Probability and Statistics

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Lecture – 62
Confidence Intervals -11

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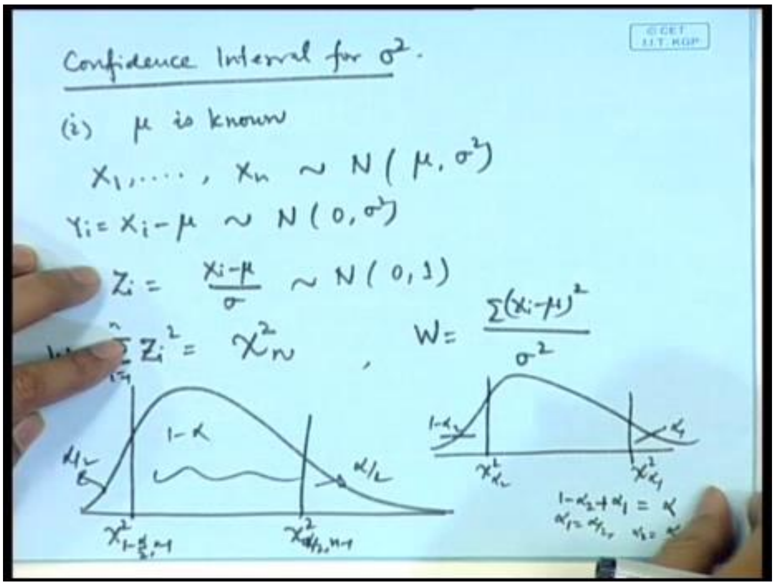

Confidence interval for; once again we have two cases; mu is known and mu is unknown, so if mu is known; let us consider here see. We are given that X 1, X 2, X n follow normal mu sigma square. So, if we consider say Y i is equal to X i minus mu and that will follow normal 0 sigma square. So if we consider say this thing divided by sigma; let us call it Z i; that is X i minus mu by sigma that follows normal 0; 1. So, if we consider sigma Z i square that will be that is called it W then that will follow a chi square distribution on n degrees of freedom.

We have seen that the sum of squares of N independent standard normal variables is a chi square distribution on N degrees of freedom. So, this since X i's are independent, this Z i's are also independent and therefore, some of squares will follow a chi square distribution on N degrees of freedom.

Now, you look at this statistics W, here W is actually sigma X i minus mu square divided by sigma square. Now if mu is known then this numerator quantity is say quantity which is based on the observations only because mu is known and the denominator is involving the parameter which is there.

So, according to our principle of pivoting quantity, this quantity can be taken as a pivot quantity because a distribution of that is distribution which is free from parameters. So, if we make use of the distribution of chi square, we consider say chi square alpha by 2 n minus 1 and chi square 1 minus alpha by 2 n minus 1. In general in chi square distribution because it is not a symmetric distribution, so one may consider say chi square alpha 1; that means, this probability is alpha 1 and chi square say alpha 2; that means, this probability is 1 minus alpha 2; such that 1 minus alpha 2 plus alpha 1 is equal to alpha, but for convenience because in that case, we may have several different values. So, for convenience one takes actually alpha 1 is equal to alpha by 2 and alpha 2 is equal to also alpha by 2 in that case this value becomes easy.

So, this probability is 1 minus alpha; this probability is alpha by 2; this probability is again alpha by 2.

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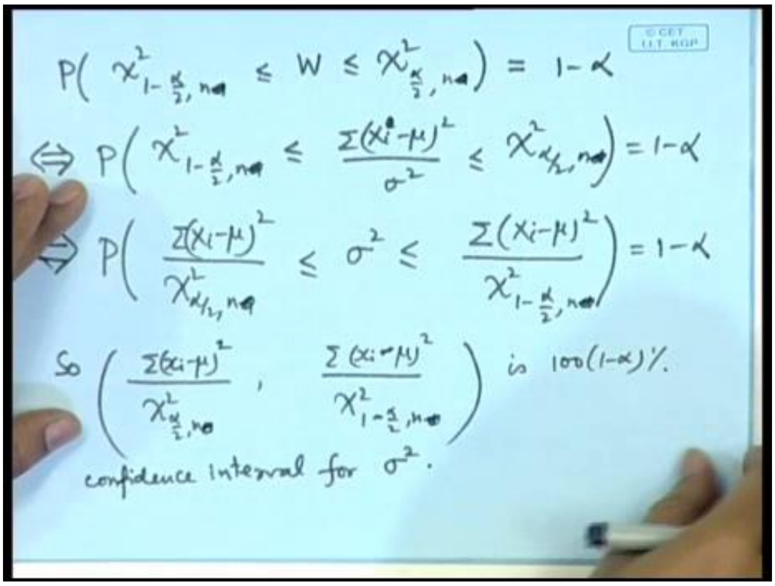

So, we can write down by statement probability that chi square 1 minus alpha by 2; n minus 1 less than or equal to W is less than or equal to chi square alpha by 2 n minus 1. This probability is equal to 1 minus alpha, so by manipulating this statement; we can get a confidence interval for sigma square in the following way. This is less than or equal to sigma X i square minus mu whole square divided by sigma square less than or equal to chi square alpha by 2 n minus 1; that is equal to 1 minus alpha.

So, if we look at this statement, this is equivalent to sigma square is greater than or equal to summation of X i minus mu whole square divided by chi square alpha by 2 n minus 1. If I look at the left hand in equality then this is equivalent to sigma square is less than or equal to summation of X i minus mu whole square divided by chi square; 1 minus alpha by 2; n minus 1. So, this statement is equivalent to probability that summation X i minus mu whole square divided by chi square alpha by 2 n minus 1; less than or equal to sigma square less than or equal to summation X i minus mu square divided by chi square 1 minus alpha by 2; n minus 1.

So, sigma of X i minus mu square divided by chi square alpha by 2 n minus 1 to sigma of X i minus mu square divided by chi square 1 minus alpha by 2 n minus 1; this is a 100; 1 minus alpha percent; confidence interval for sigma square, this is when mu is known. So, if the sample is observed then we can evaluate sigma X i minus mu square and the chi square values can be seem from the tables of a chi square distribution and the values can be substituted here.

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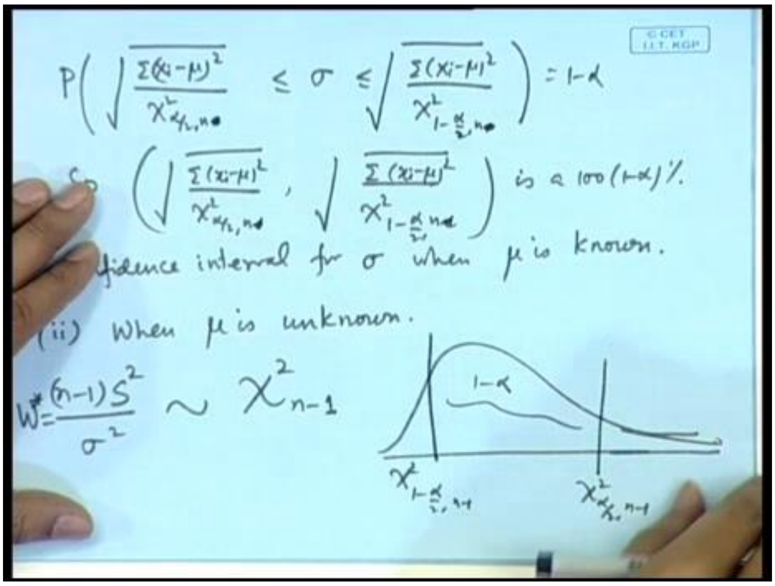

Now we observe here that this statement can also be written to create a confidence intervals for sigma also; that means, if you want a confident interval for standard deviation, so this statement can also be written as probability of a square root of sigma X i minus mu square divided by chi square alpha by 2 n minus 1; less than or equal to sigma; less than or equal to sigma X i minus mu square by chi square 1 minus alpha by 2 n minus 1. So, we get a confidence interval for sigma also that is sigma of X i minus mu whole square by chi square alpha by 2 n minus 1 to square root sigma X i minus mu square by chi square 1 minus alpha by 2 n minus 1; this is a 100; 1 minus alpha percent confidence interval for sigma; when mu is known. Now; obviously, if mu is unknown then this cannot be used because this quantity involves unknown value of mu.

So, when mu is unknown then we make use of S square. So, remember here that S square is defined as the sample variance that is 1 by n minus 1 sigma of X i minus X bar whole square. So, n minus 1; S square by sigma square follows chi square distribution on n minus 1 degrees of freedom, so you can observe here that this is a perfect quantity to be used as a pivot quantity; let me call it W star; n minus 1 S square by sigma square. So, here the numerator involves the observations and the denominator involves the parameter for which we want the confidence interval and the distribution of this quantity is free from the parameters.

Therefore, this W star can be used as a pivot quantity, so once again if we make use of the distribution of chi square on n minus 1 degrees of freedom. So, this is chi square, so I made a small error in the previous discussion; this one was chi square on n degrees of freedom; so this n everywhere not n minus 1, so this point will be n. So, the 100; 1 minus alpha percent confidence interval is sigma X i minus mu whole square by chi square alpha by 2 n to sigma X i minus mu whole square by chi square 1 minus alpha by 2 n and likewise when we consider for sigma then it is square root of the upper and the lower limits here; here the degrees of the freedom is n.

Whereas, if I am considering mu to be unknown then this is having a chi square distribution on n minus 1 degrees of freedom, so we will consider the point chi square alpha by 2 n minus 1 and chi square 1 minus alpha by 2; n minus 1.

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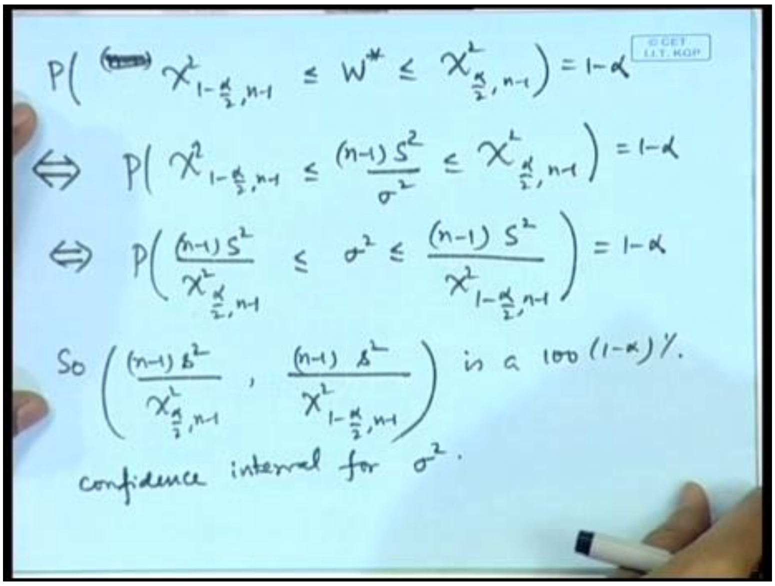

So, this probability is 1 minus alpha and we will have a statement probability of n minus 1. So we firstly write here chi square; 1 minus alpha by 2 n minus 1 less than or equal to W star, less than or equal to chi square; alpha by 2 n minus 1 is equal to 1 minus alpha. So, this statement is then equivalent to probability of chi square 1 minus alpha by 2 n minus 1 less than or equal to n minus 1, S square by sigma square less than or equal to chi square, alpha by 2 n minus 1; that is equal to 1 minus alpha.

That is equivalent to now from here, if we look at the right hand in equality; it is equivalent to sigma square greater than or equal to n minus 1 S square by chi square alpha by 2 n minus 1. If we look at the left hand in equality that is equivalent to sigma square is less than or equal to n minus 1 S square by chi square 1 minus alpha to n minus 1. Therefore, this entire statement is equivalent to n minus 1; S square by chi square alpha by 2 n minus 1 less than or equal to sigma square, less than or equal to n minus 1 S square by chi square 1 minus alpha by 2; n minus 1 that is equal to 1 minus alpha. So, n minus 1 S square by chi square alpha by 2 n minus 1 to n minus 1 S square by chi square 1 minus alpha by 2 n minus 1, this is a 100 1 minus alpha percent confidence interval for sigma square.

So if a sample is observed, we can evaluate the value sigma X i minus X bar whole square and get the sample variance and the chi square values can be seen from the tables of the chi square distribution and a confidence interval can be obtained. Once again you observe here that we may take the square root of the statements here and then we get a confidence interval for sigma also.

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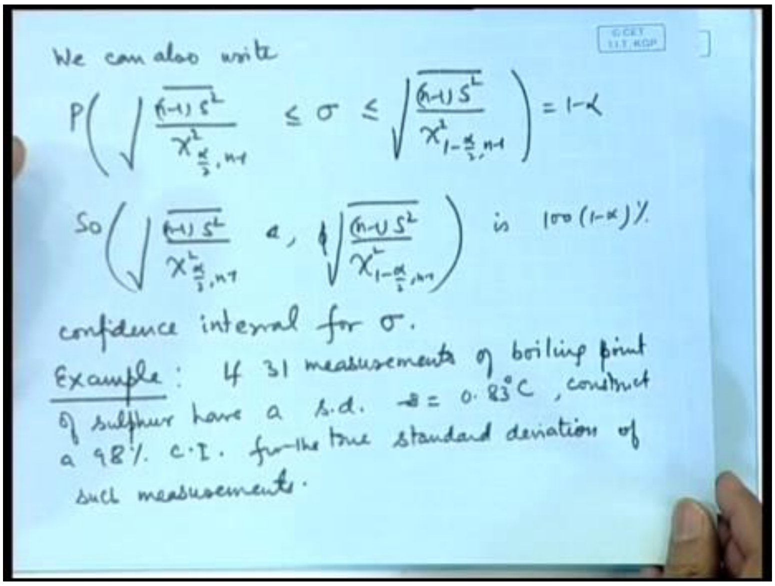

So, we can also write probability of square root n minus 1 S square by chi square alpha by 2 n minus 1 less than or equal to sigma less than or equal to root n minus 1 S square by chi square 1 minus alpha by 2 n minus 1. So, this root n minus 1 S square by chi square alpha by 2 n minus 1 to root n minus 1 S square by chi square 1 minus alpha by 2 n minus 1, this is 100; 1 minus alpha percent confidence interval for sigma. Let me do one example here; if 31 measurements of boiling point of sulphur have a s.d; that is s equal to 0.83 Celsius construct a 98 percent confidence intervals for the true standard deviation of such measurements.

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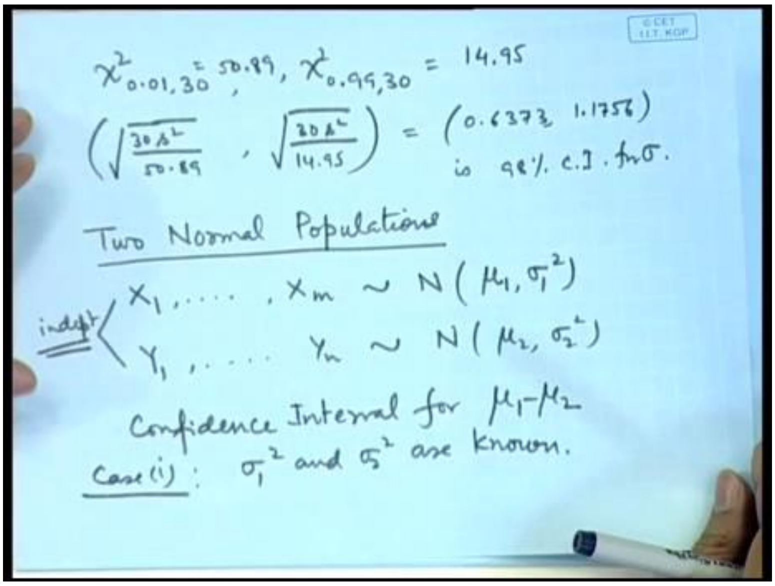

So, here we need the chi square value on 0.01 n minus 1 is 30 here. So, chi square 0.99, 30, so these two values can be seen from the tables of the chi square distribution to be this value is 14.95 and this value is 50.89. So, we calculate root 30 s square divided by 50.89 and root 30 s square by 14.95, where s value is given by 0.83. So, if we substitute these values; we get the interval to be 0.6373 to 1.1756, so this is a 98 percent confidence interval for sigma.

Now many a times in place of one normal population, we may have two normal populations and in that case we will require the confidence intervals say for the difference of the means we may require for the ratios of the variance says etcetera. We may also have populations where the proportions are important, so we may need a confidence interval for a binomial proportion, we may need a confidence interval for the difference of the proportions etcetera. So, let me consider in the first case the normal populations, so we have two normal population problems. So, we may have a different type of models; one could be that I have independent random samples say X 1, X 2, X n following normal say mu 1; sigma 1 square and Y 1, Y 2, Y n is another sample from normal mu 2; sigma 2 square and these two populations are such that the sampling is done independently. So, one problem could be to create confidence interval for say mu 1 minus mu 2 because we may be interested in the comparative difference between the two means.

For example, it could be that the first set of observations is based on certain patients and we are looking at the average effectiveness of a certain drug. So, the average effectiveness is mu 1 and the variability is sigma 1 square and suppose there is another drug and whose effectiveness is measured by Y 1, Y 2, Y n and the average is mu 2 and the variability is sigma 2 square. The sampling is done independently it means that the patients on which the drugs are have been applied their different groups and then definitely we are interested to compare the effectiveness of the two, that is what is a difference between mu 1 and mu 2. So, an important problem is to look at the confidence interval for mu 1 minus mu 2.

So, let us look at this problem; there may be again different type of cases, it could be that sigma 1 square and sigma 2 square are known; they are unknown, but equal they may be completely unknown or the sampling may be correlated. So, we will consider all of these cases one by one. Let us take case when sigma 1 square and sigma 2 square are known, so in this case we take help of the sampling distribution of X bar and Y bar.

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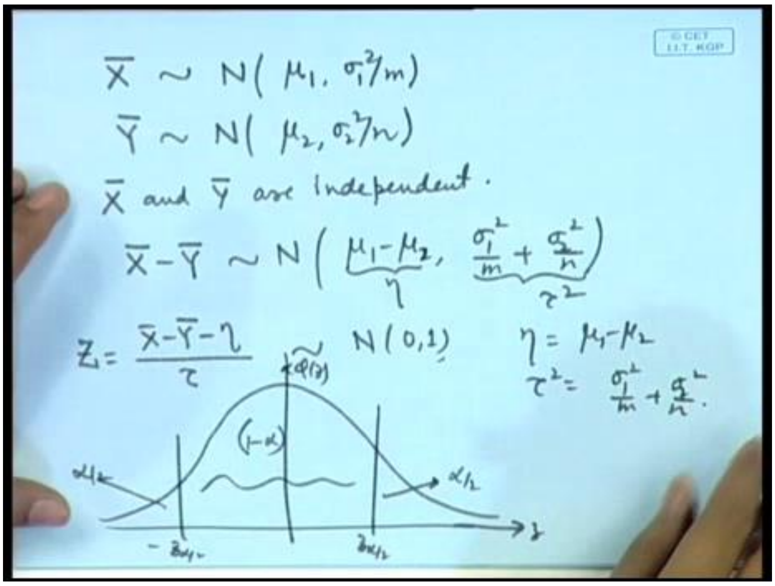

So, X bar follows normal mu 1, sigma 1 square by m; if we consider Y bar then that will follow normal mu 2 sigma 2 square by n. Another thing is that since sampling is independent X bar and Y bar are independent, so we can consider X bar minus Y bar that will have normal mu 1 minus mu 2 and the variance will be sigma 1 square by m plus sigma 2 square by n. Let us call this quantity say tau square, so we have X bar minus Y bar and this quantity we can call say eta; minus eta divided by tau, that will follow normal 0, 1 here; eta is mu 1 minus mu 2 and say tau square is equal to sigma 1 square by m plus sigma 2 square by n.

Now if we observe this random variable then this is involving the observations and the parameter under discussion; that means, we wanted a confidence interval for the parameter mu 1 minus mu 2 and that is appearing here and the distribution of this quantity is free from the parameters. Therefore, this can be considered as a pivot quantity and as before we can make use of the points of the standard normal distribution to write down a confidence interval. So, if this is the phi Z that is the density of the standard normal distribution; then Z alpha by 2 is this point such that this probability is alpha by 2 then on this side, we have minus Z alpha by 2; that is this probability is alpha by 2, so intermediate probability is 1 minus alpha.

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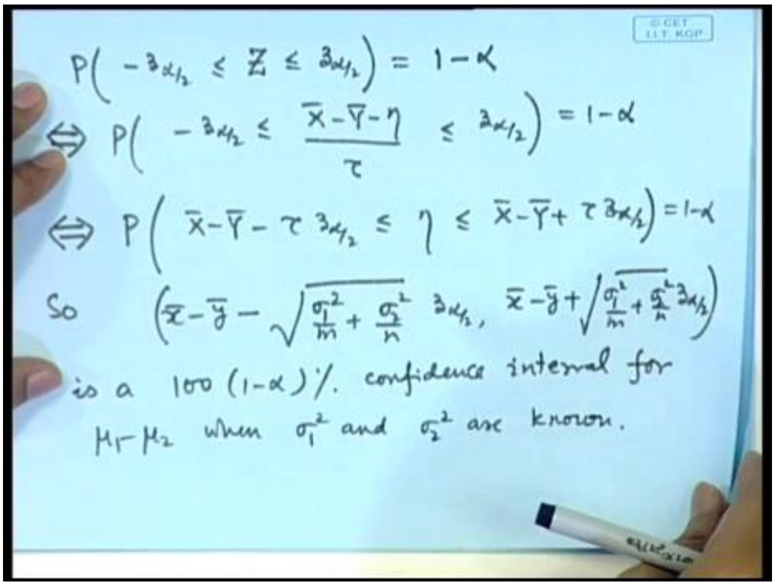

So, we are able to write down by statement probability of minus z alpha by 2; less than or equal to Z less than or equal to z alpha by 2 is equal to 1 minus alpha. So, this is equivalent to probability of minus z alpha by 2; less than or equal to X bar minus Y bar minus eta divided by tau less than or equal to z alpha by 2 is equal to 1 minus alpha.

So, if we multiply by tau and then utilize this condition that eta is greater than or equal to X bar minus Y bar, minus tau z alpha by 2 and eta is also less than or equal to X bar minus Y bar plus tau times z alpha by 2. So, the statement is equivalent to X bar minus Y bar minus tau; z alpha by 2 less than or equal to eta, less than or equal to X bar minus Y bar plus tau z alpha by 2; that is equal to 1 minus alpha. So, we get x bar minus y bar minus square root of sigma 1 square by m plus sigma 2 square by n; z alpha by 2 to x bar minus y bar plus square root of sigma 1 square by m plus sigma 2 square by n; z alpha by 2 as a 100; 1 minus alpha percent confidence interval for mu 1 minus mu 2, when sigma 1 square and sigma 2 square are known.

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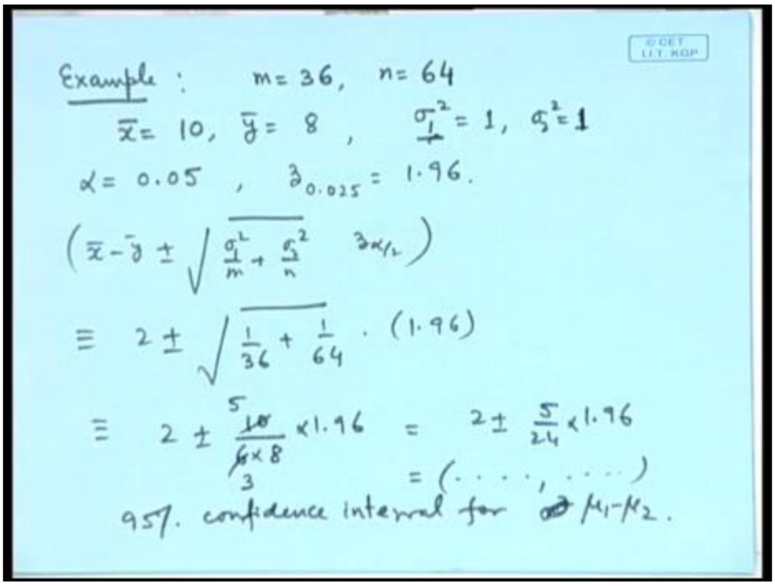

Let us take one example here; suppose we have a sample of size say m is equal to 36, n is equal to say 64, x bar value is say 10, y bar is equal to say 8, sigma 1 square is equal to say 1, sigma 2 square is equal to say 1; for convenience and let me take say alpha is equal to 0.05. Then z of 0.025 is equal to 1.96, so the confidence interval will become x bar minus y bar plus minus square root sigma 1 square by m plus sigma 2 square by n z alpha by 2. So, this confidence interval will be equal to 10 minus 8 is twice; square root 1 by 36 plus 1 by 64; 1.96. So, this value can be evaluated 64 plus 36 is 100, so 10 by 6 into 8 into 1.96; 5 by 24 into 1.96, so we get a 95 percent confidence interval for mu 1 minus mu 2.

Now this procedure especially for the normal distribution here, it suggests here that in place of mu 1 minus mu 2; suppose we are interested in say mu 1 plus mu 2 then the procedure will be similar because in place of `minus I may consider X bar plus Y bar, I may also consider any linear parametric function of mu 1 and mu 2. Suppose I consider 2 mu 1 plus 3 mu 2; then I can consider here 2 X bar plus 3 Y bar and here the variance quantity will be appropriately changing. For example, this will become 4 sigma 1 square by m plus 9 sigma 2 square by n. It may happen that we feel that the new procedure is say 3 times as much effective as the previous procedure, so in that case you would like to check whether mu 2 is equal to 3 mu 1.

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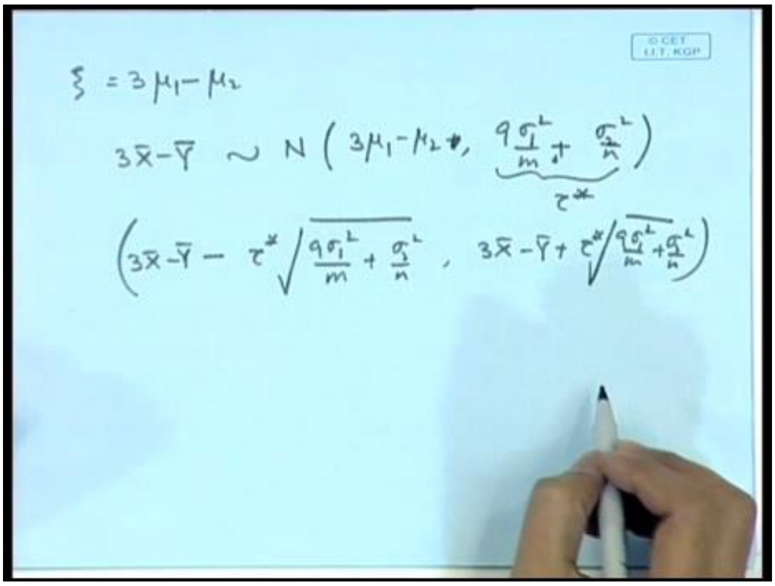

And therefore, you will like to find out a confidence interval for 3 mu 1 minus mu 2; let me call it says xi, so now we will consider 3 X bar minus Y bar. Then that will have a normal distribution with mean 3 mu 1 minus mu 2 plus and variance will be 9 sigma 1 square by m and plus sigma 2 square by n. So, the confidence interval will be appropriately changing; if I call this quantity s i; tau is star then the confidence interval will become 3 X bar minus Y bar, minus tau star; square root of 9; sigma 1 square by m plus sigma 2 square by n to 3 X bar minus Y bar plus tau star; root 9 sigma 1 square by m plus sigma 2 square by n; that means, in this particular situation for any linear parametric function of mu 1 and mu 2, I can calculate the confidence interval.

In the fourth coming lecture, I will be discussing the case when sigma 1 square and sigma 2 square are unknown and what type of problems that may lead to.