Probability and Statistics
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Lecture - 44
Distribution of Order Statistics

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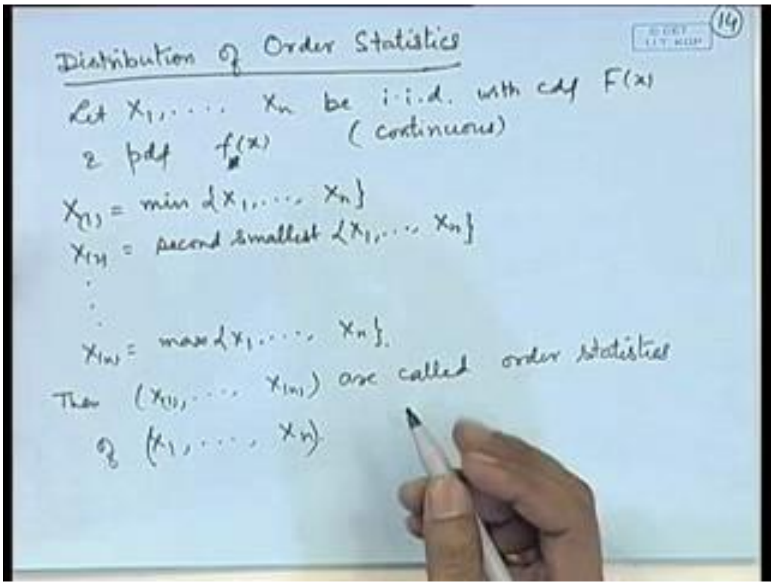

What are order statistics? Let X 1, X 2, X n be i. i. d with some cdf say F x and pdf; say f x and the let me assume it to be continuous. We define X 1 to be the minimum of X 1, X 2, X n; X 2 to be second smallest of X 1, X 2, X n. So, like that X 3 will be third smallest and so on; X n will be the maximum of X 1, X 2, X n. Then X 1, X 2, X n these are called order statistics of X 1, X 2, X n in many practical aspects, the order statistics are quite important. For example, the raw observations may be something; suppose the raw observations denote the marks by the students, but we may be interested in the ordered observation that who is getting the highest marks; who is a second highest etcetera.

If we are selecting certain candidates on the basis of certain scores, so we will be interested in selecting the best 10. So, we will be interested in say X n, X n minus 1 up to X n minus 9 say; so the top 10 students. So, in general we are interested in order statistics and therefore, the distributions of the order statistics. Suppose you want to find out the distribution of X n, so we may use a direct approach.

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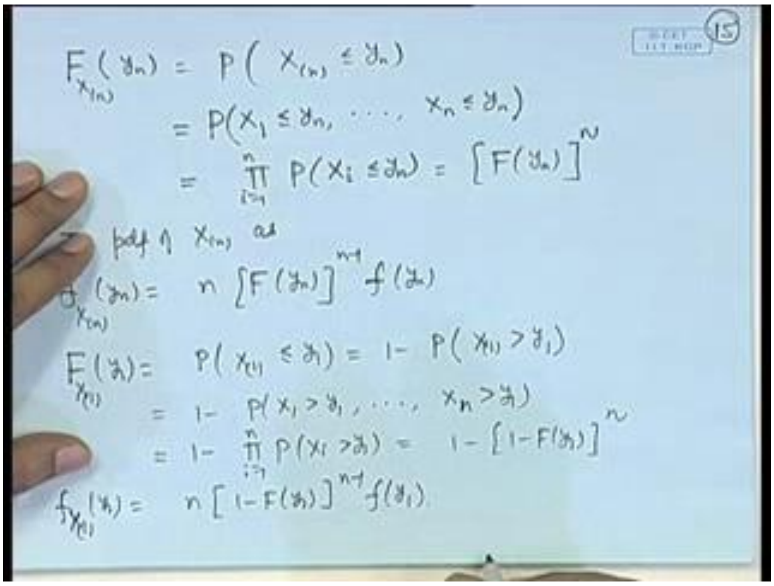

For example we look at F of X n; let me use a notation here, say Y i is equal to x i; for i is equal to 1 to n. So, the distribution of the largest that is probability of; X n less than or equal to y n, now notice here this is maximum being less than or equal to y n; this event is equivalent to that each of the X i is less than or equal to y n. Now we make use of the fact that the random variables are independent and identically distributed all of them have the cdf; F x. So, each of these values is F of y n and therefore, we get this as F of y n to the power N; if we are assuming the distributions to be a continuous then we can differentiate it and get the pdf of X n as n times F y n to the power n minus one F of y n.

So, using a direct cdf approach the distributions of the largest can be determined. In a similar way we can obtain the distribution of the minimum also; that is probability of X 1 less than or equal to say y 1; this we can write as 1 minus probability of X 1 greater than y 1. Once again if we are saying that the minimum is greater than y 1; it means that each of the observations is greater than y 1.

At this stage we can use the independence 1 minus product X i greater than y 1 and since each of this is identically same. So, it is 1 minus 1 minus F of y 1 to the power n, so the cdf of by smallest can be determined and the pdf of the smallest can be determined by differentiating this with respect to y 1; that gives n times 1 minus f y 1 to the power n minus 1; f of y 1. So, the distribution of the largest and by smallest order statistics can be determined using the cdf approach; however, this approach will be complicated; suppose we want to determine the distribution of third order statistics or say the joint distribution of fourth and seventh order statistics.

So, in that case if we are assuming the continuous random variables; we make use of the Jacobean method. So, let us consider the joint distribution of Y 1, Y 2, Y n.

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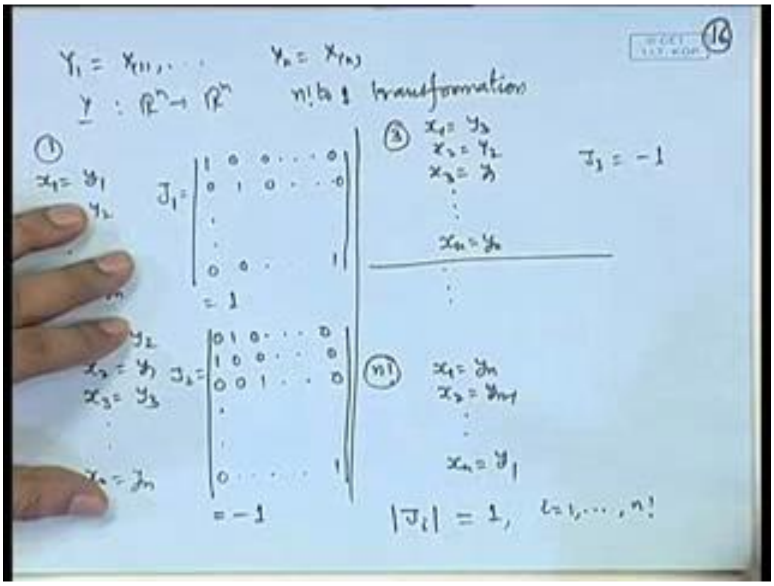

So, here the Function Y 1 is equal to X 1, Y n is equal X n; now this is a transformation from r n to r n, but it is a many one transformation. In fact, n factorial combinations of X 1, X 2, X n give the same values of Y 1, Y 2, Y n. Let us consider say n is equal to 2; suppose I say X 1 is equal to 1, X 2 is equal to 1.5, then Y 1 will be 1, Y 2 will be 1.5. We can take X 1 is equal to 1.5 and X 2 is equal to 1; once again Y 1 and Y 2 will remain the same; that means, two sets of X 1 and X 2 give the same value of minimum and maximum. In a similar way, suppose I have three values X 1, X 2, X 3 then six different combinations of X 1, X 2, X 3 will give the same values of Y 1, Y 2 and Y 3.

So, in general Y is R n to R n; this is n to n factorial 2, 1 transformation. So, we can partition r n into n factorial different regions so that from each region it is a 1 to 1 transformation. In the region 1 for example, you may have x 1 is equal to y 1, x 2 is equal to y 2 and so on x n is equal y n. In the region 2; we may have x 1 is equal to say y 2; x 2 is equal to y 1, x 3 is equal to y 3 and so on x n is equal to y n. In the region 3; it may be x 1 is equal to say y 3, x 2 is equal to y 2, x 3 is equal to y 1 and so on; x n is equal y n and so on. In the n factorial region, we may have x 1 is equal to say y 1, x 2 is equal to y n minus 1 and so on; x n is equal to y 1.

Let us look at the Jacobean in each case, the Jacobean here is the determinant of the identity matrix 1, 0, 0, 0, 0, 1, 0 and so on 0, 0, 1 which is simply equal to 1. If you look at the Jacobean in the second case; this is 0, 1, 0 and so on 1, 0, 0 and so on 0, 0, 1 and so on. Note here that this is obtained from j 1 by interchanging the first and second row or first and second column, so the value of this will be minus 1. So, likewise the Jacobean in each case will be either plus 1 or minus 1 because it is obtained by permuting rows of identity matrix. So, Jacobean absolute values for each of them will be 1 only.

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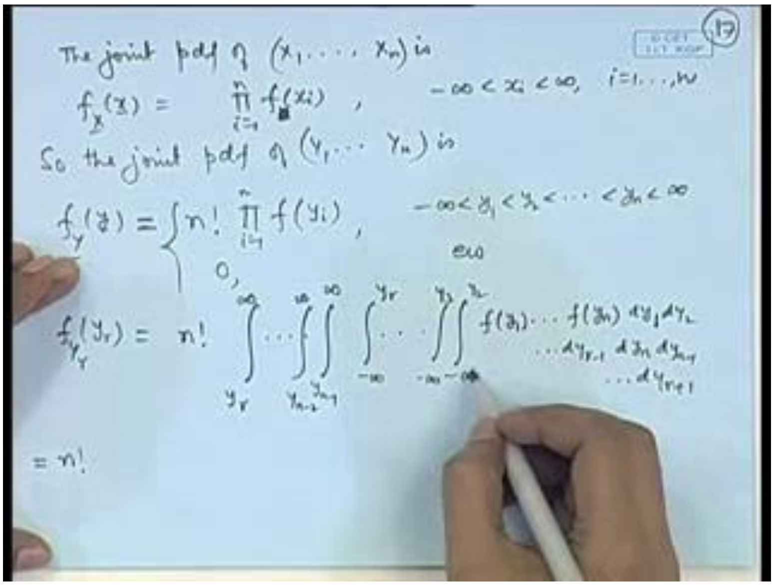

Now, the joint distribution of X 1, X 2, X n is product of f x i; i is equal to 1e to n. So, in general the regions are, so the joint pdf of Y 1, Y 2, Y n that is order statistics f y. Now in the first region, we will substitute x i is equal to y i multiplied by 1, in the second region we will multiply by 1 and substitute X 1 is equal to Y 2, X 2 is equal to Y n etcetera. Notice here that in each of the cases its only a permutation of y 1, y 2, y n, since it is a product of all these terms; it will always give total product of f of y 1, f of y 2, f of y n.

So, this becomes product of f of y i; i is equal to 1 to n and we have n factorial regions; so it is n factorial times and the region is y 1 is less than y 2 less than y n less than; we may have the cases where y 1 is equal to y 2 or say Y s is equal to Y s plus 1 for certain

s. Since we are assuming the continuous distributions, the probability of two variables being equal is 0 and we can ignore this. So, we can see here that the joint distribution of y 1, y 2, y n can be retained, now suppose we are interested in the distribution of say rth order statistics say; what is f y r; then we need to integrate y 1, y 2, y r minus 2 and y r plus 1, y r plus 2 up to y n.

So, we can device a scheme for integration it will be integral f of y 1, y 2, f of y n. So, we can integrate with respect to Y 1 from minus infinity to Y 2 with respect to Y 2 from minus infinity to Y 3 and so on; up to y r minus 1; minus infinity to y r. Then we can integrate y n from y n minus 1 to infinity; y n minus 1 from y n minus 2 to infinity and so on; d y r plus 1 will be from y r to infinity.

So let us look at the evaluation here; if we integrate with respect to y 1; f of small f of y 1 gives capital F of Y 1 and when we substitute, the limits here at minus infinity it is 0 and at Y 2; it is capital F of Y 2.

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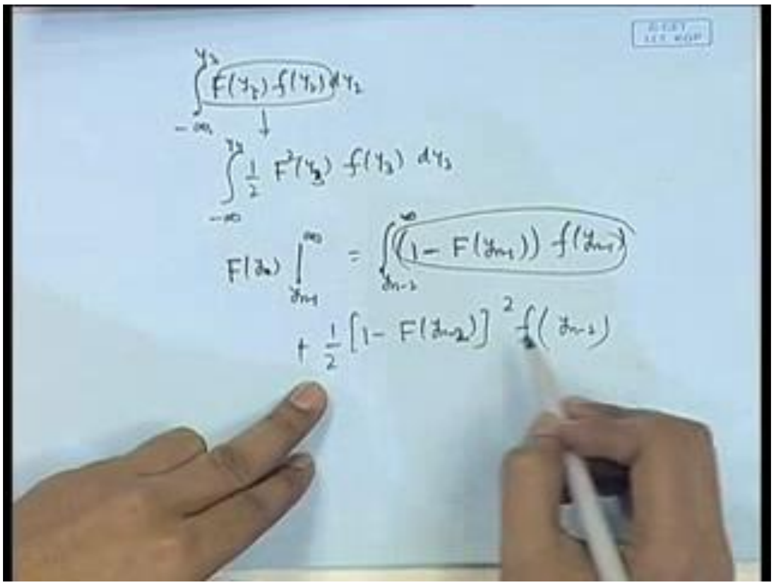

So, this gives us capital F of y 2 in to small f of y 2 in the next stage, this will be integrated from minus infinity to y 3. Now notice here is that if we integrate this will get half F square y 2 and again if we substitute the limits; I will get it as half F square Y 3 and at minus infinity this will be 0. So, at the next stage the integrant will be this term from minus infinity to y 4.

So, next stage it will give F q of y 4 and here it will be 1 by 2 in to 3. Now this step will be continue up to F of y r minus 1. So, the last step will give us up to r minus 1 factorial F of y r to the power r minus 1 because in the first stage; it is capital F of y 2, in the second stage it is F square by 2, in the next stage it is F cube Y 4 by 3 factorial. So, at the r minus 1 of the stage; this will give F of Y r to the power r minus 1 by r minus 1 factorial.

Now, let us look at the other set of variables to be integrated; when we integrate f of y n then that will gives us capital F of y n. Now the region of integration is from y n minus 1 to infinity, now at infinity this is 1, so it is 1 minus F of y n minus 1. So, at the next stage the integrand is 1 minus F of y n minus 1 into f of y n minus 1. This we are integrating from y n minus 2 to infinity. Again we notice here is that the integral will be 1 minus F of y n minus 1 is square by 2 with a minus sign. So, at infinity this will become 0 and at y n minus, this will become this term.

So, that the next stage again and this will give us cube divided by 3 into 1.

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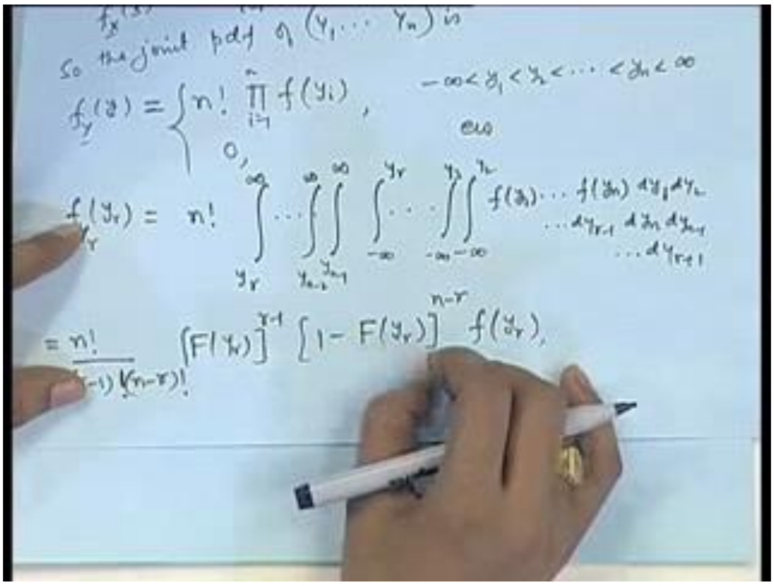

Like that we have to continue up to d y r plus 1, so this will give us n minus r factorial; 1 minus F of y r to the power n minus r; multiplied by f of y r. So, we are able to determine the distribution of the rth order statistics, here the particular case we can see, suppose the random variables are uniformly distributed on the interval 0 1; then this will become Y r, this will become 1 minus y r to the power n minus r; which is nothing but a beta distribution. So, that is one of the origins of or you can say applications of beta distribution.

Likewise if you want to integrate living variables say Y r and Y s; so the joint pdf of say two order statistics.

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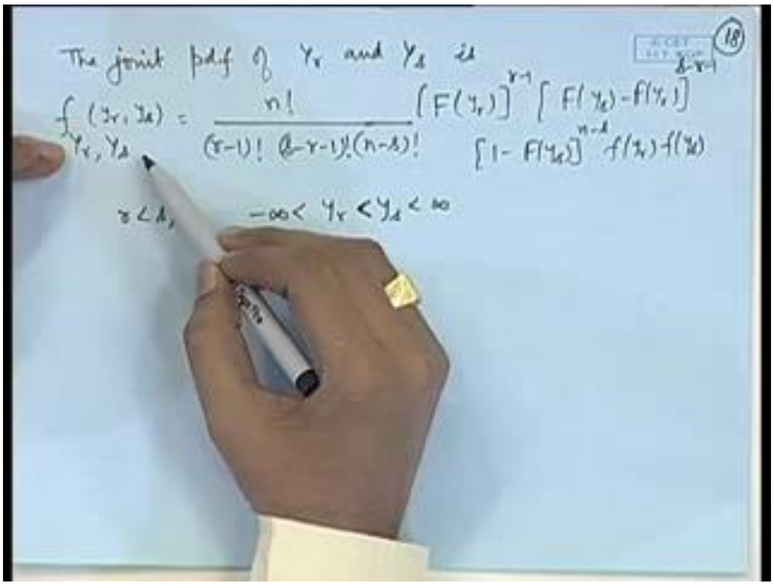

Say Y r and Y s, so that is determined as f of Y r Y s; y r; y s that is equal to n factorial divided by r minus 1 factorial; s minus r minus 1 factorial n minus s factorial; F of y r to the power r minus 1; F of y s minus f of y r to the power s minus r minus 1; 1 minus F of y s to the power n minus s; F of y r y s; here I am taking r to be less than s. So, y r will be less than y s. In particular, we may write the joint distribution of by smallest and the largest from there we can determine the distribution of the range that is y n minus y 1 etcetera. So, in the next lecture we will be considering various applications of the transformation see you.

Thank you.