Probability and Statistics
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Lecture – 32
Problems on Special Distributions – II

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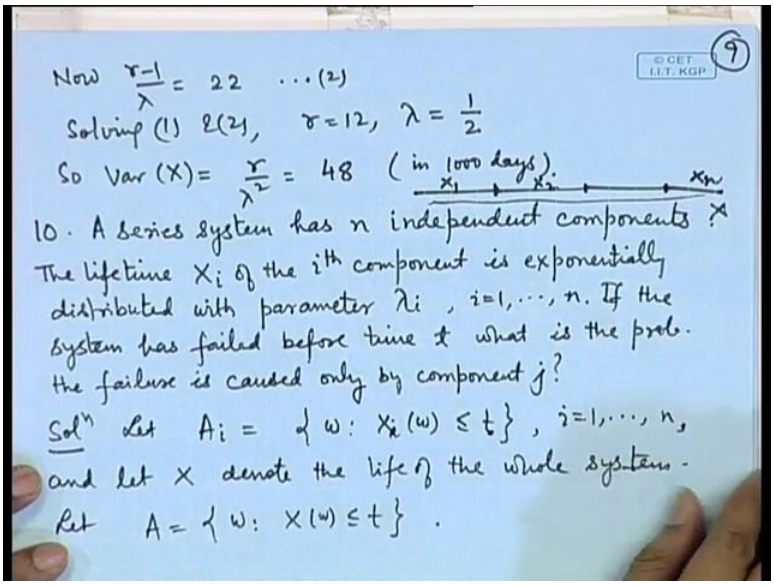

Let us consider a series system: a series system has n independent components. The lifetime, so the components are attached and the life's here are X 1, X 2, X n; suppose the total life of the system is X. So, individual life times are assumed to be exponentially distributed with parameters lambda I; so lambda 1, lambda 2, lambda n. If the system has failed before time t what is the probability that the failure is caused only by component j? Basically we consider it as a weak link.

So, let A i denote the event that the ith component fails before time t. So, X i omega less than or equal to t. Consider X to be the life of the whole system, so A is the event that the entire system fails before time t, we are asked to find out the probability that the system has actually failed. So, what is the probability that the failure is caused by the component j? So, it is actually the conditional probability that the jth component fails before time t, because A i denotes the event that ith component fails before time t.

So, here we are interested in that the jth component fails and all components other than the jth component do not fail; this A 1 compliment A 2 compliment etcetera, except A j compliment A. So, it is the simultaneous occurrence that is jth components fails and all the components other than the jth component are working at time t. So, what is the conditional probability of this event given that actually the system has failed?

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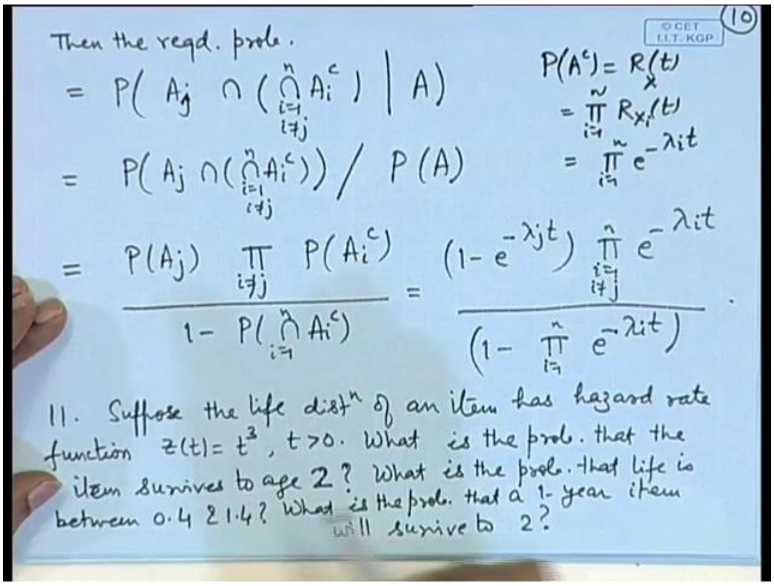

So, we apply the formula for the conditional probability; so, probability of some event e given and event f. So, it is equal to probability of e intersection f divided by probability of f.

Now, if you look at this event, here it means one of the component fails and intersection with the event that the system fails. Since it is a series system one of the components fails necessarily implies that the system has failed. So, this probability this event is a subset of the event A, therefore the intersection will give me only the event which is described here that is A j intersection with the intersection of A i compliment, where i is not equal to j. Now at this stage we make use of the assumptions that the components are independently working. That means, failure are not failure of a individual component does not affect the failure or not failure of any other component.

So, here we can apply the formula for the probability of the intersection of the events for independent events. So, it becomes probability of A j into probability of these events which again can be split as the product of the probabilities divided by probability of A. Now what is the probability of A? This means that system has failed before time t. So, it is equal to 1 minus the reliability of the system at time t, now reliability of a series system is nothing, but the probability of the; and the product of the reliabilities of the individual terms.

So, here probability of A compliment that is equal to reliability of the system at time t, which is equal to product of the reliabilities of individual components. Now each of the lives is exponentially distributed, so reliability of the ith component is e to the power minus lambda i t product i is equal to 1. So, that is the term coming here corresponding to probability of A compliment. In the numerator probability of A j that is the jth component fails before time t, it is 1 minus e to the power minus lambda j t and here it is reliabilities of the all other components except the jth component. So, it is product of e to the power minus lambda i t where i is not equal to j. So, this is denoting the conditional probability that the failure is caused by the component j alone; that means, the system has failed that we know what is the conditional probability that the failure was caused by the jth component.

Let us look at applications of Weibull distribution. So, suppose the life of an item has hazard rate function Z t is equal to t cube. So, if you recall we considered that the hazard rate function is of polynomial type, that is alpha, beta, t to the power beta minus 1, if and only if the distribution of the life is Weibull distribution with parameters alpha and beta. So, here if the life distribution is given to be Z t is equal to t cube, it is exactly of the form of a hazard rate function of a Weibull distribution.

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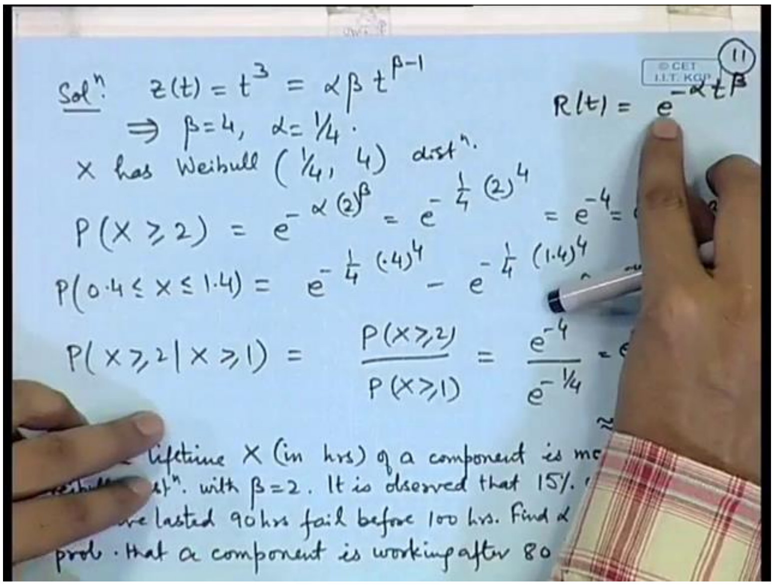

So, we can determine what are the parameters of the Weibull distribution here; so if we compare it with alpha beta t to the power beta minus 1, it gives alpha is equal to 1 by 4 and beta is equal to 4; that means, the life of the item has Weibull distribution with parameter alpha is equal to 1 by 4 and beta is equal to 4.

So, what is the probability that the item survives to age 2. So, probability of X greater than or equal to 2, the reliability function of the Weibull distribution at t is e to the power minus alpha, t to the power beta. So, here if we substitute the values of alpha and beta and t is equal to 2, then the value turns out to be e to the power minus 4 that is 0.0183 which is quite a small. What is the probability that the life is between 0.4 and 1.4? So, we are interested in finding out X line between 0.4 to 1.4; so naturally this can be written as probability that is the reliability at 0.4 and the reliability at 1.4 the difference of the reliabilities; so we make use of this formula that is at time t the reliability here is e to power minus alpha t to the power beta. So, after substituting the values of alpha, beta and t, t s 0.4 and 1.4 after simplification it turns out to be 0.61.

What is the probability that a one year item will survive to age 2; that means, it given that the item is working at age one, what is the probability that it will work till age 2? So, it is the conditional probability of X greater than or equal to 2, given that X is greater than or equal to 1. Once again we make use of the formula for the conditional probability, this is event E this is event F. So, probability of E given F is equal to probability of E intersection F divided by probability of F. Once again notice that here E is a subset of F, so in the numerator we will have probability of event e divided by probability of f. So, here this is turning out to be the reliability of the system at time 2 and this is the reliability of the system at time 1. So, in the formula for the reliability we substitute the values of alpha, beta and t and we get e to the power minus 4 and e to the power minus 1 by 4, and after simplification this 0.0235.

Let us look at another application of Weibull distribution. So, life in hours we denote by the random variable X, of a component is modeled as a Weibull distribution with beta is equal to 2; it is observed that 15 percent of the components that have lasted 90 hours fail before 100 hours, find the value of alpha, what is the probability that a component is working after 80 hours? Now here notice that for Weibull distribution the parameter beta has been specified, the parameter alpha has not been specified, but certain condition is given, so we can use this condition to determine the value of alpha.

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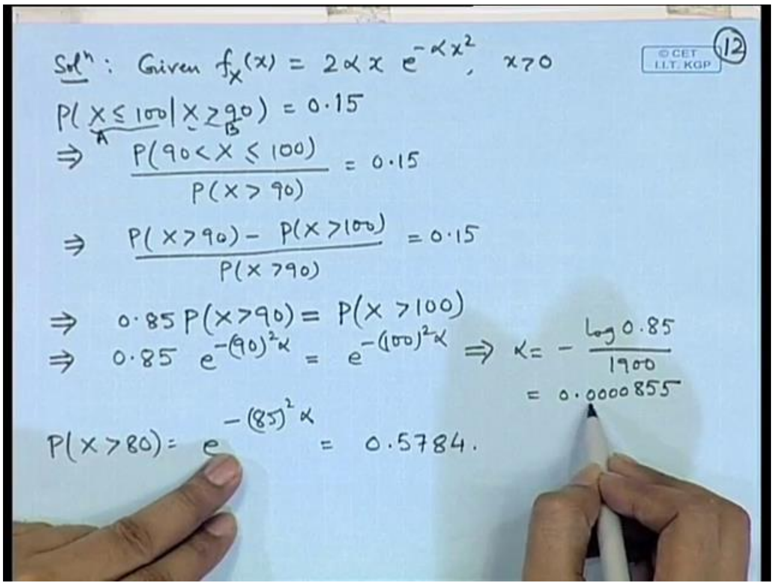

So, the Weibull distribution density is alpha, beta x to the power beta minus 1, e to the power minus alpha, x to the power beta; if we substitute the value of beta is equal to 2, then the density reduces to the form given here, that is 2 alpha x e to the power minus alpha x square.

Now, it is given that 15 percent of the components which lost 90 hours fail before 100 years; that means, they fail before 100 years that is the X less than or equal to 100 given that they work till 90 hours that is X is greater than 90, this proportion is 0.15. So, consider these events, this is event say A and this is the event B. So, probability of A intersection B becomes that X lies between 90 into 100 and probability of B is X greater than 90. So, the numerator is difference at the reliabilities at 90 and 100 hours divided by the reliability at 90 hours. So, we can simplify this terms. So, it is 0.85, probability X greater than 90 is equal to probability X greater than 100; substitute the values of the reliability function of the Weibull distribution as e to the power minus alpha, t to the power beta.

So, if we use this then the reliability here at X greater than 90 to the power minus alpha t to the power beta. So, beta is equal to 2 here, and here it is e to the power minus 100 square alpha that is t is equal to 100.

So, now after certain simplification this value of alpha turns out to be quite a small 0.0000855. So, what is the probability that a component is working after a t hours; that means, the reliability of the component at 80 times. So, by applying this formula that is e to the power minus alpha, t to the power beta, the value turns out to be 0.5784. So, these are some of the applications of the Weibull distribution, we have already seen the applications of exponential gamma etcetera also.

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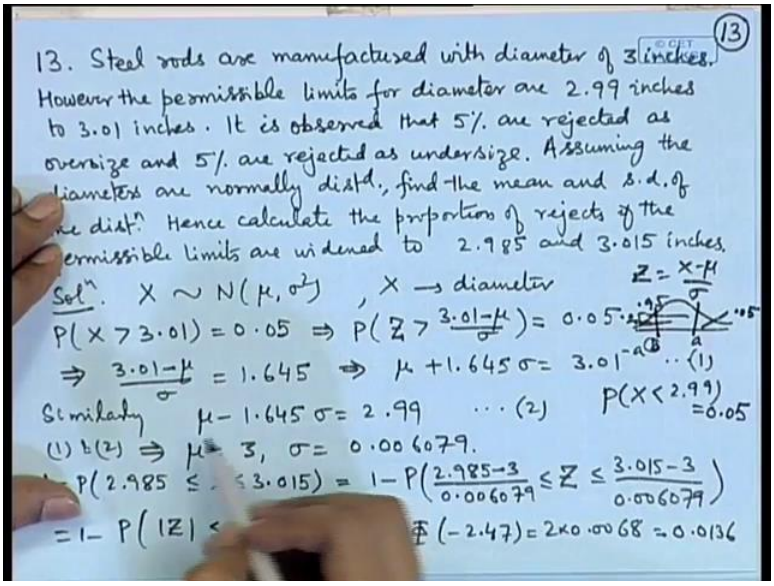

So, now we look at application of the normal distribution. So, consider steel rods which are manufactured with the diameter of 3 inches, now in any industrial process the specifications are given. So, here the specifications are that the diameter should be 3 inches for the rods. However, in the actual manufacturing there will be some deviation; that means, it may be 2.999 inches it may be 3.001 inches etcetera. So, in any industrial production the producer or manufacturer are the customer he specifies that what should be his desired specifications rather than telling exactly 3, because that may never be made. So, the permissible limits for the diameter are taken for example, from 2.99 inches to 3.01 inches; that means, if a rod is manufactured and if it is diameter turns out to be less than 2.99, it is not considered to be meeting the specification; if it is more than 3.01 inches then also it is not meeting the specifications.

Now, it is observed that 5 percent of the steel rods which are produced by this process they are rejected as oversize; that means, they are having diameter more than 3.01 inches and 5 percent are rejected as undersize; that means, they are having the diameter less than 2.99 inches. So, assume that the diameters are normally distributed, find the mean and standard deviation of the distribution. Hence calculate the proportion of rejects if the permissible limits are widened to 2.985 and 3.015 inches.

So, let us look at X as the diameter of the rod produced. So, it is given that it is normally distributed. So, let us assume that it is normally distributed with certain mean mu and certain variance sigma square. So, it is given that 5 percent of the rods have diameters more than the upper limit, they are oversized. So, probability that X is greater than 3.01 is 0.05. So, now, we have seen that the probabilities related to any general normal distribution can be transformed to probabilities related to standard normal random variable. So, here the standard normal variable can be obtained by looking at Z as X minus mu by sigma.

So, this reducing to X minus mu by sigma greater than 3.01 minus mu by sigma; so this is z; this is nothing but the point on the normal distribution such that. So, this 3.01 minus mu by sigma let us call it is say a, this is the point such that beyond this you have 0.05 probability or before that you have 0.95 probability. So, see the tables of the normal distribution this point a is 1 0.645. So, we are getting an equation 3.01 minus mu by sigma is equal to 1.645.

So, after simplification the equation reduces to a linear equation in variables mu and sigma. In a similar way if we consider probability of X less than 2.99 is equal to 0.05, then after transformation probability of z less than 2.99 minus mu by sigma is 0.05; that means, what is the point say b such that this probability is 0.05 now by the symmetric property of the normal distribution this point will be actually minus a. So, we will get 2.99 minus mu by sigma is equal to minus 1.645. So, after simplification it leads to the equation mu minus 1.645 sigma is equal to 2.99. So, if we solve these 2 linear equations in 2 unknown mu and sigma, we get mu is equal to 3 which is because it is given that the distribution is symmetric and here the assumptions of the problem here we are assuming that it is normally distributed. So, mu must be 3, because that is the target here and sigma turns out to be a pretty small value 0.006.

Now, under this value of mu and sigma, if we extend the permissible limits by little more that is by 0.015 that is from 2.985 inches to 3.015 inches, then how many or what is the proportion of rejecting of the steel rods? So, we calculate the probability of accepting the rod; that means, X lies between 2.985 to 3.015 and 1 minus that if we take this is the probability of rejection. So, this probability we can transform to a standard normal distribution by subtracting 3 and dividing by the sigma.

So, after certain simplification actually these terms turns out to be 2.4675 and this term is a minus of the same term. So, it is 1 minus probability of modulus Z less than or equal to 2.4675. So, it is 2 times the c d f of the standard normal variable at the point 2 point minus 2.4675. So, from the tables of the normal distribution we can see and this value turns out to be 0.0136. So, the probability of rejecting is nearly 0.01 that is 1 in a 100 will be rejected.

So, you can see here that initially 10 percent are rejected if we are having 0.01 as the; that is on either side of 3 we are having 0.01 as the acceptable limit, if we widen little bit more then only 1 percent are getting rejected so this is pretty fast; this point we had seen earlier also, so that in the normal distribution a large probability is concentrated around the mean.

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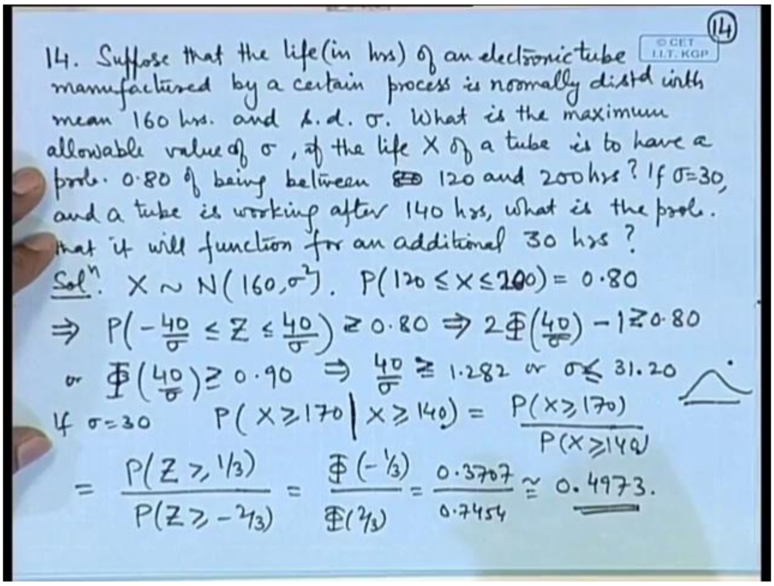

Suppose that the life in hours of an electronic tube manufactured by a certain process is normally distributed with mean 160 hours, and a standard deviation sigma; what is the maximum allowable value of sigma, if the life of a tube is to have probability 0.8 of being between 120 and 200 hours? If sigma is equal to 30 and a tube is working after 140 hours, what is the probability that it will function for an additional 30 hours? Here one natural question one may ask is that here we are looking at the life of the tube. So, life is a non negative number or a positive real number. So, how it can be normally distributed, because normal distribution is from minus infinity to infinity; that means, it takes any real value.

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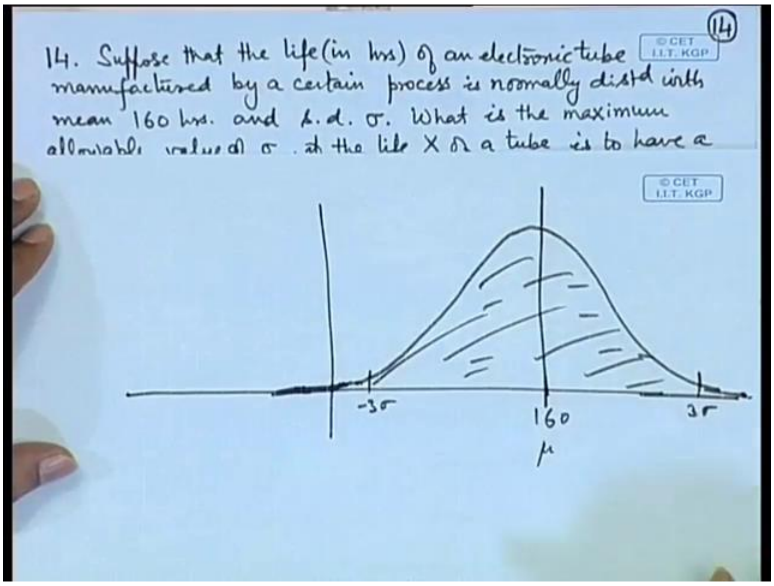

So, now this can be explained like this that the life may be positive, but suppose you are having a here mean is 160 hours. So, since we know that the normal distribution most of the probabilities concentrated within 3 sigma limits; that is minus 3 sigma to plus 3 sigma, actually more than 99.9 percent of the probability is concentrated here. So, actually the values before 0 will not be coming into picture. So, this is only a theoretical approximation to the practical situation, theoretical normal distribution is having values from minus infinity to plus infinity, but in practice the values will be concentrated in 3 sigma limits around mu.

So, in this particular case if X is denoting the life, then X follows normal distribution with mean 160 and variance sigma square; now it is given that the life is between 120 to 200 hours with probability 0.8. So, if we transform X to standard normal by subtracting 160 and dividing by sigma, then this is reducing to probability of Z lying between minus 40 by sigma to 40 by sigma, we want this probability to be at least 0.80. So, now, here we can simplify this term it is phi of 40 by sigma minus phi of minus 40 by sigma, we make use of the property that phi of t plus phi of minus t is equal to 1. So, this is reducing to twice capital phi of 40 by sigma minus 1 greater than or equal to 0.80.

So, from the tables of the normal distribution we see the point, that is the probability is more than 0.90 such that phi of 40 by sigma is greater than or equal to this. So, we can calculate this and sigma turns out to be less than or equal to 31.2. Now if sigma is equal to thirty what is the probability of an item working till additional 30 hours, which has already word up to 140 hours? So, it is probability X greater than or equal to 170 divided by X greater than or given that X is greater than or equal to 140. So, it is the conditional probability and it will be turning out as the ratio of probability X greater than or equal to 170 divided by probability X greater than or equal to 140.

So, after transforming to the standard normal probability function, this value can be evaluated and it is approximately 50 percent of the probability; that means, if the item has already worked for 140 hours, the probability that it will work for another 30 hours is nearly half.

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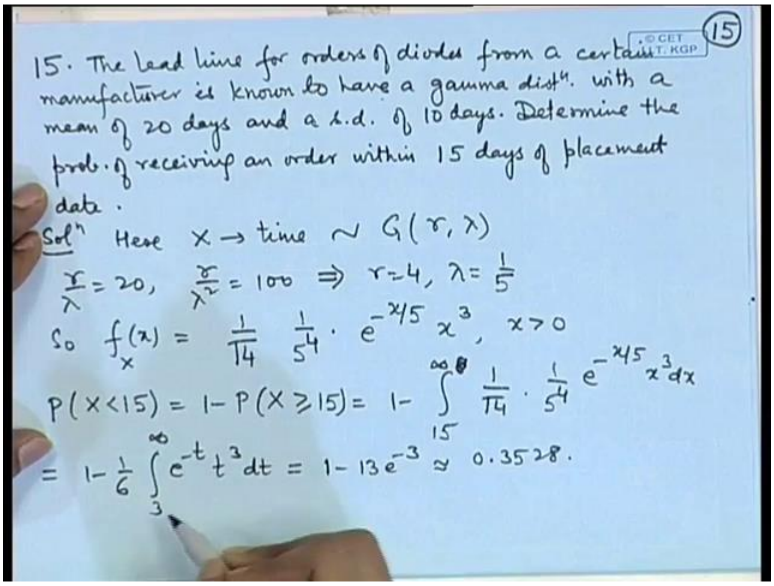

The lead time of orders of diodes from a certain manufacturer is known to have a gamma distribution with a mean 20 days and a standard deviation 10 days; determine the probability of receiving an order within 15 days of placement date.

So, let X denote the time. So, this is following a gamma distribution with parameter R and lambda; then the mean of a gamma distribution is R by lambda that is given to be 20 and the variance R by lambda square is given to be 10 square that is 100. So, after the solving these 2 equations we get R is equal to 4 and lambda is equal to 1 by 5. So, the probability density function of the time is 1 by gamma R lambda to the power R, e to the power minus lambda xm x to the power R minus 1. After substituting the value of R and lambda we get the form of the probability density function as this. So, probability that X is less than 15 is 1 minus probability X greater than or equal to 15.

So, that is 1 minus 15 to infinity the probability density function, here we can make the transformation X by 5 is equal to t, then it is reducing to a simple integral 3 to infinity e to the power minus t t cube minus dt, this can be evaluated using integration by parts and the term is equal to 1 minus 13 e to the power minus 3, which is approximately 0.35. So, under these conditions the probability of receiving an order within 15 days is nearly one-third. So, today we have seen various applications of discrete and continuous distributions. In the next lecture we will consider the distributions of the function of random variables. So, we will stop here.