it requires less computational time because the solution is available in front of you and one can understand what will be the variation of dependent parameter with the independent variables because the functional form of variation of dependent parameter or independent parameters will be known to us. That is the advantage of analytical solution also the second advantage is its computational rigourity or computational expense will be very less because you will be always getting the solution in front of us on the pen and paper.

Now, next is that once we will be getting the analytical solution by comparing a partial differential by solving a for solution of the partial differential equation, sometimes where in an actual realistic situation the situation may be very very complicated and one may not be getting an analytical solution in that case what is the way out, the way out is to go for a numerical simulation the numerical simulation may be very computational intensive process and many process industries they will not be having the resources to undertake a full-fledged you know multi-dimensional numerical solution. So, what is generally done in an analytical solution is sought initially for so, that the one can get the solution at least that will be a first iteration of solution. The basic design can be based on the analytical solution then it can be refined or fine tuned by detailed numerical simulation.

So, therefore, the importance of analytical solution is still remaining and it is much sought after and in this particular course we will be looking in we will be will what we will be doing is that we will be defining partial differential equations we will be looking into the how to you know broadly classify these equations into set of different partial differential equations like parabolic in a elliptical hyperbolic so and so forth. We will be looking into the various characteristics of this is partial differential equations and we will be mainly dealing with the linear PDE's and then we will be looking into the solution by separation of variable technique. We will be looking into the various types of you know boundary conditions as well as the governing equation and then we will be looking into it its solution.

So, therefore, this course will be of immense help to the process engineers in general. So, that they can model their system at the very fast and a first iteration and then go for the detail design later on.

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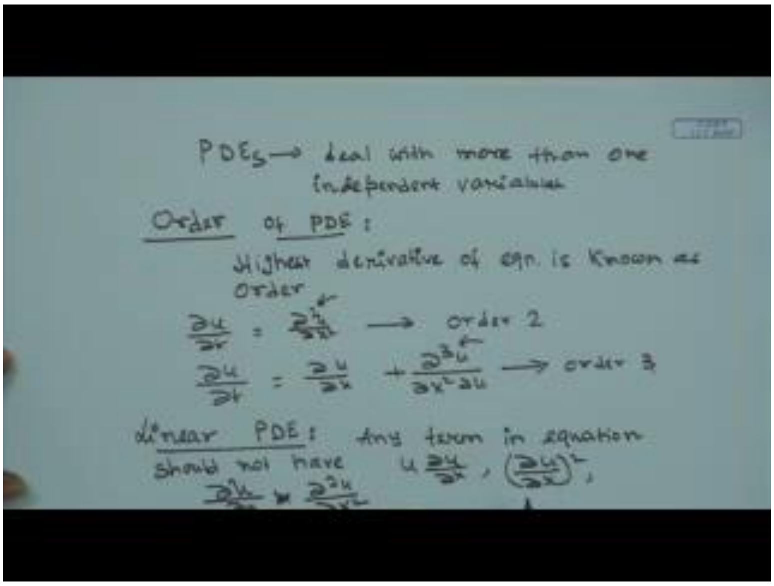

So, let us look into what is the definition of Partial Differential Equations. The Partial Differential Equations are the equations that deal with more than one independent variables then only the governing equation is called the Partial Differential Equations. Now let us look into what is called the Order of Partial Differential Equation, order is the highest derivative of the equation is known as the order is known as order. For example, del u del t is equal to del square u del x square in this equation the order is 2 because the highest derivative that will be you know appearing here is 2, then del u del t is equal to del u del x plus del cube u del x square del u in this partial differential equation the order is 3 because the highest derivative that is appearing in the equation is 3.

Next we will be looking into the Linear and Homogeneous Linear Partial Differential Equation. What is Linear Partial Differential Equation, if we have we if we do not have any term in the governing equation any term in equation should not have you know multiplication of derivatives or the dependent variable u del u del x higher order terms del u del x square.

So, del u del x multiplied by del square u del x square this time of delta square del y square or something like this. So, therefore, any term in the governing equation should not have the multiplication higher order terms of the dependent variables you know power higher power of dependent variables or their derivative or any multiplication of the dependent variable and their derivative if these terms are not present in the governing equation then the (Refer Time: 10:24) partial differential equation is called the Linear Partial Differential Equation.

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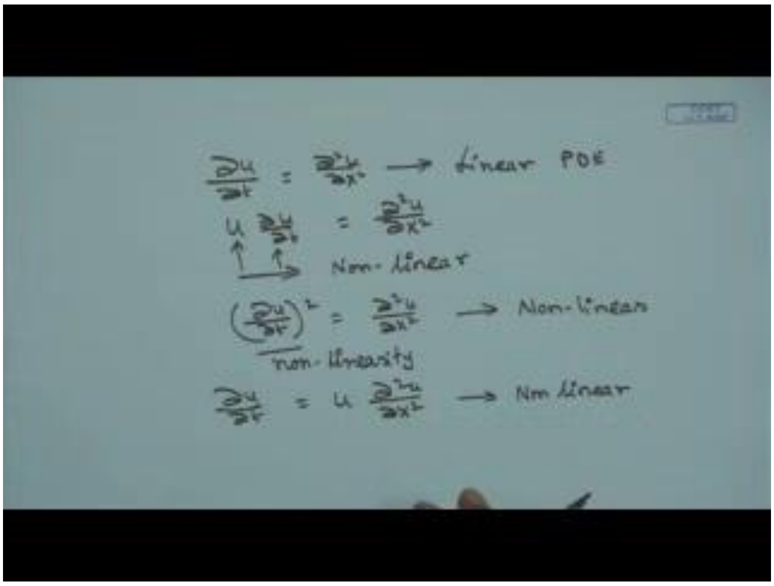

So, let us look into some of the examples. The examples let us say, del u del t is equal to del squared u del x square this equation does not have any you know multiplication with that of term like u del u del t or del u del t del square del x square something like that. So, it is a linear partial differential equation say linear PDE u del u del t is equal to del square u del x square. Now in this equation there is a multiplication of the dependent variable and its derivative. So, this makes the equation Non-linear. So, the term on the left hand side is the Non-linear a term, del u del t square is equal to del square u del x square again this equation is a Non-linear Partial Differential Equation and this is the source of Non-linearity, del u del t is equal to u del square u del x square. Now in this equation again this there is the product of dependent variable and its derivative. So, therefore, it is a Non-linear equation.

So, therefore, the product of dependent variable and or its and its derivative or the higher power of the dependent variable or its derivative if they are present then any power except one if that is present in the governing equation then the partial differential equation is a Non-linear Partial Differential Equation.

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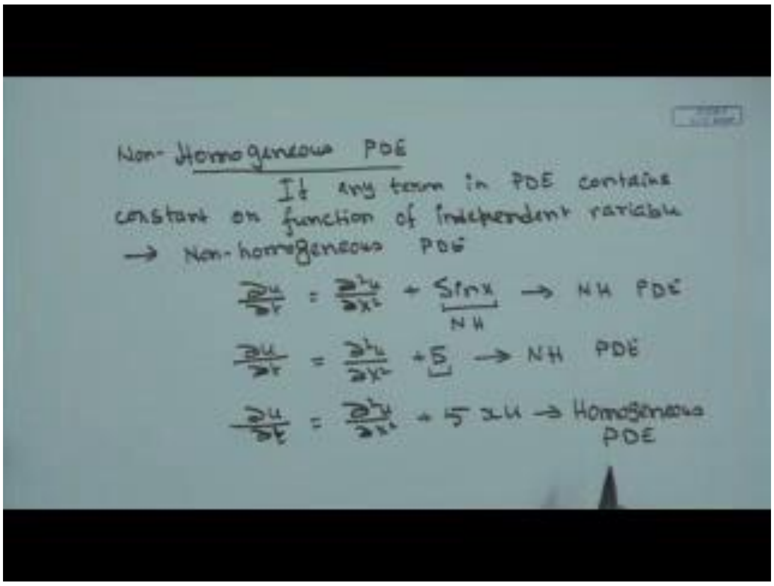

So, next we will be looking into what is called Non-homogeneous and Homogeneous Partial Differential Equation. So, Homogeneous PDE now if any term if any term in PDE contains constant or function of independent variable only. So, this is the definition of Non-homogeneous Partial Differential Equation if any term in PDE contains constant or function of independent variables then it is a Non-homogeneous Partial Differential Equations. For example, del u del t is equal to del square u del x square plus sin x. So, these in this example this is the Non-homogeneous term and there is a Non-homogeneous Partial Differential Equation. Del u del t is equal to del square u del x square plus 5. Now this constant term is the source of Non-homogeneity and this is a Non-homogeneous Partial Differential Equation.

So, if you have term like this del u del t is equal to del square u del x square plus 5 x u, but now in this case all the terms containing the a dependent variable. So, this term here it is del u del t time derivative of dependent variable in this term it is a double derivative of u with respect to x and here you will be having a dependent variable u. So, this is a Homogeneous Partial Differential Equation.

So, if there is a term in the governing equation which does not contain the dependent variable then only it will be a Non-homogeneous Partial Differential Equation and the Non-homogeneous term can be a constant or it can be any function of the independent variable only, it should be stripped of the dependent variable then it is called a Non-homogeneous Partial Differential Equation.

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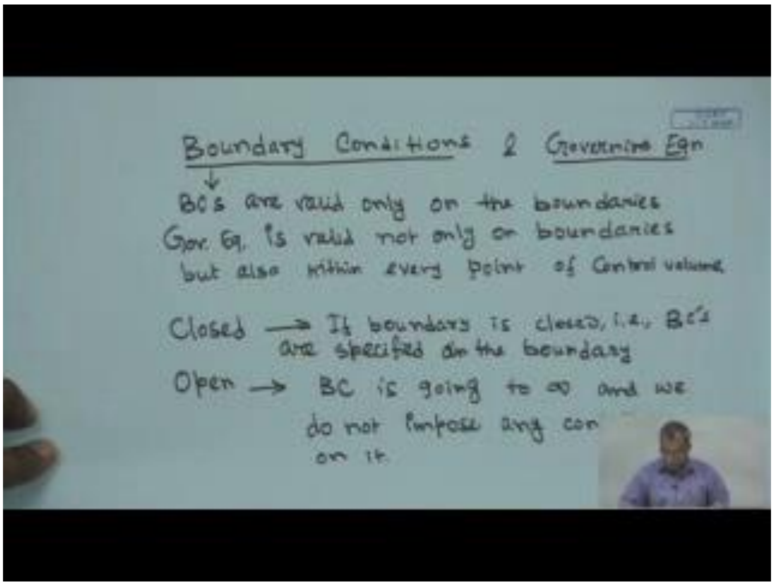

Now, once the differential equations are defined then you have to define the boundary conditions. The boundary conditions are very very important in case of differential partial differential equation. Now if it should be it should be known to you that boundary conditions are valid only on the boundary of the control volume where we are solving the problem, but the governing equation is valid governing equation means the differential equations or partial differential equations that is valid not only on the boundary, but also within every point of the control volume. So, that is very important to differentiate between the boundary conditions and governing equation boundary conditions BC's are valid only on the boundary, but governing equations governing equation is valid not only on boundaries, but also within every point of Control volume.

So, there is a difference between the boundary conditions and governing equations boundary conditions are not valid inside the control volume they are valid only on the boundaries there are two types of boundaries in general, Closed boundaries and Open boundaries, if boundary is defined is a boundary is closed; that means, it is surrounded by a region of interest the boundary conditions specified on the boundary that means, the boundary conditions are specified entirely on the boundary, then the boundary conditions is known as the Closed boundary. What is the Open boundary, Open boundary is that boundary conditions where boundary conditions is going to infinity and we do not impose any condition on it. So, that is the difference between the Closed and Open boundary.

Now, let us look into the different types of boundary conditions one may come across during the solution of Partial Differential Equations. Now there are various categories of boundary conditions now will be discussing each of this categories in detail. So, first category is known as the Dirichlet Boundary Condition.

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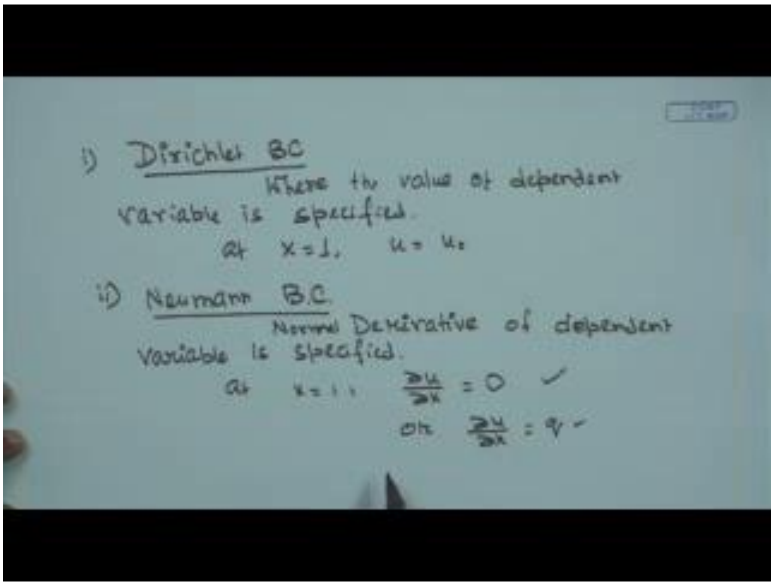

In case of what is the Dirichlet boundary conditions, Dirichlet boundary conditions is that boundary where the value of dependent variable is specified here we specify the value of the dependent variable for example, at x is equal to 1 u is equal to u naught. So, we specify the value of u as u naught. So, there is a there is an example of Dirichlet Boundary Condition.

The next boundary condition is Neumann Boundary Condition. What is the Neumann Boundary Conditions? In case of Neumann Boundary Condition the derivative of the dependent variable is specified at that boundary. So, when the derivative of dependent variable is specified then it is known as the Neumann Boundary Conditions. For example, at x is equal to 1 it is typically the Normal derivative x is equal to 1 del u del x is equal to 0. So, this is a typical type of you know boundary conditions that is present there and or del u del x is equal to q. So, either it is 0 or q so the value of the normal the derivative of the dependent variable is specified at that boundary. So, this is known as the Neumann Boundary Condition.

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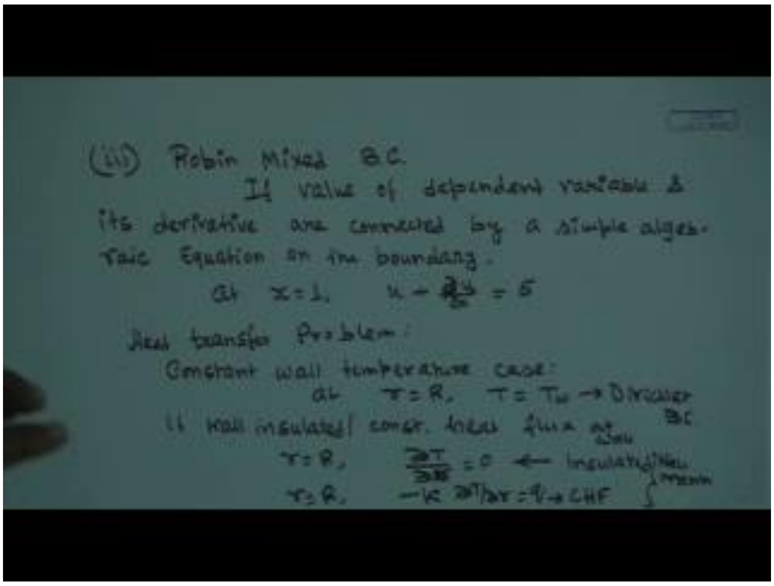

So, what is the third one, third one is a mixed it is also called a Robin Mixed Boundary Condition. For example, this is a mixture of Dirichlet and Neumann Boundary Condition if value of dependent variable and its derivative are connected by a simple algebraic equation; equation on the boundary then this is known as the Robin Mixed Boundary Condition.

For example at x is equal to 1 u plus del u del x is equal to let us say some value 5. So, this is an example of Robin Mixed Boundary Condition and you can give an example a Heat transfer problem where all these boundary conditions may appear. For example, if we are talking about a constant wall temperature problem, constant wall temperature case. So, in that case at the boundary let us say at r is equal to capital r t is equal to constant t w. So, this is a case of Dirichlet Boundary Condition at if wall is insulated or there is a constant heat flux present in the system heat flux at wall then r is equal to r del t del x del t del r will be equal to 0.

So, that is the insulated boundary condition insulated and r is equal to r if minus k del t del r is equal to q there is a constant heat flux condition both are Neumann Boundary Condition both are example of Neumann Boundary Condition then will I will be giving an example of Robin mixed in case of heat transfer problem. If the boundary is open to atmosphere then whatever the heat that will be coming at the edge of the boundary by conduction that will be taken away by convection by the external ambient surroundings then it is a robin Mixed Boundary Condition, minus k del t del x. So, let us say at any boundary let us say at x is equal to l minus k del t by del x is the amount of heat that is conducted at that boundary is taken away by h times h is the heat transfer coefficient t minus t infinity. So, this is a Robin Mixed Boundary Condition.

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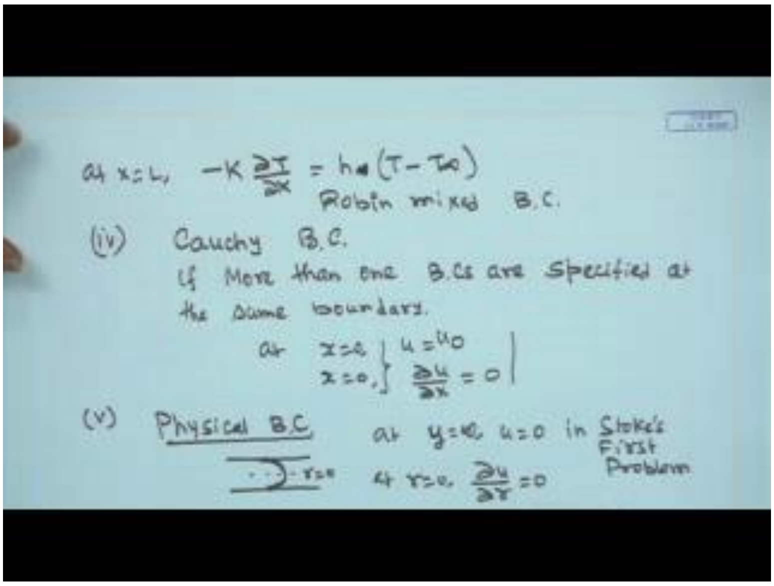

So, these three are the typical boundary conditions one will be coming across in heat transfer, mass transfer and feed mechanics. So, in another type another two types of boundary conditions are quite common in various engineering problem. For example, let us look into the fourth type of boundary condition. So, this one is the Cauchy Boundary Condition. In case of Cauchy Boundary Conditions if two boundary conditions are specified if more than one-boundary conditions are specified at the same boundary then it is known as the Cauchy Boundary Condition. So, more than if more than one boundary conditions are specified at the same boundary then it is a Cauchy Boundary Condition for example, at x is equal to 0 u is equal to 0 at u is equal to let us say u naught at x is equal to 0 we have del u del x also may be specified, let us say these will be equal to 0. Both the boundary conditions are situated at the same boundary. So, it is a Cauchy Boundary Condition.

So, next boundary condition that will be talking about is a physical boundary condition. In most of the in some of the cases the boundary conditions are not known a prior from the boundary condition will be defined from the physics of the problem. So, it will be known as the Physical Boundary Condition. So, we can give one example for example, there is a stagnant fleet where the where you have kept a plate and the plate is executing a simple harmonic motion or it may be moving in the x direction with a uniform velocity u naught. So, what happens the velocity next to the plate in the stagnant fleet will be having a velocity close to the wall? So, there is if the wall or the plate is moving in the x direction with u naught velocity the fleet element next to the plate will be having a velocity of u naught. Then as you go along in the y direction then its velocity keeps on decreasing because of the viscous effect and at infinite distance at a very large distance the velocity will be assuming the (Refer Time: 28:15) velocity that will be equal to 0.

So, therefore, in this particular problem we are defining a boundary at x equal to at y equal to infinity, at y equal to infinity. So, velocity is equal to 0 this particular boundary condition is evolving out of physics of the problem. So, this is an example of physical boundary condition. So, at u is equal to at y is equal to infinity u is equal to 0 if we keep a plate in a stagnant fleet and the plate moves in the forward in the x direction with a velocity u naught, this is known as the Stokes First Problem.

So, y equal to at y equal to infinity u is equal to 0 in Stokes First Problem another condition is if there is a flow through a pipe and you have a velocity you know laminar velocity profile then at the midpoint at r is equal to 0 we do not we know the boundary conditions that r is equal to capital r velocity is equal to zero, but what is the condition at the so that is called as the No Slip Boundary condition. R is equal to capital r velocity is equal to zero, but what is the condition at r is equal to 0 at the center point at the center point the velocity will be having some finite value, but del u del r will be equal to 0.

So, the stress is the minimum there and therefore, at r is equal to 0 we have del u del r will be equal to 0. Now nobody has specified this boundary condition. So, this boundary condition is coming out of physics of the problem. So, this is known as a physical boundary condition. So, typically this five-boundary condition will be appearing in various engineering applications of related to the context of Partial Differential Equations.

So, I will stop in this class. So, in the next class I will be talking up the how to characterize classify and characterize the various Partial Differential Equations and how will come to know by looking into the Partial Differential Equation. How in which category or class it will fall into.

Thank you very much.