Probability and Statistics
Prof. Somesh Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 25
Special Continuous Distributions – II

In the previous lecture we have introduced negative exponential distribution and we gave its origin as the distribution of the weighting time for the first occurrence in a Poisson distribution.

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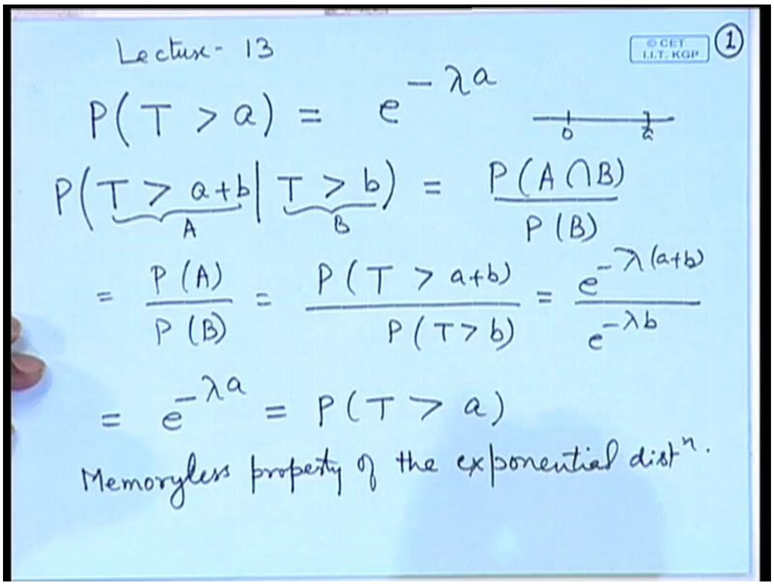

So, we considered probability of T greater than T, if T is have an exponential distribution. So if we consider probability of say T greater than a number a then it is equal to e to the power minus lambda a. Now if we consider; so this means that the first occurrence has not taken place till time a; that means, in the interval 0 to a, it has not taken place.

So, if we consider probability of say T greater than say a plus b, given that T is greater than b. So, if we consider this as in event A, this as event B then it is equal to probability of A intersection B divided by probability of B. Now A inter section B so here A is the subset of B therefore, this becomes probability of A divided by probability of B that is probability of T greater than a plus b divided by probability of T greater than b. So, by the formula that we have developed this one will become equal to e to the power minus lambda a plus b divided by e to the power minus lambda b. So, after simplification it becomes e to the power minus lambda a, which is nothing but the probability of T greater than a.

So, this phenomena and represents that if the event has not taken place till time b then it will not take place till an additional time a, the probability of that is same as that the event will take place after time a, a starting from time 0. That means if we are considering the waiting time for certain thing, then the probability of waiting for an additional time is same irrespective of the a starting point. So, like geometric distribution this is called the Memoryless property of the exponential distribution.

You can see that exponential distribution is analogous to the geometric distribution of the discrete case, in the geometric distribution we were considering Bernoullian trails and we were waiting for the first success or first occurrence. So, here also it is a Poisson process and we are waiting for the first occurrence. So, in a sense this exponential distribution is a continuous analogue of the geometric distribution.

So, if we consider say occurrence to be say failure of certain component in a mechanical system, and if we know that the system has not failed till that time then the probability of failure after a certain time is same irrespective of the a starting point where we have taken. So, this type of thing is interesting, because many times we buy say second hand item, second hand transistors, second hand computer or calculator because if it as still working, then the probability of its failure after a certain time will remain the same. So, this is because of the memory less property of the exponential distribution.

Let us also consider some modifications of the exponential distribution, because here we are starting from the time 0, but many times for example; you consider certain item which we are purchasing from the market, then the market when your purchasing then the shop keeper gives you a guarantee for certain time like 1 year or 2 year etcetera. That means the item is not suppose to fail before; that because if it is failing he is taking it back, so it is again as if you are a starting a fresh. So, this is considered as a shifted exponential distribution by certain constant. Another thing we notice here is that when we consider the form lambda e to the power minus lambda T, where lambda was the rate of the Poisson distribution then the mean of this turns out to be 1 by lambda. And since there is a single parameter if we right say 1 by lambda say something like sigma, then this will become 1 by sigma e to the power minus T by sigma.

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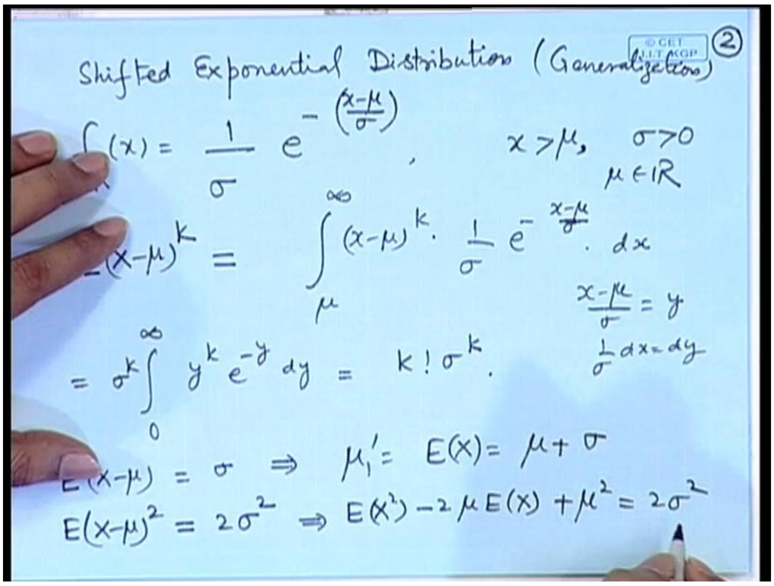

So in the popular form with the shifted origin we can consider the form of exponential distribution else. So, we call it shifted exponential distribution, the density we can write as 1 by sigma e to the power minus x minus mu by sigma. So, here x is greater than mu and of course, sigma is positive.

And advantage of this form is that in place of time if we are considering some other representation for x, x need not to be time all the time, so if it is something else then mu can be negative also and then also this distribution remains valid. So, you can consider this is a generalization of the original exponential distribution. If we consider this particular form, then you can see it is easier to calculate the movements of x minus mu or x minus mu by sigma. So, let us consider expectation of x minus mu to the power k, which is of course, not necessarily the central movement because we have not shown that mu is the mean.

However, for convenience we have considering this. So, it is equal to x minus mu to the power k 1 by sigma e to the power minus x minus mu by sigma, d x from mu to infinity. So, we may put x minus mu by sigma is equal to y that is 1 by sigma d x is equal to d y, then this is equal to 0 to infinity, sigma to the power k, y to the power k, e to the power minus y d y that is equal to gamma k plus 1 r k factorial sigma to the power k. So, in particular if we consider expectation of X minus mu to the power 1 this is equal to sigma, this means mean of the shifted exponential is mu plus sigma, if we are considered lambda their then it would have been mu plus 1 by lambda.

So, definitely we would be interested in the variance here. So, expectation of X minus mu square is equal to 2 sigma square, which gives us expectation of X square minus 2 mu expectation X plus mu square is equal to 2 sigma square.

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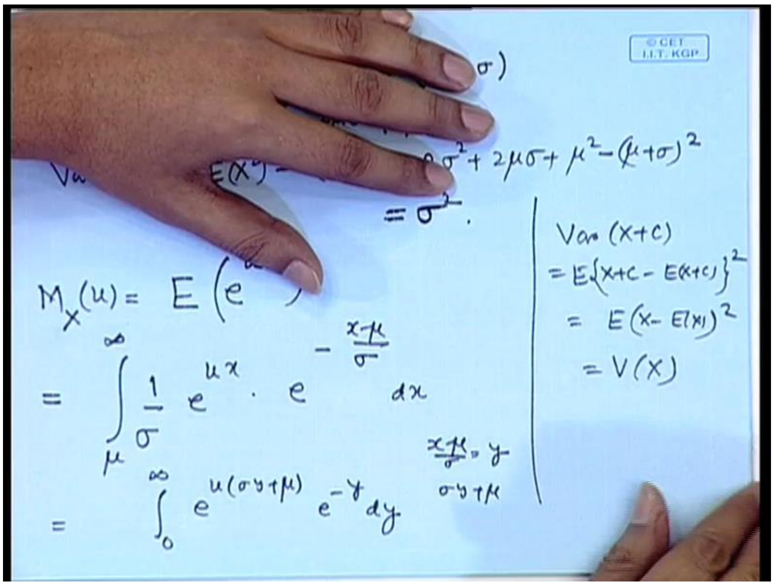

Further, simplification gives expectation of X square is equal to 2 sigma square minus mu square plus 2 mu expectation x is mu plus sigma. So, this becomes equal to 2 sigma square plus 2 mu sigma, minus mu square plus 2 mu square. So, it becomes plus mu square. So, if we calculate the variance of X, that is equal to expectation X square minus expectation X whole square that is equal to 2 sigma square plus 2 mu sigma plus mu square minus mu plus sigma whole square which is simply sigma is square.

So, variance has not changed by shifting; this is true because the fact that variance is independent of the shift in the origin, the variance of x plus c is expectation of x plus c minus expectation of x plus c whole square. So, this is equal to expectation of x minus expectation x whole square which is the variance of x. So, the variance is unaffected by the change in the origin. We can consider here the moment generating function of this and also the moment generating function of the original exponential distribution. So, let us look at this, moment generating function at a point u; it is equal to expectation of e to the power u x. So, this is equal to 1 by sigma e to the power u x, e to the power minus x minus mu by sigma d x mu to infinity.

So, we can consider here x minus mu by sigma is equal to y, the transformation that we made here. So, it is becoming 0 to infinity e to the power u and x becomes sigma y plus mu e to the power minus y d y.

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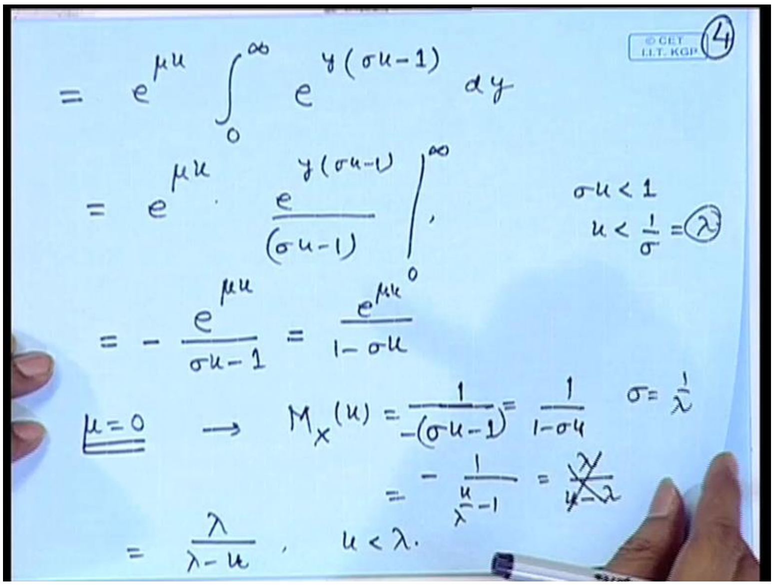

So, this can be simplified; it is equal to e to the power mu u, e to the power y sigma u minus 1, d y from 0 to infinity; e to the power mu u, and this becomes e to the power y sigma u minus 1 divided by sigma u minus 1; provided sigma u is less than 1 or u is less than 1 by sigma, which was basically lambda from 0 to infinity. So, at infinity this becomes 0 because I am taking sigma u to be less than 1 and at 0 this becomes 1, so you are getting e to the power mu u divided by sigma u minus 1.

In particular if we have taken mu is equal to 0 then this M x u would have been 1 by sigma u minus 1, if we put sigma is equal to say 1 by lambda then this will become 1 by u by lambda minus 1, that is equal to lambda by u minus lambda, there is a minus sin here because when we put 0 there is a minus sin here. So, this will become actually equal to e to the power mu u 1 minus sigma u. So, this will become with a minus sin 1 by 1 minus sigma u, so minus is 0. So, this is equal to lambda by lambda minus mu, for u less then lambda. So, the moment generating function of a exponential distribution then you consider the original form that is waiting time is starting from the 0, then m g of is lambda by lambda minus mu are 1 by 1 minus sigma mu, sigma u if we consider the starting from mu then it is e to the power mu u by 1 minus sigma u.

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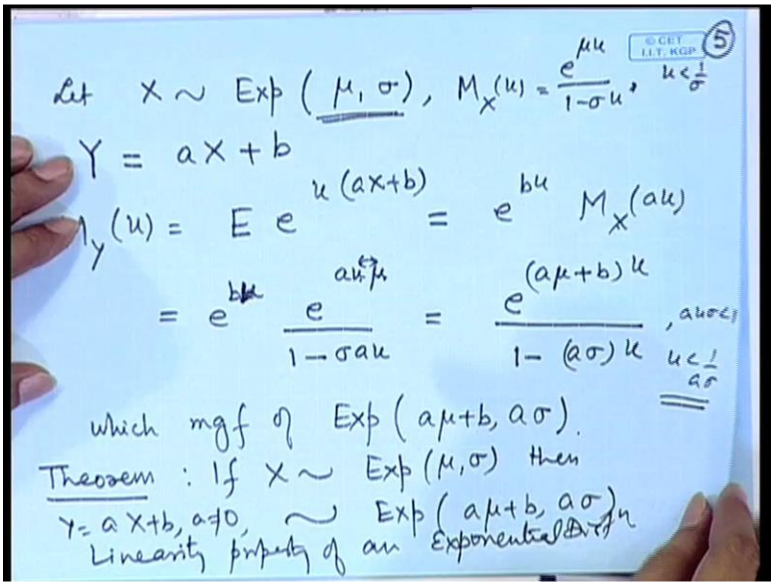

So, here we can see the relationship between these exponential distributions. So, let x follow exponential distribution with parameters mu and sigma, consider the transformation y is equal to say a x plus b, consider the moment generating function of this, then it is expectation of e to the power u into a x plus b, this is equal to e to the power b u and the moment generating function of x at the point a u. So, what is the moment generating function of x at the point u, that we derived it as e to the power mu u by 1 minus sigma u for u less than 1 by sigma. So, if we make use of this term here in place of u we can put a mu here. So, it is equal to e to the power a u mu, divided by 1 minus sigma a u. So, this we can adjust and write it has a mu plus b u see term was b u and here it is a u mu. So, mu a mu this actually you will interchange here.

So, it becomes a mu plus b and then u divided by 1 minus a sigma u; compare this the term here the moment generating function of x is e to the power mu u by 1 minus sigma u. So, here mu is replaced by a mu plus b, and sigma is replaced by a sigma, the form is the same at the same time since we have replaced a u by u in the expression for m x, this expression was valid for u sigma less than 1. So, here a u sigma must be less than 1; that means, u is less than 1 by a sigma. So obviously, you can see by comparing this that this is an m g f of; which is m g f of exponential distribution with parameter a mu plus b and a sigma.

So, this means we have proved the following theorem; if x follows exponential mu sigma then y is equal to a x plus b where a is not 0, follows exponential distribution with parameter a mu plus b a sigma, this is basically linearity property of an exponential distribution; that means, any linear function of an exponential distribution is again having exponential distribution. So, this form is more general this is a 2 parameter exponential distribution and it is more useful.

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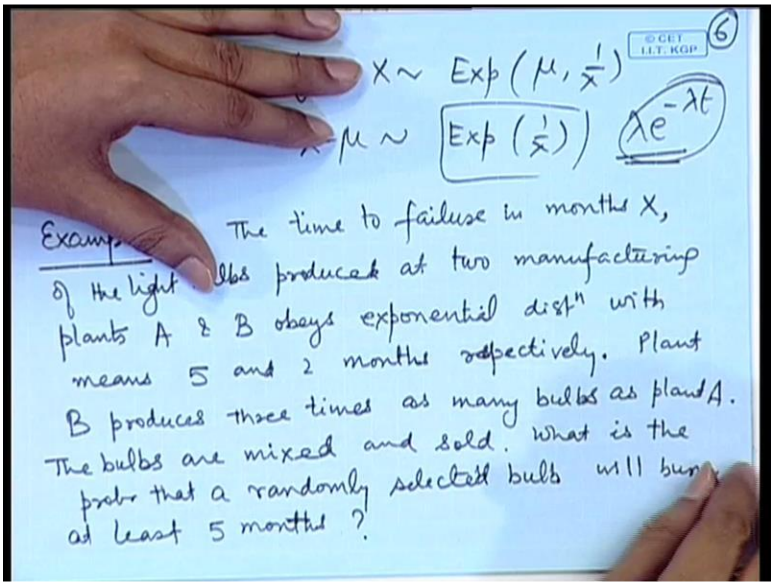

In particular we will have if X follows exponential say mu 1 by lambda, then x minus mu will follow exponential 1 by lambda; that means, by standard form with the density lambda e to the power minus lambda t. So, both the forms are useful let us take one example here for exponential distribution, the time to failure in months.

So, suppose it is X of the light bulbs produced at 2 manufacturing plants say A and B obeys exponential distribution with means 5 and 2 months respectively. Plant B produces 3 times as many bulbs as plant A. The bulbs are mixed so they look in distinguishable to a (Refer Time: 19:45) and sold. What is the probability that a randomly selected bulb will burn at least 5 months?

So, let us consider here, the distribution of x is exponential, but it is not the same for all the bulbs. For a certain proportion of bulbs the distribution is exponentially with mean 5. That means, the density will be 1 by 5 e to the power minus x by 5, and for certain bulbs the mean time is 2 months, so the density will be 1 by 2 e to the power minus x by 2.

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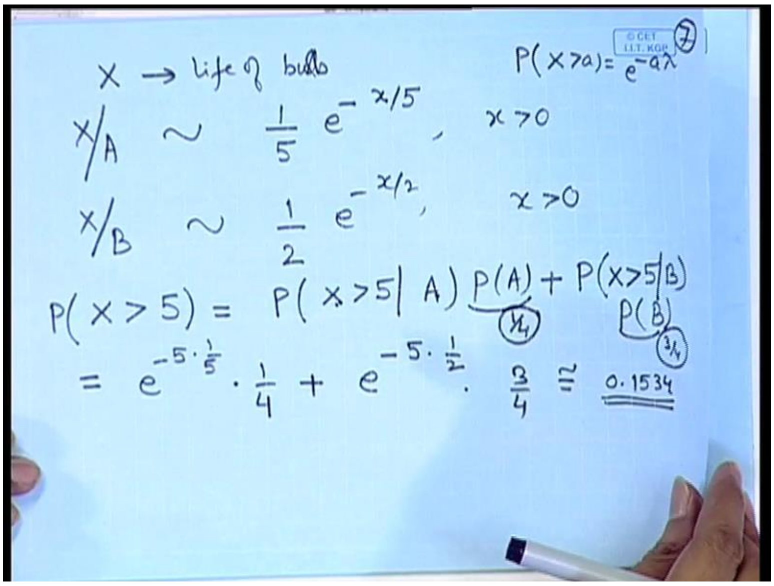

So, if we are considering the X as the time are life of bulb, then X given that it is produce by plant A, this as a density 1 by 5 e to the power minus x by 5 for x greater than 0, and of course 0 otherwise. If we are considering the bulb produced by plant B then the density function is half e to the power minus x by 2 for x greater than 0. We are interested in a randomly selected bulbs life to be more than 5 months. So, here we can apply the theorem of total probability. So, probability of x greater than 5 given that it is produce by plant A, into the probability of being produced at plant A plus probability of x greater than 5 given that it is produced at plant B into probability of plant B.

So, this is equal to; now if we are considering probability of x greater than 5 from this one then consider the general formula probability of x greater than a is equal to e to the power minus a lambda. So, here lambda is equal to 1 by 5 and a is equal to 5. So, this becomes e to the power minus 5 into 1 by 5 in to probability of A. Now what is probability of A? It is given that B produces 3 times as many bulbs as A. So, the probability of the bulb being selected from A may be 1 by 4, and the probability of this may be 3 by 4. So, this is into 1 by 4 plus e to the power minus 5 here the parameter is 1 by 2, 3 by 4. So, this can be simplified and it is 0.1534.

So, probability that a randomly selected bulb is working beyond 5 months is only 0.15 approximately. Which may looks slightly surprising, because you say that there are 2 plants and from one of the plants it is 2 months and from another plant it is 5 months so it should be nearly half but it is not so, because the plant B gives more supply compare to the plant A and in the plant B the probability of having more than 5 months is much a smaller. Because it is e to the power minus 5 by 2 and here it is e to the power minus 1, so this number is obviously larger then this number.

Thank you.