Probability and Statistics
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Lecture – 27
Special Continuous Distributions – IV

In the last lecture I introduce the concept of reliability of a system at a given time t.

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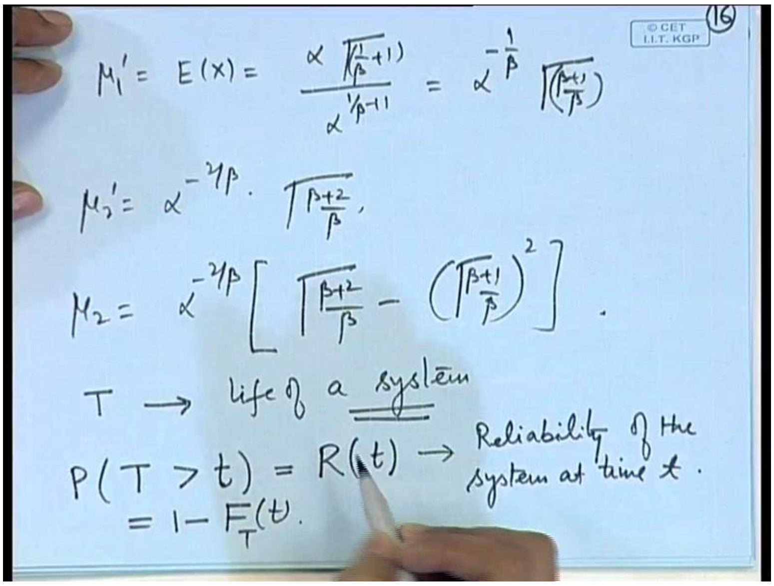

So, if T is a random variable which denotes the life of a system then probability that the system is functioning at time t is equal to probability of T greater than small t and we denote it by capital R t that is the survival function or the reliability function of the system at a given time t. So, by the definition of the cumulative distribution function R t is equal to 1 minus F t.

So, this reliability function is quite important in the study of lives of mechanical systems, in electronic system; that means in engineering science as where ever we are dealing with the various kind of instruments, equipments etcetera we are interested in the survival probability.

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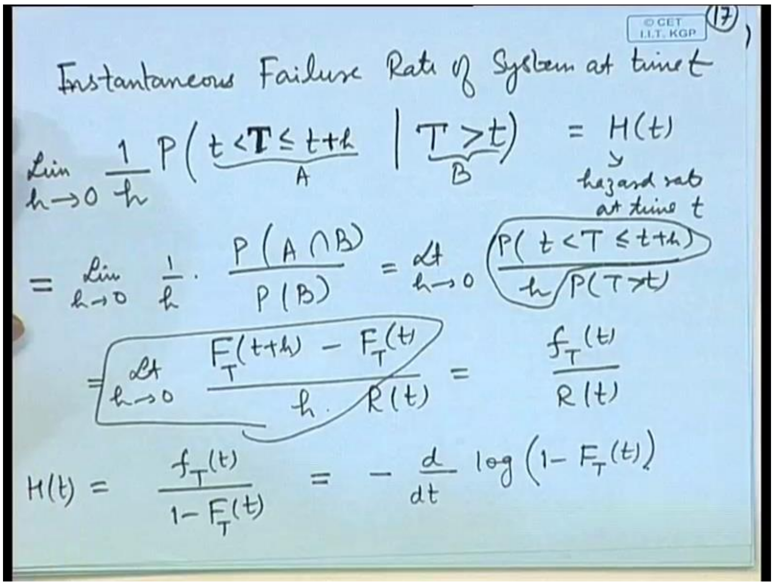

So, we further introduced a concept called Instantaneous Failure Rate of a System at time t which I called hazard rate at time t, which is the probability of the system failing within a short time immediately after time t; so the rate of failure. So, we divided by h and we calculated the expression for that and it turns out to be the density function divided by the reliability function or the density function divided by 1 minus the cumulative distribution function of the system. And there is an inverse relationship also which we showed.

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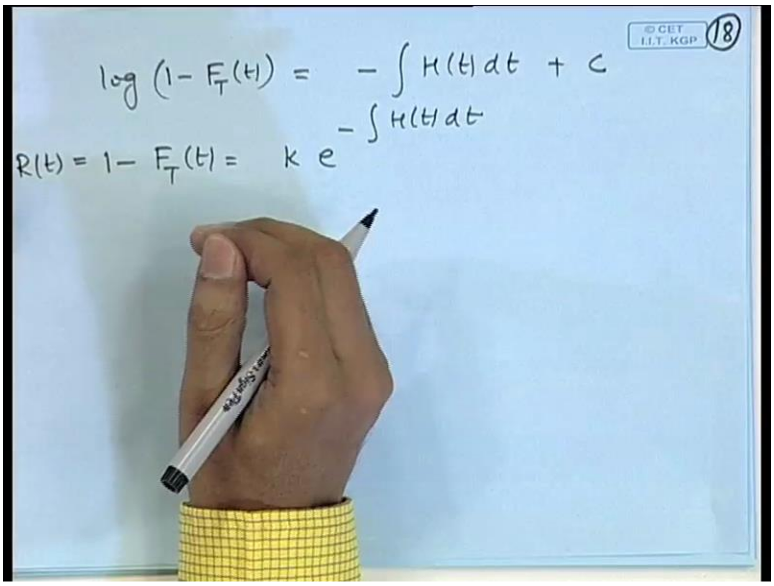

That the reliability of the system is equal to a constant times e to the power minus integral of H t dt where the constant has to be determined from the initial condition.

So, now we will show that this reliability function uniquely determines the distribution, the distribution uniquely determines the reliability function, the reliability function introduces or you can say uniquely determines the hazard rate, the hazard rate uniquely determines the reliability function etcetera. Notice here one thing that we are dealing with the continuous distribution, because we are talking about the lives of the systems. Although the reliability concept can be determined for probability of x greater then x can be determined for any discrete distribution also, but right now we are concentrating on this one.

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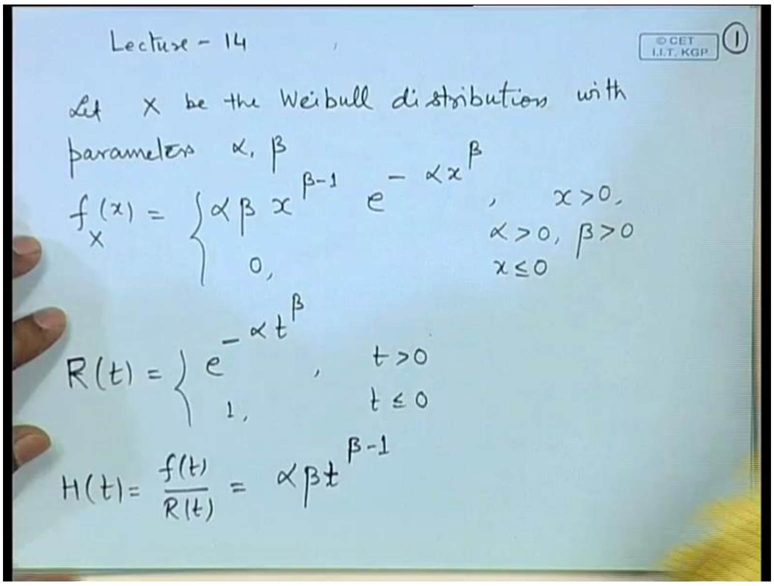

So, here let us consider say the distribution let x be the Weibull distribution with parameters alpha beta. So, we have the density function as alpha beta x to the power beta minus 1 e to the power minus alpha x to the power beta; where x is positive and alpha and beta are positive parameters, the density is 0 for x less than or equal to 0.

Now, for this we calculated the reliability function to be; we firstly calculated the cumulative distribution function which was equal to 1 minus e to the power minus alpha x to the power beta. Therefore, if we write down the reliability function it is equal to e to the power minus t to the power beta for t greater than 0 and it is one for t less than or equal to 0. So, if we calculate the hazard rate for this, this is equal to f t divided by R t. So, when you take the ratio you are left with alpha beta t to the power beta minus 1.

Notice here that beta if it is taking a positive integral value then this is a polynomial function. On the other hand if beta is a fraction then suppose beta is a number less than 1 then it will become an increasing hazard rate as t increases. If we are given H t is equal to this then we can use the formula for the calculation of reliability from here in the following way.

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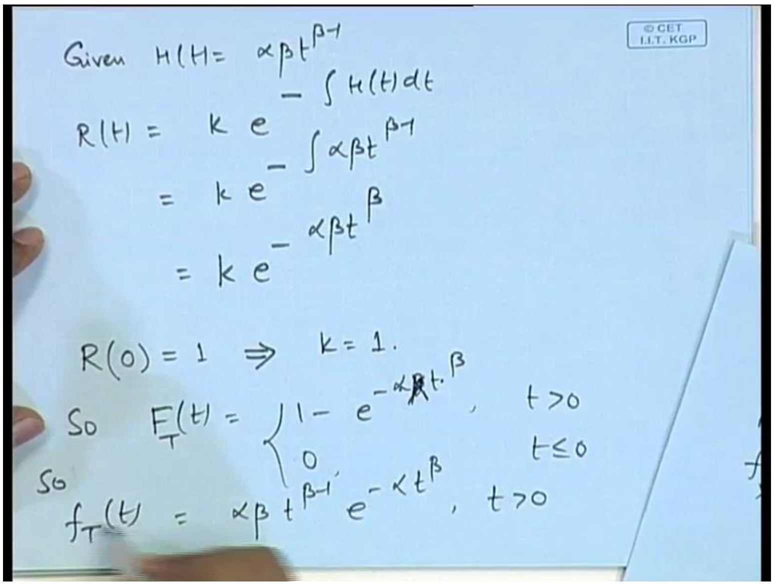

Given H t is equal to alpha beta t to the power beta minus 1; R t is equal to k e to the power minus integral H t dt that is equal to k times e to the power minus integral alpha beta t to the power beta minus 1 that is equal to k times e to the power minus alpha beta t to the power beta.

So, if I say put say R 0 is equal to 1 then this will give us k is equal to 1. So, f t will become equal to 1 minus e to the power minus alpha beta t to the power beta, for t greater than 0 and 0 for t less or equal to 0. So, the density function is equal alpha beta t to the power beta minus 1 e to the power minus alpha this will be e to the power minus alpha t to the power beta which is nothing but the probability density function of the Weibull distribution.

So, this discussion shows that for Weibull distribution the form of the hazard rate is of a polynomial type of function- alpha beta t to the power beta minus 1. Of course, it will depend upon what are the values of beta whether beta is a real value or it is a positive integral value etcetera. But one thing is clear if beta is greater than 1 then this will be increasing with t and if beta is less than 1 then this will be decreasing with t. So, you can have systems where the hazard rate may be increasing with time or it may be decreasing with time. For example, you consider a very new system. Now, we know that a very new system generally it is likely to fail more, but once it is in operation; for example, a new car or a new computer, but after certain time it will work once it has stabilised the life will be more. However, after again certain stage the life may start decreasing.

Let us consider the special case here beta is equal to 1, in the Weibull distribution which was corresponding to the exponential distribution. If we take beta is equal to 1 in the hazard rate this become simply alpha that is a constant. Now this is interesting.

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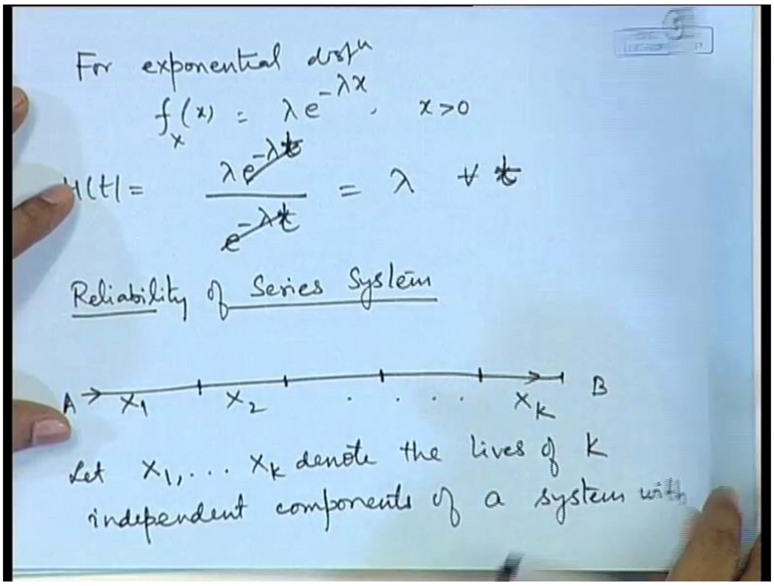

In the case of exponential distribution for exponential distribution; that is if I am taking say f x is equal to lambda e to the power minus lambda x, the hazard rate is equal to lambda e to the power minus lambda x the reliability function was e to the power minus lambda x. That cancels out this is lambda for all x. That means the hazard rate function; let us put t here in place of x, hazard rate function of the exponential distribution is independent of the time; that means, constant failure rate. Now this is the point which I wanted to emphasise that earlier when we are talking about the memory less property of a exponential distribution. That means, if a system is functioning it has not failed then the failure rate remains the same. So, that is why the systems which are having exponential life are more stable in nature and they are good basically for the users.

On the other hand if the H t is given to be lambda we can calculate using this formula, so we will get the k e to the power minus lambda t and k will again become one by using the initial condition r 0 is equal to 1 so it becomes lambda e to the power minus lambda t as the density function.

We also discuss the reliability of complex systems. Many times the entire system is made up of several components which may be connected in parallel or in a series. So, let us consider the reliability of series system. So, suppose we have say k components with say lives X 1, X 2, X k and they are connected in a series. That means the entire system will work if and only if all the components are working. So, let X 1, X 2, X k denote the lives of k independent components of a system with say life x.

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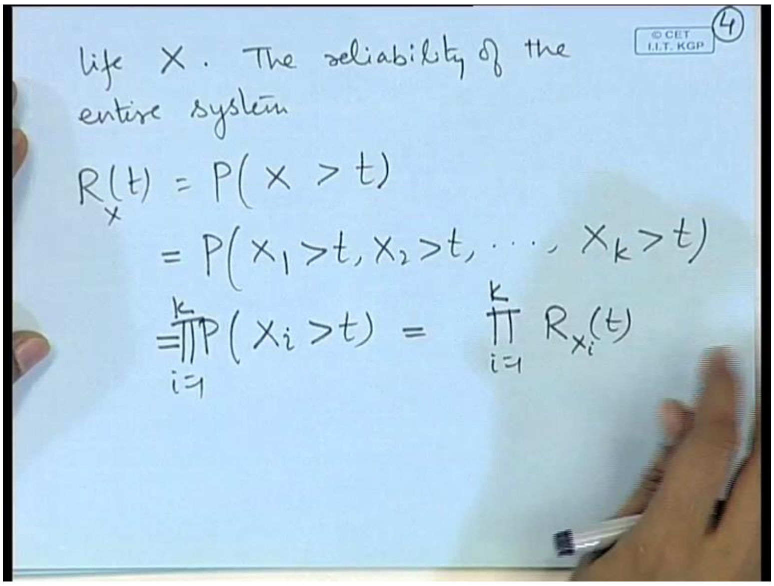

So, the reliability of the entire system that is R t is equal to probability of x greater than t. However, the system will be working if and only if each of the components X 1, X 2, X k are working at time t. That means, this is equal to probability of X 1 greater than t X 2 greater than t and so on X k greater than t.

We are assuming that the components are working independently; therefore the probability of the simultaneous occurrence will be equal to the product of the probabilities. Now this is nothing but the reliability of the rth system; ith system. So, what we get here is that the reliability of a compound system which consists of several components connected in a series is equal to the product of the reliabilities of the individual components.

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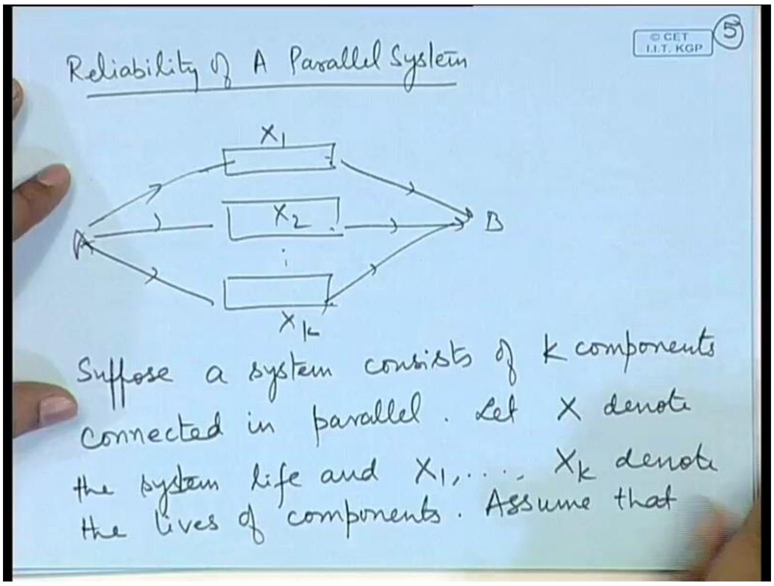

In a similar way we may have systems which are connected in parallel. Let us look at reliability of a parallel system. So, there are several links from say A to B and the system will function if any one of the components is functioning; let us call lives as X 1, X 2, X k etcetera.

Suppose a system consists of k components connected in parallel. So, let X denote the system life and X 1, X 2, X k denote the lives of components. Once again assume that the component lives are independent.

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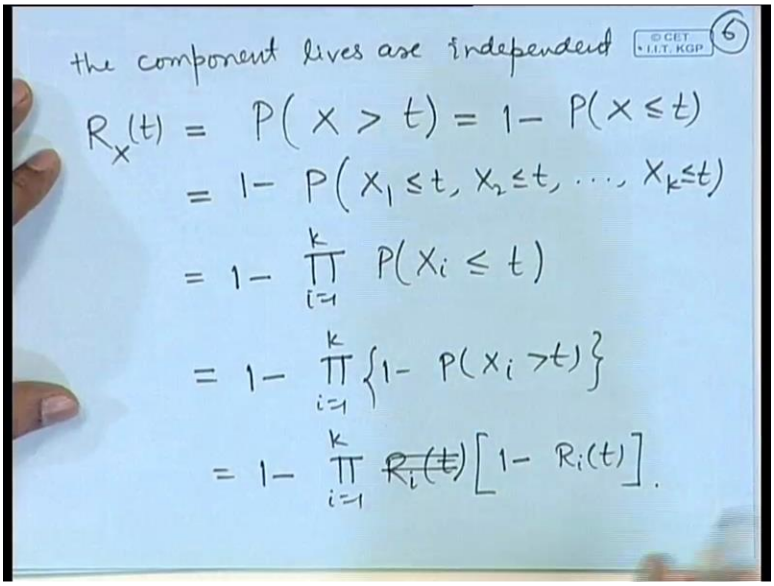

Now if we consider the reliability of the entire system at time t. Now this will function if any of the systems is functioning. So, we can write it as 1 minus probability of x less than or equal to t. That means, the system fails before time t. Now the systems will failure before time t if each of the components fails. That means, it is equal to probability of X 1 less than or equal to t, X 2 less than or equal to t and X k less than or equal to t. This is equal to 1 minus; now once again we can make use of the independence of the lives of the individual components. So, it becomes product of the probabilities X i less than or equal to t, which is nothing but 1 minus probability of X i greater than t.

Now, this is nothing but the R i. Therefore, the reliability of a system consisting of parallel components is expressed in terms of r x t is equal to 1 minus product of the reliabilities of the individual, sorry this is 1 minus so this is wrong expression 1 minus R i t.

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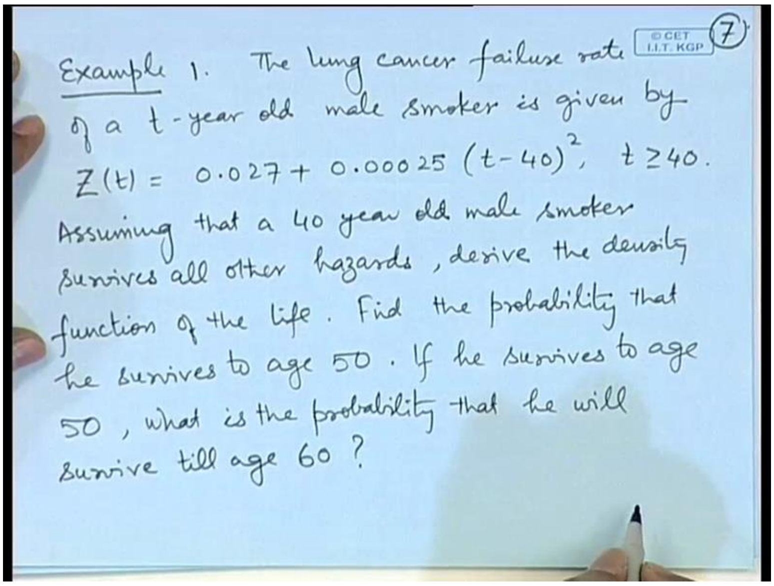

Let us look at one application: the lung cancer failure rate that is hazard rate of a t-year old male smoker is given by Z t is equal to 0.027 plus 0.00025 t minus 40 square for t greater than or equal to 40. Assuming that a 40 year old male smoker survives all other hazards derive the density function of the life. Find the probability that he survives to age 50. If he survives to age 50, what is the probability that he will survive till age 60?

So, this is the shifted Weibull distribution kind of thing, because in the Weibull distribution we have seen the hazard rate function is given by alpha beta t to the power beta minus 1; that means starting from time 0, now if you compared with this here it is shifted; so t minus 40. So, we are assuming that the person is already 40 years old after that the hazard rate function is given by this. So, from here you can make out that the distribution of the life of the t-year old male smoker is given by Weibull distribution.

So, we can make use of the relationship between the reliability function and the hazard function to calculate the reliability function hence the cumulative distribution function and hence the probability density functions of the life.

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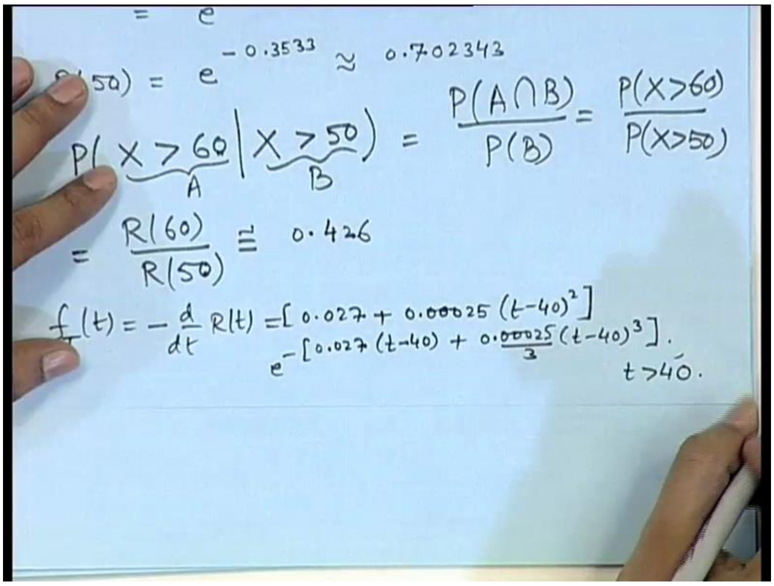

So, consider here R t; R t is equal to e to the power minus Z s d s. Now here we have written in definite integral there but here since it is already given so we can put the limit as 40 to t. That is equal to e to the power minus 0.027 t minus 40 plus 0.00025 by 3 t minus 40 cube. So, one question was asked find the probability that he survives to age 50; that means the reliability at t is equal to 50.

So, once we have the reliability function we can calculate R 50, which is obtained by substituting t is equal to 50 here. So, after evaluation it turns out be e to the power minus 0.3533 which is approximately 0.702343. That means, if he is a smoker and he is already 40 years old the probability of his surviving beyond 50 is only 70 percent. That means, 30 percent chance is that he will not be able to survive.

If he survives to age 50 what is probability that he will survive till age 60; this is conditional probability that he is having life up to 60 at least up to 60 given that he has survive till 50. So, if we consider this event as A and this event as A then it is probability of A intersection B divided by probability of B and that is equal to probability X greater than 60 divided by probability X greater than 50; which is the ratio of the reliabilities at t is equal to 60 and 50 and after some simple evaluation this turns out to be 0.426. Which means that if he survive till h 50 then the probability of surviving till age 60 is 0.4. If you see here the probability of surviving till age 50, 0.7; but if you survive till 50 then probability of surviving till 60 is 0.42.

The probability; the density function of the life can be obtained simply by looking at the derivative of the reliability function; because the reliability function is 1 minus the cumulative distribution function. So, the density function is obtained as minus d by dt of R t which is equal to 0.027 plus 0.00025 t minus 40 square e to the power minus 0.027 t minus 40 plus 0.00025 by 3 t minus 40 cube that is t is greater than 40; which is nothing but the probability density function of a shifted Weibull distribution.

Let us consider one more example here.

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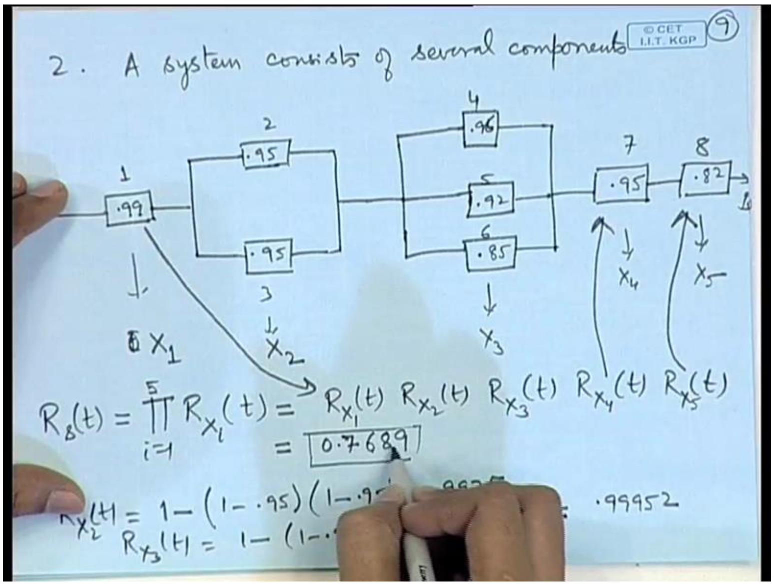

A system consists of a several components as shown below. So, the numbers this represents the reliability of the component to be functioning, so 0.99 is the reliability of component 1 to be functioning 0.95 for the second one. So, this second assembly here is consisting of two components connected in parallel. So, the system this particular component will be working provided any of them is working.

Let us name it 2 and 3. The next part consists of three components connected in parallel with the reliability is 0.96 for component four, 0.92 for component five and 0.85 for component six. The next one consists of a single component 0.95. And the next one consist of a single component suppose this is A, this is B. So, the entire assembly is consisting of A to B; eight components out of which some of them are connected in parallel and some of them are connected in the series. Let us name this system as a 1, let me call it system life as X 1, this system life as X 2, this system life as X 3 this as X 4, this as X 5.

So, if we are interested to calculate the reliability of the entire system. Now this is a series so it will be the reliability of the x ith system i is equal to 1 to 5; 1, 2, 3, 4, 5 systems which are connected in parallel. So, this one will become equal to r X 1 t. Now X 2 t is again a compound system it consist of two parts connected in parallel, the probability that the system is functioning or the reliability of this is given by the formula for the reliability of a parallel system 1 minus product of 1 minus of the reliabilities.

So, if we use this one the reliability of the second system will be equal to 1 minus let me firstly write R X 2 t, R X 3 t, R X 4 t and R X 5 t. Where R X 1 t is given by this number, r X 4 is given by this number, r X 5 is given by this number. If we look at r X 2 t it is equal to 1 minus 1 minus 0.95 into 1 minus 0.95 which is equal to 0.9975. Similarly, if we consider the reliability of the third system that is R X 3 that is equal to 1 minus 1 minus 0.96 1 minus 0.92 1 minus 0.85 which is equal to 0.9995.

If we substitute these values in this formula we get 0.7689. This shows that the reliability of a complex system can be calculated by using the formula for the reliabilities of the individual systems which are connected either in parallel or connected in a series form. One more thing we should notice here the reliabilities of the individual systems are quite high for example, 0.99, 0.95, 0.95, 0.82 etcetera.

But if you are looking at the reliability of the compound system it is 0.7689 which is much smaller; which shows that when we multiply the probabilities they become a smaller, because the each of the number is less than 1 that is the effect. So, although individual systems may have high reliability, but when we conduct them in a say series system the reliability will become much smaller. If we connect them in a parallel the reliability will increase, because any of them working will make ensure that the system is functioning.

So, we will stop here.

Thank you.