**Probability and Statistics**
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# Lecture - 72
# Large Sample Test for Variance and Two Sample Problem

In the last lecture I have explained how to test hypothesis for parameters of a normal distribution for example, if I have available a random sample from a normal distribution with mean mu and variance sigma square, then how to conduct the tests for mean, that is mu or for sigma square. We have considered various cases; that means, when we are testing the mean is equal to certain quantity or it is less than or equal to certain quantity, then whether variance is known or unknown in the different 2 cases we have 2 different tests, one is based on a normal distribution and another is based on a t distribution. Similarly when we are testing for the variance, then we are assuming mean may be known or mean may be unknown and you have a 2 different chi square tests.

We also discuss the tests for a proportion that is binominal distribution for the large n. So, we approximated it by a normal distribution and we considered the test statistic as x minus n p naught divided by square root of n p naught q naught, where p naught is the value which is to be tested against at the null hypothesis point. In the case of the variance testing, we used a chi square test; that test is quite sensitive to normality assumption, if the assumption of the normality is not satisfied then the test may be somewhat bad, it may give a false result. However, if we observe the distribution of S square that is used their which is chi square under the normality assumption, even if we do not have the normality assumption for large n the distribution of S can be approximately normal distribution.

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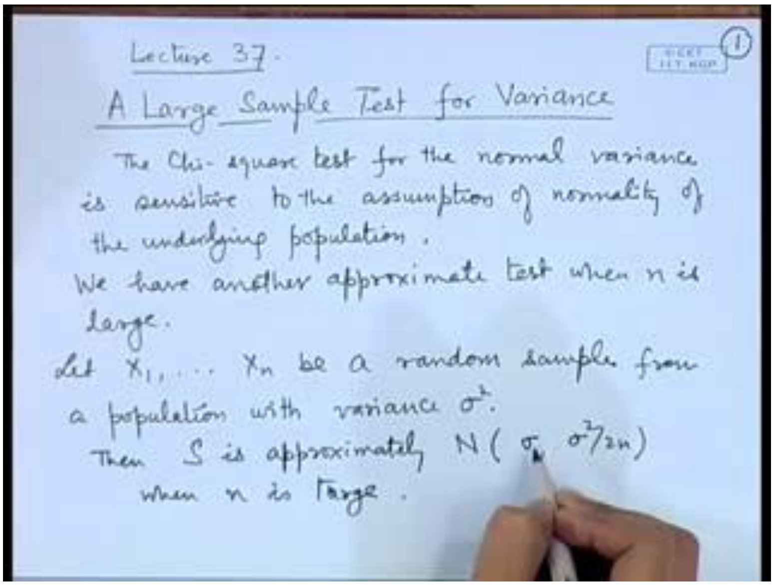


So, the chi square test for the normal variance is sensitive to the assumption of normality of the underlying the population. So, we have another approximate test when n is large. So, let X 1, X 2, X n be a random sample from a population with variance sigma square then S is n approximately normal sigma, sigma square by 2 n when n is large.

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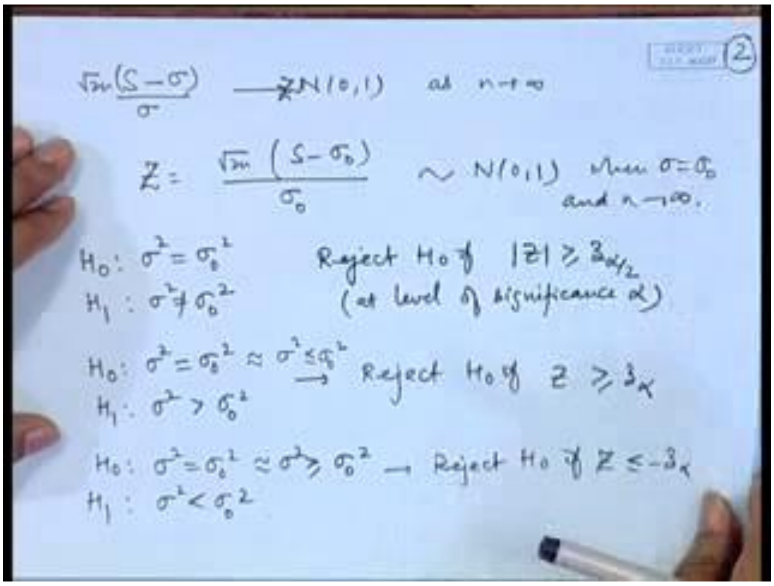


So, since S square is unbiased estimate for sigma square, and S is not unbiased, but asymptotically it is unbiased for sigma; therefore we may make the following thing that S minus sigma divided by sigma by root 2 n, then this is approximately normal 0, 1 as n tends to infinity distance to a. So, if we define our random variable Z is equal to root 2 n S minus sigma naught, divided by sigma naught then this follows normal 0, 1 when sigma is equal to sigma naught and n tend to infinity.

So, if we want to test about say sigma square is equal to sigma naught square, again it say sigma square is not equal to sigma naught square, we can make a test based on this Z variable. So, we can use the test as reject H naught if modulus Z is greater than or equal to z alpha by 2. So, at level of significance alpha, the test is that you reject H naught when the absolute value of Z is greater than or equal to z alpha by 2.

Similarly we can write one sided test for example, if I have H naught, sigma square is equal to sigma naught square against say sigma square is greater than sigma naught square, then we will reject H naught if Z is greater than or equal to z alpha. Of course this hypothesis is equivalent to writing sigma square is less than or equal to sigma naught square. Similarly if I am considering sigma square is equal to sigma naught square against one sided alternative sigma square less than sigma naught square, then we will reject H naught if Z is less than or equal to minus z alpha and again this is equivalent to sigma square greater than or equal to sigma naught square.

Let me give one example here also the framing of the hypothesis is important here, since we are controlling the probability of the type one error that is rejecting the H naught when it is true therefore, rejecting H naught is always a considered as a strong conclusion. Therefore, the hypothesis for which we want to give an importance we can put it in H 1 of course, that may not be possible if we are testing for the equality, because then inequality cannot be put in the alternative hypothesis. I will give one example here where this role of null and alternative hypothesis is important and one has to take a justify decision that in what way we should write down the null and the alternative hypothesis.

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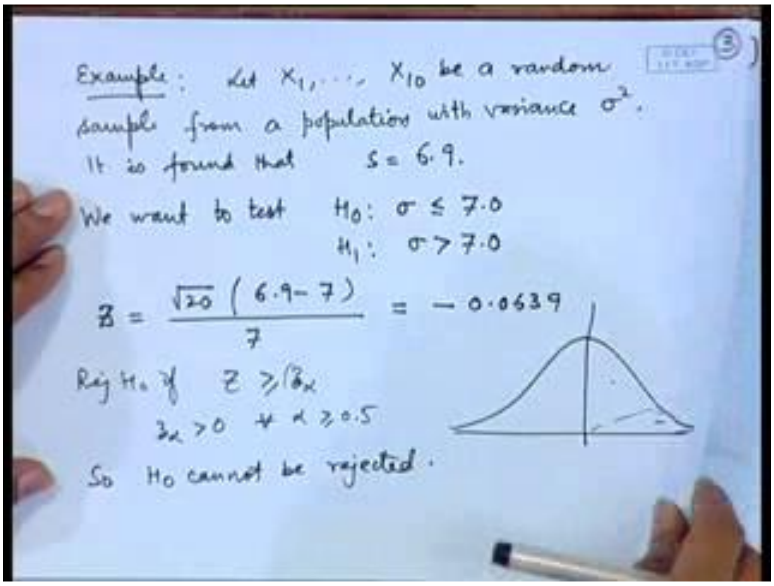


Let us consider let X 1, X 2, X 10 be a random sample from a population with variance sigma square, it is found that S square S is equal to 6.9 say. Now we want to test say H naught whether sigma square is less than or equal to 7, against H 1 sigma is greater than 7. So, let us create this Z variable here that is root to n is root 20, we have 10 observations here S is 6.9 minus 7 divided by 7. So, this can be evaluated easily it is minus 0.0639. Now if we consider say we have to reject H naught if Z is greater than or equal to say Z alpha. Now for any level of alpha up to say 0.5, the value of z alpha will always be non negative because Z is a symmetric distribution about 0. So, all these point these are all positive. So, this value will never be negative, this z alpha is positive for all alpha greater than or equal to 0.5; so H naught cannot be rejected.

Now one may feel that since we have rejected H naught, we cannot reject H naught; that means, there is a strong support to the hypothesis that the variance is smaller than 49 or the standard deviation is smaller than 7, but this is slightly miss normal here, we should be careful in the choice of the null and the alternative hypothesis.

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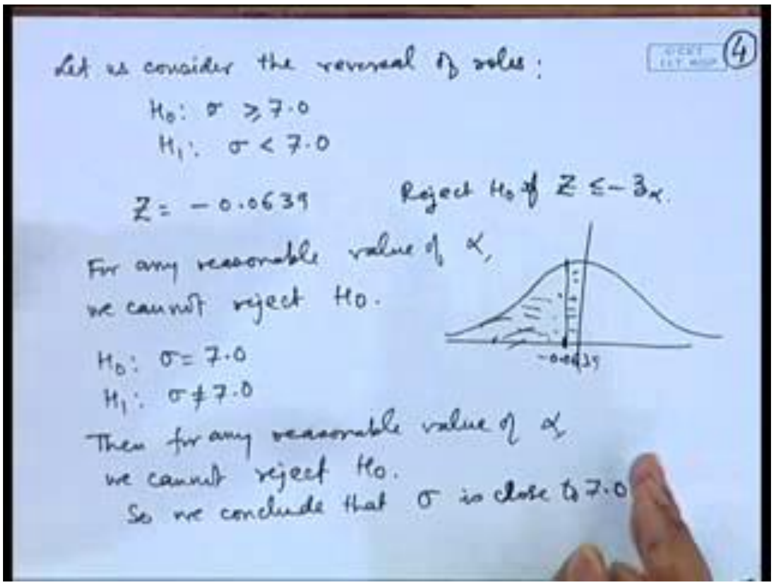


Let us consider say the reversal of roles, let us consider say H naught as sigma is greater than or equal to 7.0 against H 1 sigma is less than 7.0. Now if you consider your Z value that is minus 0.0639 and we are saying reject H naught if Z is less than or equal to minus z alpha. Now you can see for any reasonable level of significance, because these are the negative values. So, we are saying z is equal to minus 0.06 that is somewhere here; that means, there is a very small probability above this.

So, for a large value of alpha the hypothesis of H naught cannot be rejected, because all these values see this value 0.00639 this is were much higher, so for any reasonable value of alpha we cannot reject H naught. What does it mean, does it mean that sigma is greater than or equal to 7? Because in the previous test we concluded that sigma is less than or equal to 7 from here. So, I mention this point earlier that we frame the hypothesis in such a way that H naught is actually a weaker hypothesis or you can say accepting H naught is a weaker conclusion therefore, we put stronger (Refer Time: 11:58) on the alternative hypothesis.

So, if we see carefully probably the reason is that the Z value is smaller; that means, there is a reason to suspect that if we consider the hypothesis sigma is equal to 7, against sigma is not equal to 7 then for any reasonable value of alpha, we cannot reject H naught. So, we conclude that sigma is close to 7; that means, the hypothesis of sigma is equal to 7 cannot be rejected; that means, sigma is actually not significantly different from 7. So, this is an example of explaining that we should frame the null and alternative hypothesis in a judicious way as well as we should analyze the result for possible rejection of various kind of hypothesis, because we should not conclude falsely here for example, if we look at these and since this cannot be rejected, we are getting that actually sigma is greater than 7, but it is not significantly greater because when we consider sigma less than or equal to 7 then also this hypothesis could not be rejected. So, the only conclusion that we can draw is that actually sigma is closer to 7.

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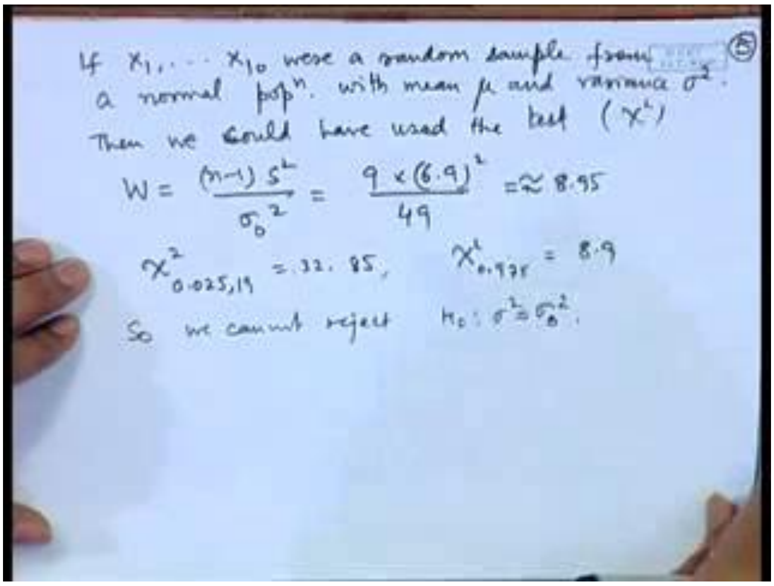


In this case suppose normality assumption was there, if the normality assumption was there then we may consider if X 1, X 2, X 10 were a random sample from a normal population with say mean and variance sigma square, then we could have used the chi square test, that is W is equal to n minus 1 S square by sigma naught square, that is 9 into 6.9 square by 49 that is approximately this value can be calculated as. So, this will be closer to 9, but slightly less than 9, some 8 point may be approximately let me put it here.

If we see the chi square value here, from the tables chi square say 0.025 that is 32.85, chi square say 0.0975 that is 8 point something. So, this is closer to this thing 8.95 something, this is 8.9 here. So, we can see here that neither W greater than this is satisfied nor W less than this is satisfied. So, we cannot reject hypothesis sigma square is equal to sigma naught square here also. So, the main point which I was trying to make here is that, here the hypothesis of equality cannot be rejected under any alternative.

Now we consider 2 sample problems; that mean, we may have to compare the means of 2 different distributions, the variances of 2 different distributions etcetera.

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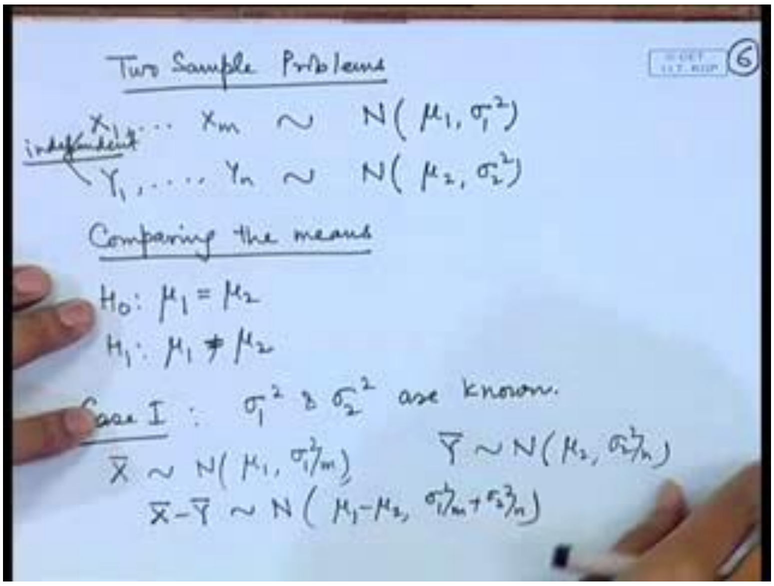


So let us consider the case when there is a normality assumption. So, 2 sample problems. So, in the beginning let us consider the normal model let X 1, X 2, X n, X n be a random sample from normal distribution with mean mu 1 and variance sigma 1 square and let Y 1, Y 2, Y n be a sample from another normal population with mean mu 2 and variance sigma 2 square. This type of situation occurs when many times we are interested in comparing 2 different setups for example, we may have 2 brands of certain product and we want to compare their average lives, average performance we may have say 2 ethnic populations, we may like to consider their say average expenditure on health care per month, we may have 2 different countries, we may like to concede compare the average longevity or average life of people in the 2 different countries.

In such cases the appropriate model is that we are considering 2 independent random samples, one from one normal population with certain mean and variance and another for from another normal population with certain mean and variance of course, the assumption of normality could have been replaced by some other populations also, but we are considering certain procedures which are applicable to the normal populations. Now suppose we want to compare the means, now if we want to compare the means; that means, we may have a hypothesis of the type say mu 1 is equal to mu 2 against say mu 1 is not equal to mu 2 or one sided hypothesis like mu 1 is less than or equal to mu 2 against mu 1 is greater than mu 2 etcetera.

Now there are various cases here, so first case is we may have the information about sigma 1 square and sigma 2 square, this may be through some prior experiment or from past data, the values of the variances may be known. In that case the testing problem becomes much simpler, we may consider X bar as normal mu 1 sigma 1 square by n, and Y bar follows normal mu 2 sigma 2 square by n. Now X bar and Y bar are independent because these 2 samples are considered independent therefore, if we consider the distribution of X bar minus Y bar that can be normal mu 1 minus mu 2, sigma 1 square by n plus sigma 2 square by n.

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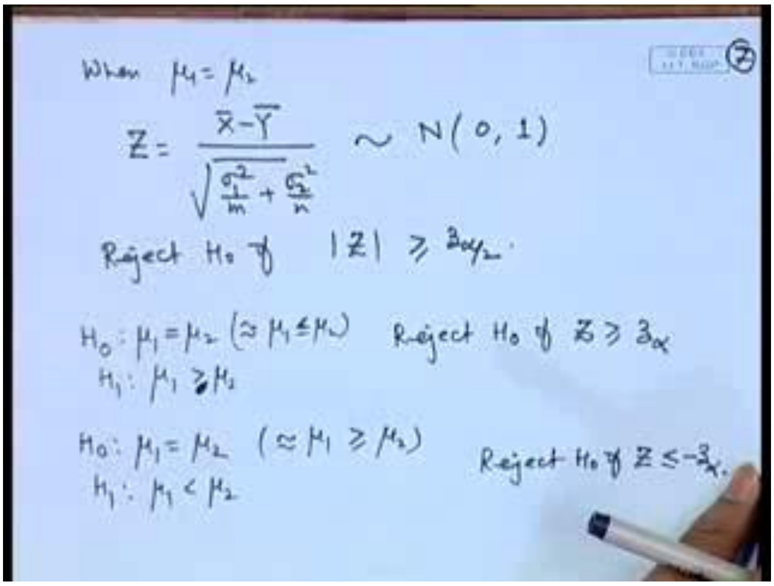


W Under the null hypothesis when mu 1 is equal to mu 2, this mu 1 minus mu 2 term becomes 0 and therefore, easily we can construct the test function. Then mu 1 is equal to mu 2 the function X bar minus Y bar divided by square root of sigma 1 square by n, plus sigma 2 square by n; this follows a normal distribution with mean 0 and variance unity.

So, this is the perfect test statistic to be utilized for testing this hypothesis. So, we may reject H naught, if modulus of Z is greater than or equal to say z alpha by 2; so this is the test for level of significance alpha. We can answer the question for the one sided hypothesis testing problems also for example, if I have mu 1 is equal to mu 2, against same mu 1 greater than mu 2 of course, this one is also equivalent to writing mu 1 less than or equal to mu 2, because the rejection region is dependent upon the alternative hypothesis. So, we may take the rejection region as reject H naught if Z is greater than or equal to z alpha.

Likewise we may right for the one sided hypothesis, where mu 1 is less than mu 2, mu 1 is greater than or equal to mu 2. So, here you will reject H naught if Z is less than or equal to minus z alpha. We will consider examples of this little latter. Firstly, let us consider the other cases also. So, if the variances sigma 1 square and sigma 2 square are unknown, then we cannot utilize this test statistic in that case we will have to put certain estimates for sigma 1 square and sigma 2 square.

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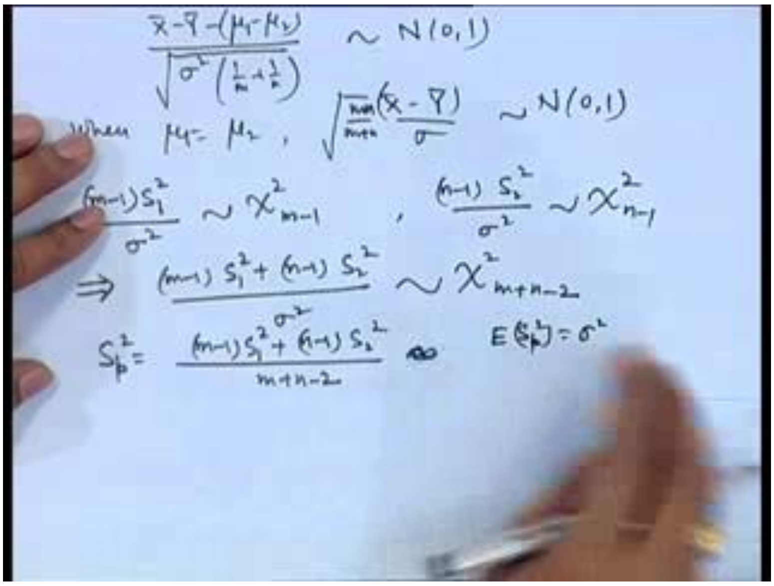


Once again here we have 2 different cases; so we have case 2, when sigma 1 square is equal to sigma 2 square is equal to say sigma square which is unknown. So, here we are assuming that the variances are not known, but they are known to be equal, this type of situation occurs in the cases such as we are considering measuring 2 different instruments using certain device and our measuring instrument, since it is the same measuring instrument the variability in the measurements will be same, but since it is measuring 2 different instruments therefore, the value which is being measured that is the means may be different. So, you may have different means, but the variances may be same. So, we may not know what is the variance, but since it is we are using the same instrument then the variability is likely to be the same.

For example if we have a we conduct a test and the test paper has a certain difficulty level therefore, if we conduct on 2 different sets of a students, the average marks may defer, but the variability may be the same because the test procedure is the same, we are using the same test. So, in such a situation we may make use of that X bar follows normal mu 1 sigma square by n and Y bar follows normal mu 2 sigma square by n. So, here X bar minus Y bar, minus mu 1 minus m 2 divided by sigma square 1 by n plus 1 by n is square root, this follows normal 0 1 once again the independence is utilized here. Now when mu 1 is equal to mu 2 we will get X bar minus Y bar square root m n by m plus n by sigma this follows normal 0, 1 distribution. Now we will consider estimation of sigma here because we cannot utilize the sigma in the test statistic. So, we look at the estimates of sigma square from both the samples and we may merge them.

So, if we look at S 1 square, then S 1 square is an estimate for sigma square. In fact, we have m minus 1 sigma 1 square, S 1 square by square this follows chi square distribution on m minus 1 degrees of freedom, and n minus 1, S 2 square by sigma square follows chi square distribution on n minus 1 degrees of freedom. Since the samples are independent the this statistics and this statistics are independent and therefore, by the additive property of the chi square distributions we have m minus 1, S 1 square plus n minus 1 S 2 square by sigma square follows chi square distribution on m plus n minus 2 degrees of freedom. So, if we define S p square as the pooled sample variance then this follows then expectation of S p square that is sigma square.

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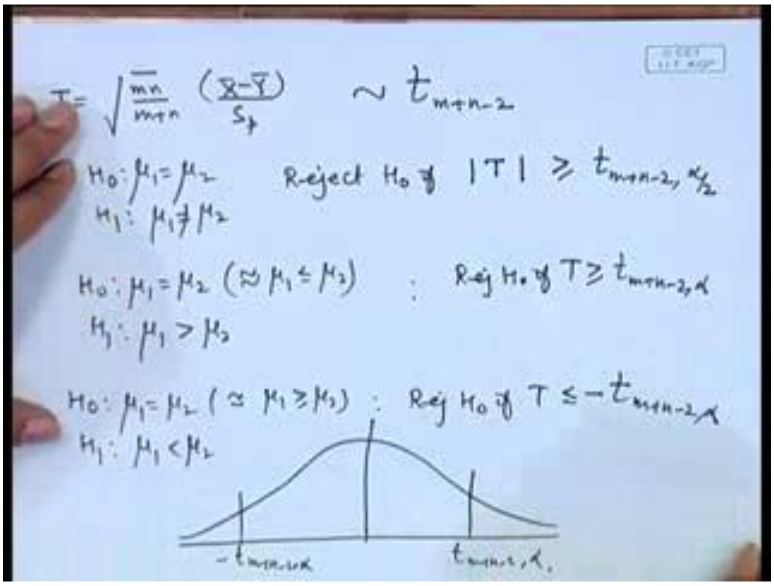

So, if we consider here the ratio we get square root m n by m plus n, X bar minus Y bar by S p that will follow t distribution on m plus n minus 2 degrees of freedom. Let us denote this by T. So, for H naught mu 1 is equal to mu 2 against mu 1 is not equal to mu 2, we can consider the test as reject H naught if modules of T is greater than or equal to t m plus n minus 2 alpha by 2. So, this is the level alpha test here.

If we have 1 sided alternatives for example, mu 1 is equal to mu 2 against say mu 1 is greater than mu 2 of course, this null hypothesis is can also be replaced by mu 1 less than or equal to mu 2, we will have the test reject H naught if T is greater than or equal t m plus n minus 2 alpha. On the other hand if we have mu 1 is equal mu 2 against mu 1 is less than mu 2 of course, this null hypothesis again equivalent to mu 1 greater than or equal t o mu 2, then we can use reject H naught if T is less than or equal to minus t m plus n minus 2 alpha, because the density of t is symmetric about 0. So, this is minus t m plus n minus 2 alpha point and this will be plus t n plus n minus 2 alpha point.

On the other hand there may be a situation where sigma 1 square and sigma 2 square may not be equal. There may be unknown as well as unequal if that is so then we may not be able to merge it here for example, here we will have sigma 1 square and here we will have sigma 2 square, likewise here also we will have sigma 1 square by m plus sigma 2 square by n. Naturally if we consider this ratio the term will not conceal out, in this case it is prudent to replace the unbiased estimators of sigma 1 square and sigma 2 square here, and see whether we can do something about the distribution of that. It is observed that that has an approximate t distribution, so in that case we use the following procedure.

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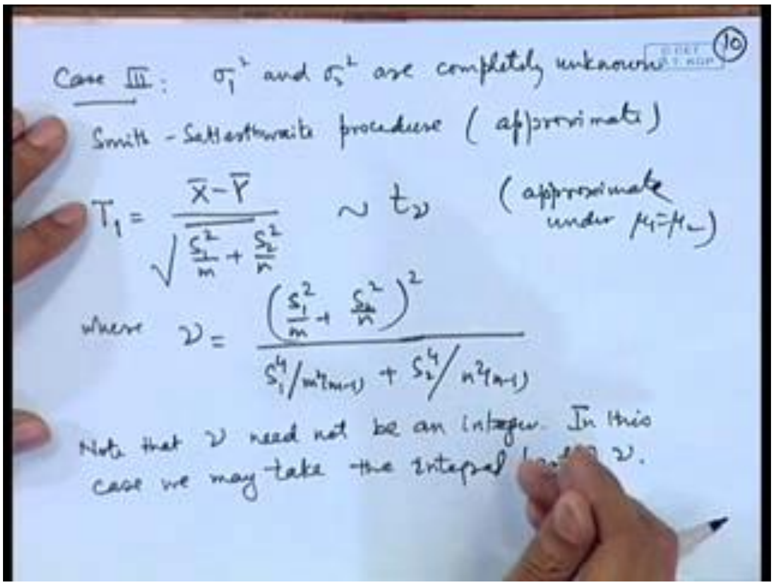


So, we have the third case when sigma 1 square and sigma 2 square are completely unknown; that means, we have no prior information about then, in that case we have the following Smith-Satterthwaite procedure. So, this is approximate procedure it is not exact, because exact distribution cannot be determined here. So, if we consider say T 1 is equal to X bar minus Y bar, square root of S 1 square by m plus S 2 square by n, then this has approximate t distribution on nu degrees of freedom under the assumption that mu 1 is equal to mu 2, where this nu value is given to be S 1 square by m plus S 2 square by n whole square divided by S 1 to the power 4 by m square into m minus 1 plus S 2 to the power 4 divided by n square into m minus 1.

Now, note here that this, nu note that nu need not be an integer, in this case we may take the integral part of nu. So, rather than taking the rounded of value which may be higher also it is better to take the lower value, because the power of the test will increase if we consider a lower degrees of freedom here.