Probability and Statistics
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Lecture – 04
Sigma-Ring, Sigma-Field, Monotone Class

Now let me define another extension of the definition of ring. And we call this structure as a Sigma- Ring, we can also write it in this fashion Sigma-Ring.

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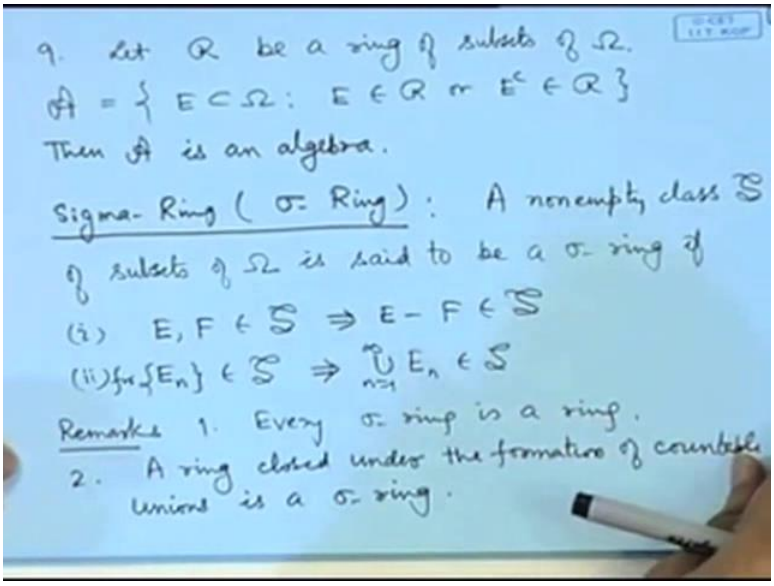

So, this is an extension of the definition of ring in the sense that a ring was closed under the operation of taking finite unions and differences, if we change the finite unions to countably when unions then it becomes the definition of a sigma- ring. So, a formal definition is a nonempty class of subsets of omega. So, let me denote this class by say is script S is said to be a sigma- ring. If it satisfies the following two properties; that given any two sets they are difference must be in the class. And secondly, if I consider any sequence then its union must belong to.

So, usually you can see that the second one is a generalization from the definition of a ring, because there we assumed only finite union. So, naturally the consequences that every sigma- ring is a ring. Secondly, if I have a ring and it is closed under the formation of countable unions then it is a sigma- ring. As you seen in that in the definition of ring we assumed only closeness under the formation of unions and the differences. However, we could show that it is closed under the formation of symmetric differences, the formation of intersections etcetera.

In a similar way if I am considering a sigma- ring then it will be also closed under the formation of intersections. So, let us write down that a statement.

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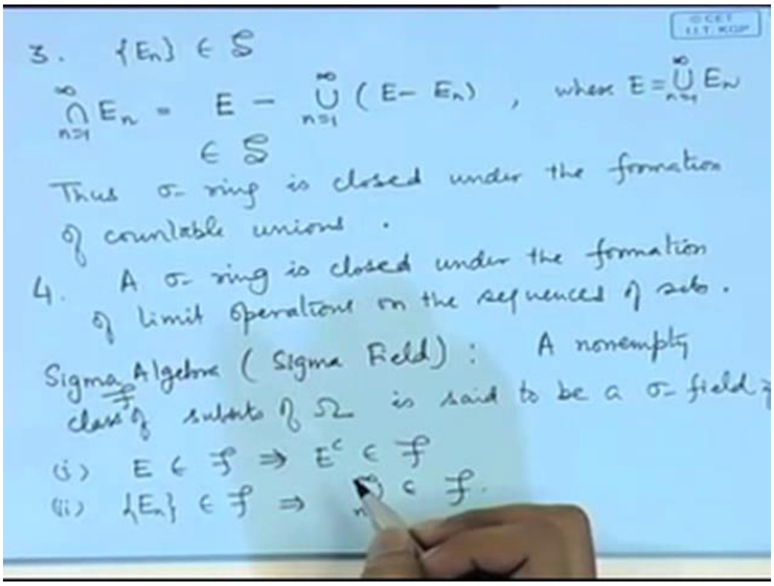

If I have a sequence say E n in S then we can write intersection E n, n is equal to 1 to infinity as E minus union E minus E n, n is equal to 1 to infinity. Where E denotes the set union n is equal to 1 to infinity. Now if is script S is a sigma- ring and if I am considering a sequence E n there, then naturally the countable union of the sets belongs to the class S. Now, E minus E n also belongs, because this is the differences of that sets. And therefore, if I again take the countable union that is again belonging to the class S and therefore E minus this is again in this. So, this belongs to S.

Now since this is alternative representation of the intersection E n this means that every sigma- ring is closed under the formation of countable unions. The previous class we introduced the concept of the limit of a sequence of sets. It was actually defined as a limit superior of a sequence of sets, limit inferior of a sequence of sets and if the two are equal then the limit exists. Now we had alternative representation of the limit superior and limit inferior in the form of countable unions of countable intersections are countable intersections of countable unions. So now, if by initially structure is a sigma- ring and it is closed under the formation of countable unions and countable intersections therefore, it is also closed under the operations of limit superior and limit inferior. And therefore, if the limit exist then under the operation of taking limits.

So, you can mention that a sigma- ring is closed under the formation of limit operations on the sequence of sets. One more extension of the definition of ring R, algebra R sigma-ring is the so called structure called sigma algebra or a sigma field. So, let us define it in the following fashion sigma algebra or a sigma field. So, a nonempty class of subsets of omega. So, let me define this class as a script F is said to be a sigma field if it satisfies that for any given set its complementation will also be in the given class. Secondly, for any sequence of sets the countable union must also be in the class.

So, you can see that it is a generalization of the definition of a field as well as its generalization of the definition of sigma- ring. It is a generalization of the definition of field in the sense that in a field we are closed under the operation of taking complementations and unions, but unions where taking to be finite, here we have taken countable unions. It is an extension from the definition of a sigma- ring in the sense that countable unions are there and the differences have been replaced by complementations. So, in some sense it is one of the largest structures among the four structures that we have defined.

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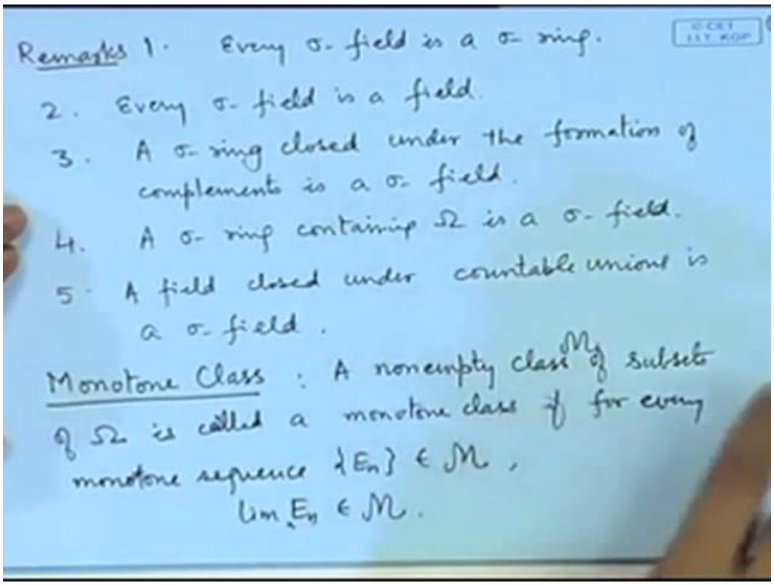

So, we can write the comments that every sigma field is a sigma- ring; every sigma field is a field, and therefore every sigma field is also a ring. We can also say that a sigma-ring closed under the formation of compliments is a sigma field. A sigma- ring containing omega is a sigma field. A field closed under countable unions is a sigma field. So, among the four structures this sigma field or sigma- ring is the most generally structure.

A related structure is that of a monotone class: a nonempty class of subsets of omega is called a monotone class if for every monotone sequence E n in this class, unit of E n belongs. We prove that a monotone ring is a sigma- ring and a sigma- ring is a monotone class. See we already said that in a sigma- ring the limit operations are valid, and therefore in the sigma field also the limit operations are valid.

So, monotone class is a particularly structure which is something like in between a ring and the sigma- ring are between a field and a sigma field. However, it is useful in the sense that gives you in a class if I just look at whether the limits are their then monotone class is confirmed and in many operations that is what we finally need. So, let me prove the following theorem.

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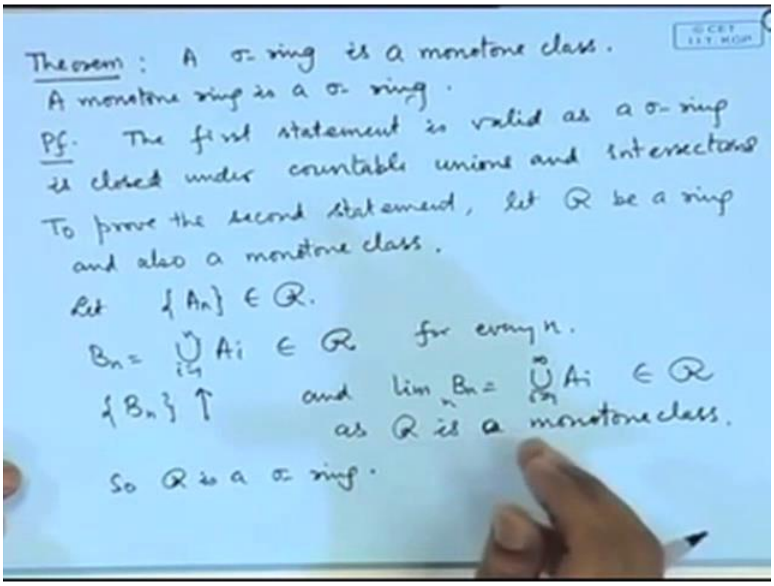

A sigma- ring is a monotone class. A monotone ring is a sigma- ring. To prove that a sigma- ring is a monotone class we notice that a sigma- ring was closed under the operations of taking infinite unions and infinite intersections. And therefore, the limit operations were valid. Therefore, a sigma- ring is naturally a monotone class because if a monotone sequences taken its union are intersection is the limit depending upon whether you have a monotonically increasing sequence or a monotonically decreasing sequence.

So, the first statement is valid as a sigma- ring is closed under countable unions and intersections. To prove the second statement let us see in the following fashion. Let R be a ring and also a monotone class. Since it is already a ring we have to only show that countable unions will belong to the given class to prove that it is a sigma- ring. So, let us consider a sequence An in R. Now if I take say Bn is equal to union of Ai, i is equal to 1 to n then it is a finite union of the sets in R, and therefore it will belong to R for every n. Now the nature of this set Bn is that; if I take Bn plus 1 then one more set will be coming. So, Bn is naturally a monotonically increasing sequence of sets, and limit of Bn will become equal to union Ai, i is equal to 1 to infinity.

Since, R is a monotone class limit of Bn will belong to R; as R is a monotone class. So, R is a sigma- ring. This result is useful in the sense that if we want to create a sigma- ring form a given class of sets then taking all the unions' etcetera may be quite complicated. Whereas if it is already a ring, if we insure that the limits of the sequence of the sets is present in the given class then it will become a sigma- ring. So, it is something like you can say a method of obtaining generated sigma- rings from a given class which is already a ring.

A somewhat simpler structure in this direction is a Semi- Ring.

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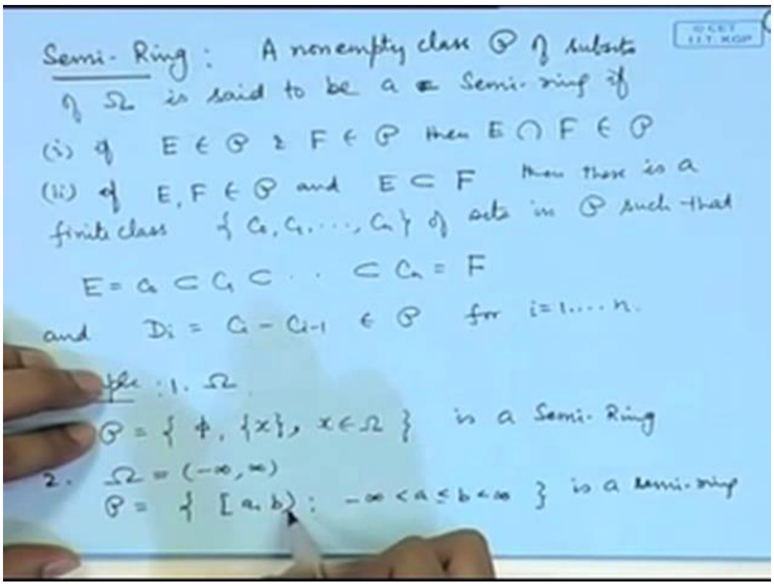

A nonempty class say P of subsets of omega is said to be a semi- ring if it satisfies the following properties: that is if a set is there then E intersection F belongs to P. That means it is closed under the formation of the taking intersections. However, a another property which looks somewhat different and the once which we have until now is that is E and F are any two subsets of omega (Refer Time: 17:03) and say one of them is a subset of the other one then there is a finite class c naught c 1 up to c n of sets in P such that E is equal to c naught subset of c 1 subset of c n which is equal to F. And the success see would difference is of Ci's must be in P.

I mention that it is a structure which is simpler compare to the structures that we define just now. In the sense that I am not assumed that union will be there our complementations will be there. You can see here through an example; if I consider say omega to be any set and P is say the class consisting of the phi set and the set of consisting of singleton sets for all x belonging to omega then this is a semi- ring. Let see how.

If I take intersection of any two sets then that will be empty set. If I take any two sets is most that is the subset of another one the second property is trivially satisfied, so this is a semi- ring. Now why I said that it is a structure which is more simpler in nature compare to a ring, sigma- ring, field or sigma field. Because now you see this set P it is not satisfying properties of none of the previous structures for example, unions of two sets are not there. So, it cannot be a ring sigma- ring field are sigma field, whereas it is a semi- ring.

Let me take another example: suppose I consider omega to be the set of yield numbers and I consider P to be the class of intervals of the form a to be; that means, semi closed intervals I am taking the left closed and right open, then this is a semi- ring. To see the structure of this suppose I am considering any two intervals of the form a to b.

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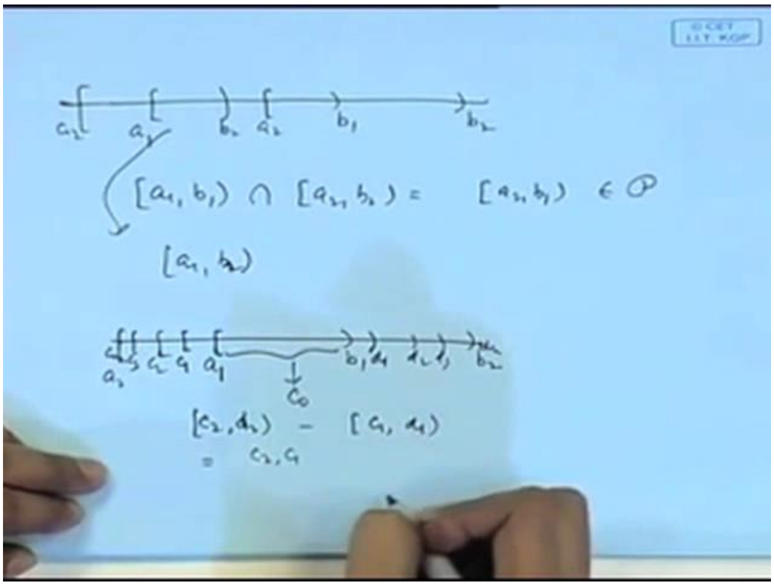

Now, if I take say a 1, b 1 and say a 2, b 2. Then if I take the intersection of this, then if there of this form then the intersection of a 1, b 1 with a 2, b 2 is of the form a 2, b 1, which is again and interval of the same form. Suppose I take these to be disjoint then the intersection will be phi which is correspondent to the choice A is equal to B here. Similarly, I may take this on this side say a 2, b 2, in that case the intersection will be a 1, b 2 which is again and interval of the same form. So, in all the cases the intersections will be existing in the class P. Now suppose I consider two intervals such that one of them is contained into another one. So, let me take a 1, b 1 contained in the interval a 2 to b 2.

Now, if I look at this then I can consider n classes which are n sets, here if you look at the second property of the definition of the semi- ring then there is a finite class of sets in P c naught, c 1, c n such that there is smallest of this is same as E and the largest of this is same as F; such that they are difference is belong to P. So, if you look at this one I can always construct sets like this. So, this set itself can be i c naught, this can be some set let me call it is a c 1 to d 1 say c 2 to d 2, c 3 to d 3 and this is say c n to d n. Then each of them is a super set of the previous one, the largest is equal to the bigger interval a 2 to b 2, the smallest is equal to the interval a 1 b 1. All of the sets are of the same form, and therefore they are satisfying the property that they are in P.

If I consider the difference of any two successive sets so for example, if I take c 2, d 2 minus c 1, d 1 then it is equal to now in this particular structure the difference is equal to c 2, d 2, so I am removing c 1, d 1 from here then it is becoming equal to c 2, c 1. So, if I look at the definition of the semi- ring then it is belonging to the class P.

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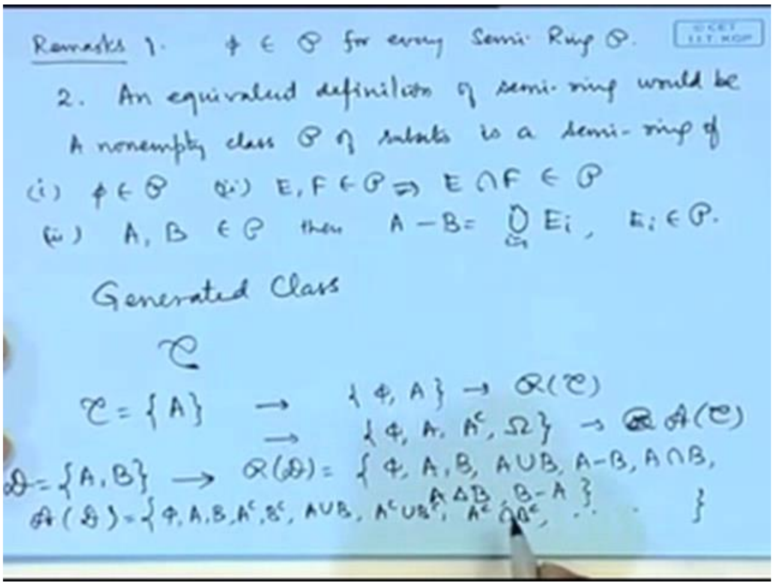

We have the following remarks: phi belongs to P for every semi- ring, because I can take two sets we equal and if I take the differences then it will be there. An equivalent definition of semi- ring would be a nonempty class P of subsets is a semi- ring if empty set is there, for any two sets intersection must be there, and thirdly if A and B belong to P then I should be able to represent A minus B as the union of a finite unions of sets in P. So, this is an alternative representation of the definition of semi- ring.

A useful concept in this context is that of a generated class. What is the generated class? So, given a class of sets if I consider the smallest ring containing c is smallest sigma- ring containing c, the smallest algebra containing c, or the smallest sigma algebra containing c, or the smallest monotone class containing c, then it is called a generated ring it generated sigma- ring, a generated algebra or a generated sigma algebra.

Now to see this suppose I look at say a set consisting of single time set A then phi A; suppose I will say this is my c then this is the generated ring of c. Suppose I take this same thing and I take phi A A compliment and omega then this is a generated algebra from C. Since this is a finite class therefore, this is also generated sigma- ring or generated sigma algebra.

To see that how this process becomes more complicated if I considered larger classes is that; suppose I consider say class say A B. Now you can see in order to generate a ring out of this I have to consider the empty set, the two sets, they are union, they are difference is; as we have said its intersection will also be there its symmetric difference will also be there. So now you can see that variety of sets B minus A must also be there because we can take the difference of these two.

So, a generated ring is of this nature which is consisting of many sets. Now you see how the process will become even more complicated if I try to generate a field out of this. If I consider a field generated form D then the number of sets is much more. I will have to take phi, A, B, then I have to take A compliment, B compliment, A union B then A compliment union B compliment, then I have take A compliment intersection B compliment and so on so forth. So, the number of sets will be much more.

And this shows that this kind of a structures will be useful in the definition of probability, because when we want to introduce the probability of different events when their simultaneous occurrence, their differences, their unions etcetera also will be needed in defining the probabilities. So, this type of definition will be extremely useful for our.

Thank you.