Probability and Statistics
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Lecture – 24
Special Continuous Distributions – I

So far we have discussed various discrete distributions and many of them we gave direct origins that what kind of experiments leads to those distributions. Apart from that there are certain distribution such as the distributions which can be represented as a power series because you observed say geometric distribution, where your getting a 10 q to the power j minus 1 into p where j is from 1 to infinity. So, there is something like a power series in negative binomial distribution also. And then you have a finite polynomial sums like in binomial distribution or in poison distribution you have certain term e to the power and then you have power series.

So, there is a general family of power series distributions that we will talk little later. Firstly, let us discuss some Special Continuous Distributions.

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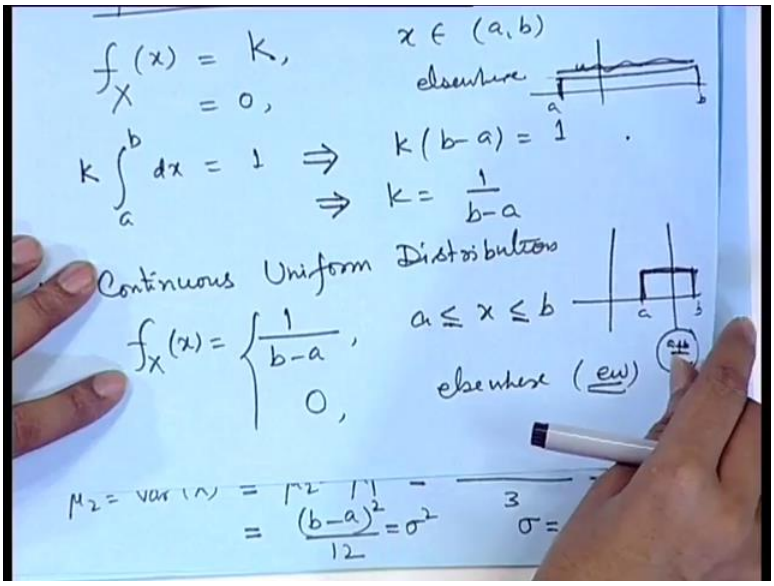

As in the case of discrete distributions, a simplest example of a continuous distribution could be where the density is uniform or density is a constant. For example, you consider a rod with a uniform where for example, you consider this sheet and here the sheet it is the thickness are say weight of this sheet at every point will be same. So, if we consider say this portion of the pen then here in this portion the density, the weight, the thickness or the width is constant.

So, if the density is a constant over a certain interval say a to b and 0 else where we call it uniform distribution. Now what should be the value of the constant that can be determined by the condition that the density must be non-negative and the integral of the density must give you 1. So, k must be non-negative and interior of k from a to b must be 1, this means k into b minus a is equal to 1; that means, k is equal to 1 by b minus a. So, a continuous uniform distribution has the probability density function given by 1 by b minus a, a is less then x then b 0 for x lying outside this; we will use the notation ew for writing elsewhere.

Of course here it is in material whether we use a strict in equalities are we may use equalities at some points less than or equal to because the probability of a point is 0. So, in a continuous distribution inclusion or exclusion of a point does not make any difference. So, if you look at the shape of this distribution suppose a and b is here, then 1 by b minus a is yes. So, you can say a Plato kind of thing the continuous uniform distribution.

We may look at some of the properties here.

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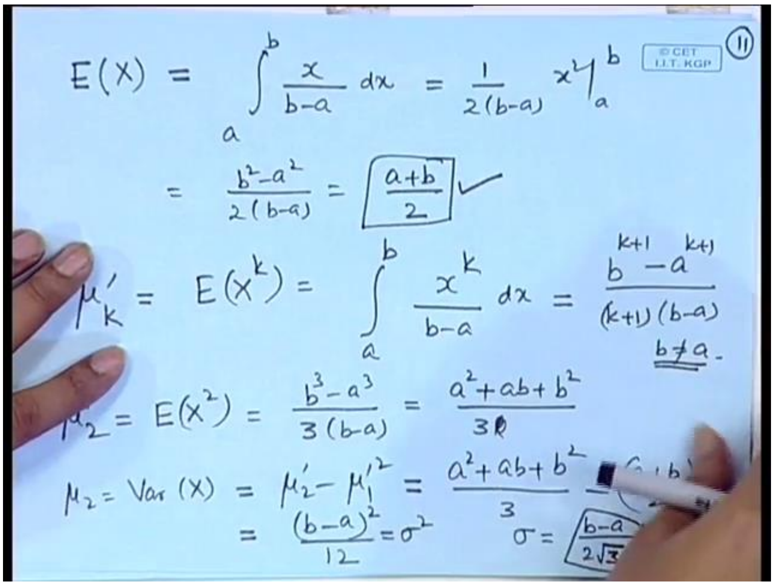

So, what is the first movement? That is expectation of X. So, it is equal to integral x divided by b minus a d x from a to b. So, that is half b minus a x is square from a to b. So, that gives us b square minus a square divided by twice b minus a, that is equal to b plus a by 2. You can easily see that it is the midpoint of the distribution, which is understandable because the density is constant and therefore, the average value must be the midpoint.

Now since it is a finite interval we can look at the movement of any order here, the movement of any order will exist x to the power k divided by b minus a d x. So, this is equal to b to the power k plus 1 minus a to the power k plus 1 divided by k plus 1 into b minus a. If we have b is equal to a then it will not be a continuous distribution. So, in particular mu 2 is equal to expectation of mu 2 prime that is expectation of x square that will be equal to b cube minus a cube divided by 3 into b minus a, that is a square plus a b plus b square divided by 3 into divided by 3. So, the variance of the uniform distribution that is mu 2 prime minus mu 1 prime is square that is a square, plus a b, plus c square by 3 minus a plus b whole square by 4.

So, after some simplification a plus b by 2 whole square; so we can make simplification here this turns out to be b minus a whole square by 12 that is sigma is square. So, the standard deviation of this distribution is b minus a divided by root 12 that is 2 root 3. So, you can see here the range of the distribution is b minus a and the variability is b minus a divided by 2 root 3, which is slightly you can say much less because 2 into root 3 is root 3 is 1.71. So, this becomes b minus a divided by 3.4 two kind of thing. So, this is much smaller than the range and it is because of the uniformity that the variability is much smaller.

However, if b and a are for a part that is if b minus a is a large number, then even though after division the variability will be high. So, for example, you have too much of flatness say to b and the value of this is like this 1 by b minus a. So, basically it will become much smaller here if you plot 1 by b minus a here. So, that shows at the variability increases if the range of the distribution increases.

One may also look at mu 3 and mu 4 and calculate the measures of esculent skewness and kurtosis; obviously, it is a symmetric distribution therefore, measure of skewness will be 0; however, measure of kurtosis will depend upon the difference between a and b as you have seen here. If the difference between a and b is smaller than the density will be plotted at a higher this one, if the difference between a and b is large then it will be quite flat, so mu 4 will be dependent upon the value of b and a.

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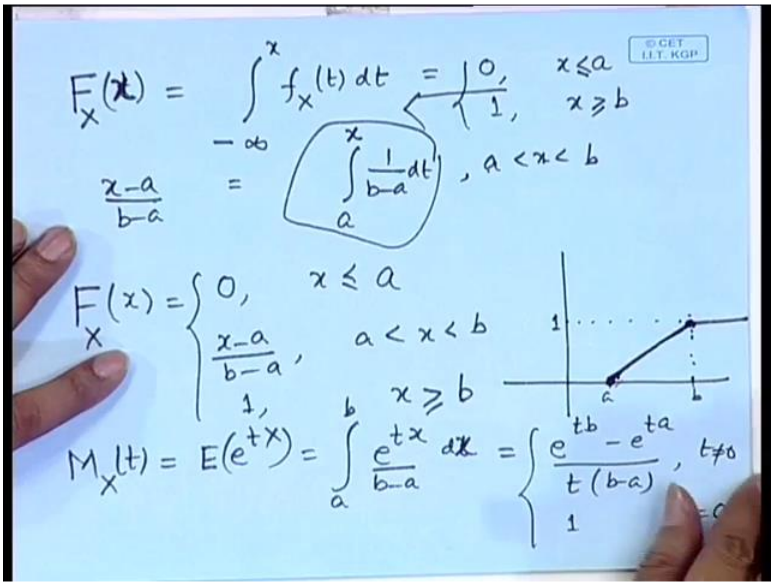

We also look at the plotting of the cumulative distribution function; that is integral minus infinity to x, f x, t, d t. Since the distribution is 0 up to a and beyond b; that means, this is going to be 0 for x less than a and it is going to be 1 for x greater then b you may put equality here also it does not make any difference. However, if I am considering x to be between a and b then it is integral from a to x of 1 by b minus a d T and this value is equal to x minus a by b minus a, therefore the CDF can be written as 0 for x less than or equal to a, x minus a by b minus a for a less then x less then b and 1 for x greater than or equal to b; you can easily observe that the function is absolutely continuous the derivative will give you the density and the end points a and b of the intervals the function is continuous.

If we plot this suppose a is here and b is here. So, up to a it is 0 from a to b. So, suppose this is the value one here. So, it is a line here and there after it becomes 1. So, it is a straight line joining the point 0 to 1 here; we may also look at the movement genetic function e to the power T x, d x divided by b minus a from a to b. So, it becomes e to the power T b minus e to the power T a divided by T b minus a; obviously, at T is equal to 0 this is not define, but at T is equal to 0 this is 1. So, this is T naught equal to 0 1 T equal to 0.

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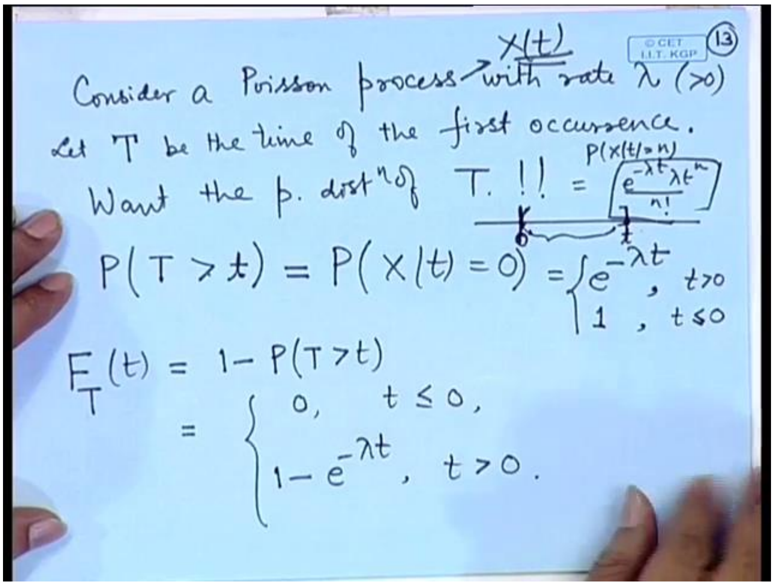

Let us consider a Poisson process; consider a Poisson process with rate lambda of course, lambda is greater than 0.

Let us denote by say T be the time of the first occurrence. So, we know that it is a Poisson process the number of occurrences will follow a Poisson distribution. So, at some point of time we start observing the process for example, we go understand at a ticket counter of certain cinema and then we want to see when the first customer arrives, we are suppose I am a traffic police person and I go to the my designated traffic crossing, now when I am standing there I start observing when is the first accident taking place. So, if we consider T is the time of the first occurrence, from the time when we start observing then T is a continuous random variable because it is the time. So, we want the probability distribution of T what is the distribution of T?

So, we can look at a event say probability T greater then say a small t, what does it mean? That means, if we are observing the process from certain time T then up to the time is small t; if you are starting from 0 up to time small t that event has not taken place; that means, in interval 0 to T the number of occurrences is 0. So, if you are considering this Poisson process as X T then the event capital T greater then a small t is equivalent to probability that X t is equal to 0 because if there is no occurrence in the interval 0 to t then capital T is definitely going to be greater than a small t and vice versa so these two events are same.

However the distribution of X T is assumed as a Poisson distribution, because we have made the assumption here that it is a Poisson process with rate lambda; that means probability X T is equal to n is e to the power minus lambda T lambda to the power n by n factorial. So, here if I put n is equal to 0 then this is given me e to the power minus lambda t. So, of course, this is statement is true if T is greater than 0 because we are observing from certain time onwards. So, this is for T greater than 0 this probability will be 1 if T is less than or equal to 0. So, if we consider the cumulative distribution function of T that is capital F of T it is 1 minus probability of P greater than T. So, from this derivation it is equal to 0 for t less than or equal to 0 and it is 1 minus e to the power minus lambda t for t greater than 0.

So, now you can observe here, we are able to derive the CDF of this continuous distribution and it is an absolutely continuous function. So, if we differentiate this we can get the density function of the time for the first occurrence during the Poisson process.

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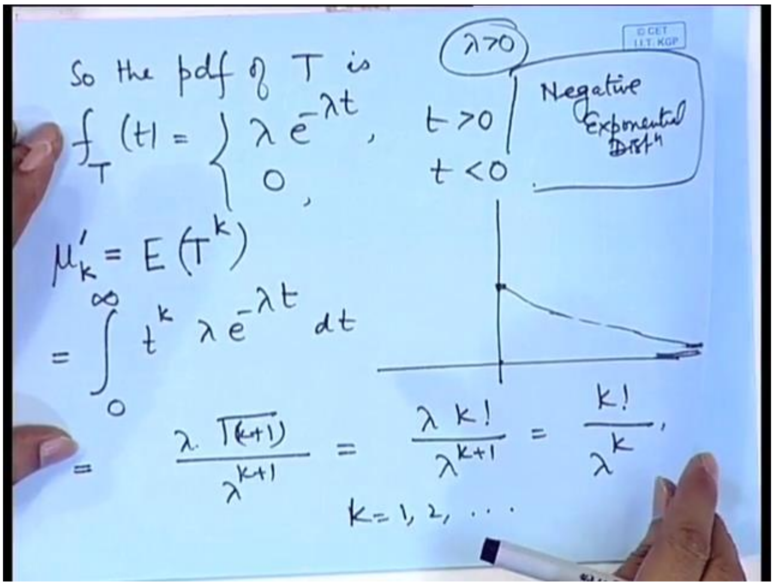

So, the probability density function of T is f T equal to lambda e to the power minus lambda t for t greater than 0 and it is equal to 0 for t less than or equal to 0, this is known as negative exponential distribution. So, this is a continuous distribution and this distribution arises as the distribution of the waiting time in a Poisson process for the first occurrence of the event whatever occurrence we are trying to observe. Naturally we would like to look at the properties of this distribution; so for example, the shape of the distribution at t equal to 0 it is 0 up to 0 and at t is equal to 0 the value is equal lambda and then e to the power minus lambda t because lambda is positive.

So, this will be less than 1 and this will be a decreasing function because and you will have decreasing to 0 as T tends to infinity. Let us look at it is movement's etcetera. So, if we consider a general movement of the kth harder it is expectation of t to the power k that is equal to t to the power k lambda e to the power minus lambda t d t 0 to infinity. Now this is nothing, but a gamma function, so this becomes lambda gamma k plus 1 divided by lambda to the power k plus 1. So, this is equal to lambda k factorial divided by lambda to the power k plus 1 that is k factorial divided by lambda to the power k for k equal to 1 2 and so on.

Of course here we can see that if k was 0 then actually it was nothing, but the integral of the density which would have been 1. So, the movements of all positive order exist here in particular we can calculate mean variance etcetera.

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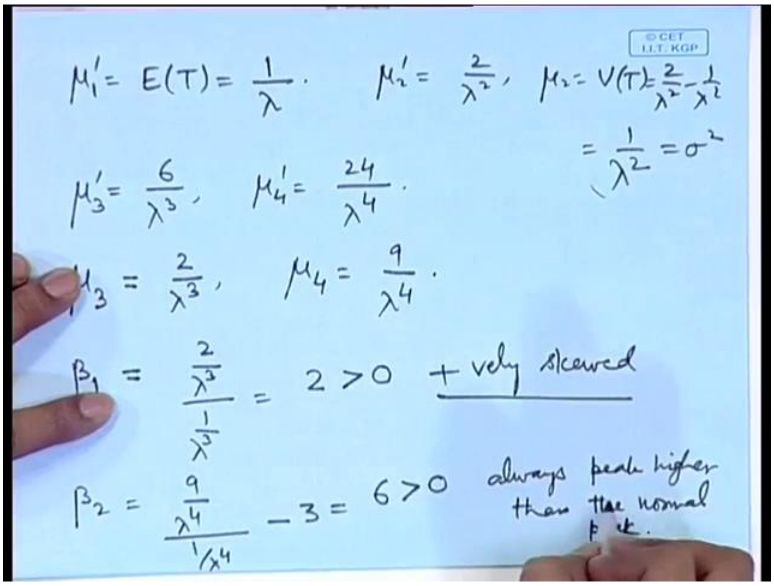

So, for example, mu 1 prime that is the mean of this distribution is equal to if I put k equal to 1 here, I will get 1 by lambda. So, if the rate of occurrence is lambda per unit of time then the waiting time the average getting time for the first occurrence is 1 by lambda, which is very natural to understand. Suppose I say that in 1 our 2 events will occur so roughly average waiting time for first occurrence will be 30 minutes. We may also look at mu 2 prime that is equal to 2 by lambda is square and therefore, mu 2 that is the variance will become 2 by lambda is square minus 1 by lambda is square that is equal to 1 by lambda is square.

So, you can observe here the variance of the exponential distribution is a square of the mean. We may calculate mu 3 prime that is equal to 6 by lambda cube; mu 4 prime will become equal to 24 by lambda to the power 4 using the relationship between the non central and central movements mu 3 is equal to 2 by lambda cube and mu 4 is equal to nine by lambda to the power 4. From here we can calculate the measures of his skweness and kurtosis. So, beta 1 is equal to 2 by lambda cube that is mu 3 divided by sigma cube. So, this is sigma is square, this becomes 1 by lambda cube that is equal to 2, which is always positive; that means, no matter what the value of lambda is the exponential distribution is always positively is skewed.

So, this you can see from the shape of the distribution also because here it is a constant. Similarly if you look at say beta 2, beta 2 is equal to 9 by lambda to the power 4 that is mu 4 divided by mu 2 is square that is divided by 1 by lambda 2 to the power 4 minus 3 that is equal to 6. So, no matter what the value of lambda is, it is always having peak higher than the normal peak.

So, this is always positively skewed and always peak high that is higher than the normal peak. Now here you observed this is slightly different from the earlier distributions; in binomial distribution, in Poisson distribution, in geometric distribution etcetera the condition for the skweness and the kurtosis was dependent upon the parameter; that means, like in the binomial distribution if P was half the distribution was symmetric, if P is less than half it was positively a skewed, if P was greater than half it was negatively is skewed. And similarly if p q has less than 1 by 6 it was leptokurtic, if p q was greater than 1 by 68 it was leptokurtic etcetera

Unlike those distributions same thing we observed in the Poisson and also in the Poisson distribution the measure of skewness was 1 by lambda, so it was positively skewed, but as lambda becomes large it approaches the symmetry. Similarly in the measure of kurtosis was also having lambda in the denominator, but here this measures beta 1 and beta 2 are constant and they will always have the same behaviour that is positively skewed and the peak higher than the normal peak.

In the next lectures we will be considering extension of this concept of the time, because here the negative exponential distribution is obtained as distribution of time for certain occurrence. So, we will consider extension of this concept in the next lecture.

Thank you.