Probability and Statistics
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Lecture – 02
Sequence of Sets

Course on Probability and Statistics, this is an introductory course I will consider a first course on the probability and statistics and it is quite useful for all branches of science and engineering.

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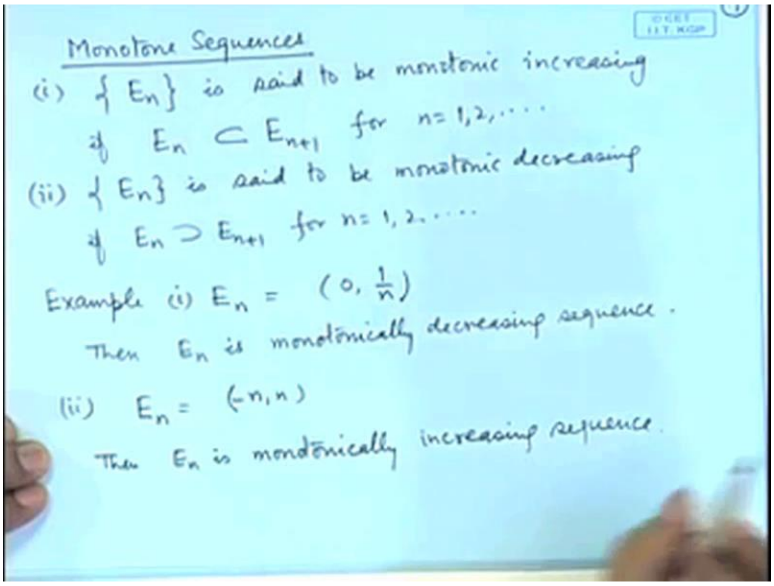

So, will always exist where called monotone sequences; so we say that a sequence E n is said to be monotonic increasing if E n is a subset of E n plus 1 for n is equal to 1, 2 and so on. In a similar way we define E n to be monotonic decreasing, if E n is a containing E n plus 1 for n is equal to 1, 2 and so on. Let us consider some example here; let E n be the interval say 0 to 1 by n then E n is monotonically decreasing sequence. Further we can consider the say sequence E n is equal to interval say minus n to n then E n is monotonically increasing. One important result which true for the monotonic sequences is that the limit always exists, I will state it in the form of a theorem here.

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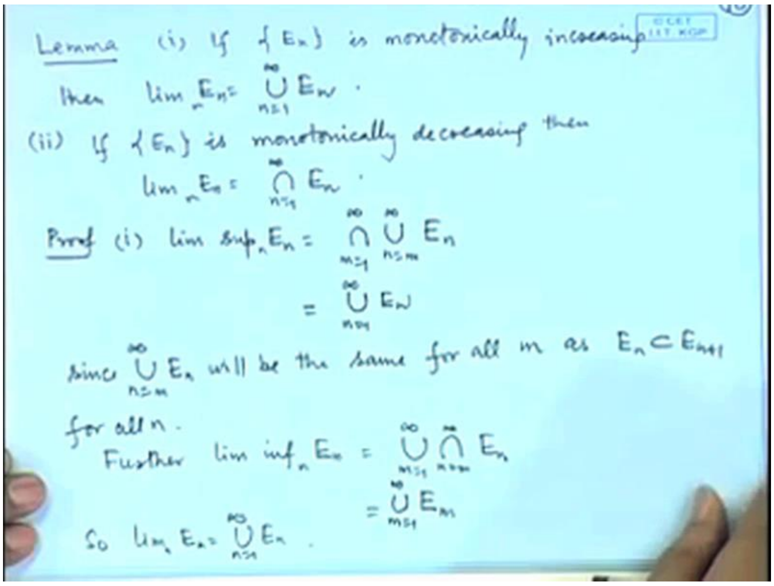

If E n is monotonically increasing then limit of E n is equal to union of E n; n is equal to 1 to infinity. In a similar if E n is monotonically decreasing then limit of E n is the intersection of the sequence of sets E n. To look at the proof let me proof the statement 1 first; if we take the E n to be monotonically increasing sequence of sets then the union n E n will be same for all n, and therefore limit superior will be same. Let me explain it further, if we consider limit superior of E n then it is equal to intersection union E n; n is equal to m to infinity, m is equal to 1 to infinity; this will be equal to simply union n is equal to 1 to infinity. Since union E n n is equal to m to infinity will be the same for all m as E n is subset of E n plus 1 for all n.

Further if we look at limit inferior of the sequence where this is union intersection E n n is equal to m to infinity. Now in the intersection E n, this is starting from E n E n plus 1 intersection E n plus 2 etcetera. Since the sequence is monotonically increasing, the first set in the sequence is the smallest set and it is contained in all the sets which are coming after this. Therefore, the intersection will be equal to the first set itself and therefore it is equal to E m; that means, we are getting limit inferior as also union of E m's.

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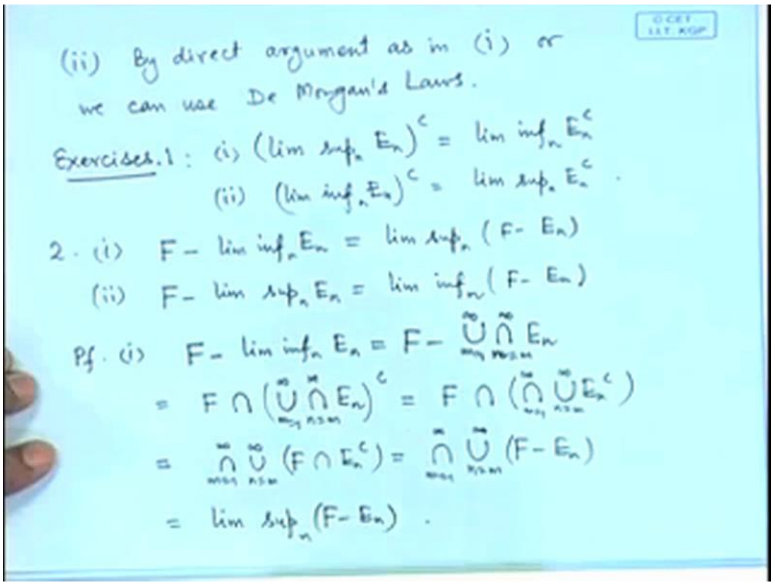

If we combine these two results the limit exists and it is equal to the union of the sets. The proof of the second statement can be given by direct argument as in 1 or we can use De Morgan’s Laws. Let me state as exercises a few more results which are related to the limit of the sequence of the sets; for example, if I consider limit superior of a sequence of sets and if I take complementation of that then it is equal to limit inferior of the compliments of the sets.

In a similar way, if I consider limit inferior of the sequence of sets and I take its compliment then it will be equal to the limit superior of the compliments of the sets. The proofs will be almost truly elfin make use of the lemma 1 which gave the representation of the limit superior and limit inferior and make use of the De Morgan’s laws. And an extension of this exercise would be; if I consider F minus limit infimum of the sequence E n then it is equal to limit superior of F minus E n. In a similar way, if I consider F minus limit superior of E n that it is equal.

To proof say the first part of this we can consider F minus limit inferior of E n and let us write down the representation of the limit inferior in terms of unions and intersections; m is equal to m to infinity, m is equal to 1 to infinity and at this we just use a set theoretic notation where A minus B is equal to A intersection B compliment. So, this becomes F intersection union intersection E n; n is equal to m to infinity m is equal to 1 to infinite compliment that is equal to F intersection intersection union E n compliment; if we apply De Morgan's laws.

Now, at this stage we can apply the distributive properties of the unions and intersections and this will give us intersection m is equal to 1 to infinity union n is equal to m to infinity F intersection E n compliment which is equal to intersection union F minus E n where n is equal to m to infinity; m is equal to 1 to infinity and this is nothing, but the limit superior of the sequence F minus E n. In a similar way we can proof the statement two of this. There are certain relations which include the characteristic functions of the limit superior and limit inferior and I will state it as a statement in the following exercise.

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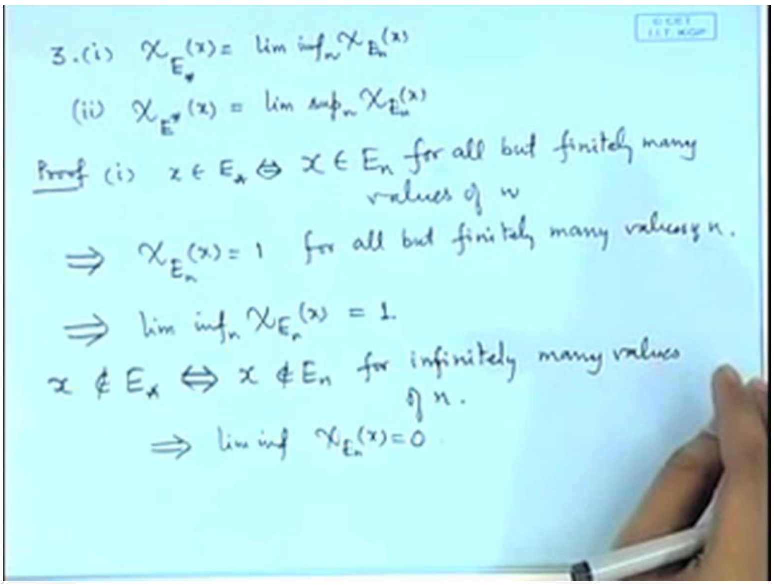

Indicator function of the limit inferior of a sequence of sets is the limit infimum of the indicator functions of the sequence of the sets. In a similar way if we consider indicator function of the limit superior then it is equal to limit superior of the sequence of the sets. We may look at the proof of say one of them, consider say x belonging to E star then this implies that x belongs to E n for all, but finitely many values of n; this implies that the indicator function of the set E n is 1 for all, but finitely many values of n which is equivalent to the statement that limit infimum of the indicator function chi and x is equal to 1 and these statements are both if and only if.

Further if I consider x not belonging to E lower star then this will imply that x does not belong to E n for infinitely many values and this implies that limit infimum of chi and x is 0. In a similar way, if we use a definition of the limit superior of a sequence of functions then we will be able to proof the second statement; useful relation which is used for in set theory is that of symmetric differences.

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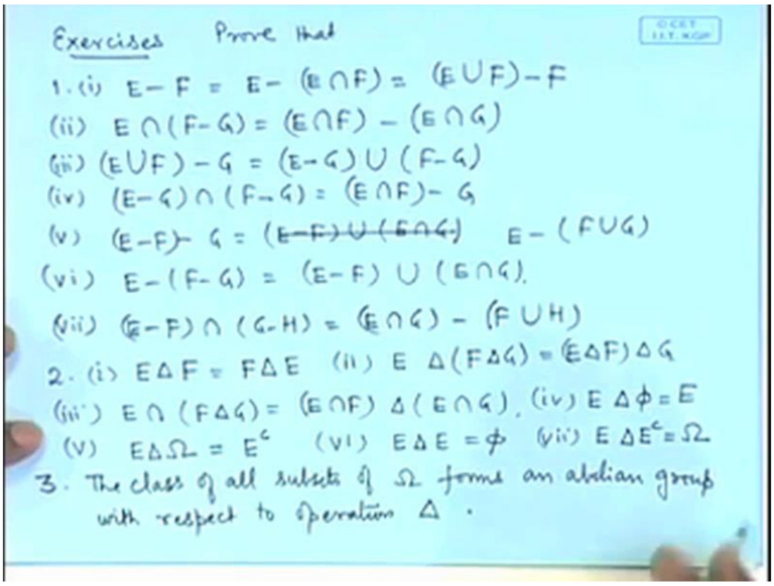

The concept of symmetric difference is defined by A delta B is equal to a minus B union B minus a. So, you can see here that it is a minus B and B minus a both are combined together and that is why it is called a symmetric difference and equivalent interpretation for this is a union B minus A intersection B.

From the Venn diagrams, we can see that if I have two sets A and B then the symmetric difference is the shaded portion. Certain relationships which are true for set theoretic operations are given in the form of exercises below prove that E minus F is equal to E minus E intersection f; it is equal to E union F minus f, E intersection F minus G is equal to E intersection F minus E intersection G, E union F minus G is equal to E minus G union F minus G E minus G intersection F minus G is equal to E intersection F minus G, E minus F minus G is equal to E minus F union E intersection G.

E minus F minus G is equal to E minus F union E intersection G, E minus F intersection G minus H is equal to E intersection G minus F union H. Certain relationships which are true for the symmetric difference are given the next exercise for example, symmetric difference of E with F is same as symmetric difference of F with E, symmetric difference satisfies associative property that is E, delta F, delta G is same as E, delta F, delta G. E intersection F delta G; it is equal to E intersection F, delta E intersection G; that means, intersection and symmetric differences are distributive. If we consider the symmetric difference of a set with the empty set then we get the same state; that means, as a group theoretic operation empty set acts as an identity operator.

If we consider with the fully space then I get the complementation. If we consider with itself then we get empty set; that means, with respect to group theoretic operation; we use its own inverse and if we consider E delta, E complement then I get the fully space. So, one can ask that the class of all subsets of omega forms and abelian group with respect to symmetric difference operation.

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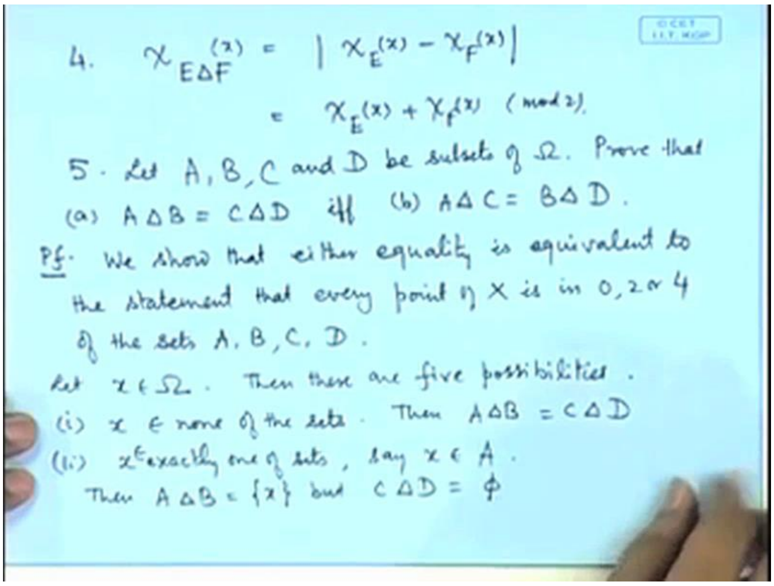

Further if we consider the indicator function of the symmetric difference then it is equal to the absolute difference between the indicator functions of the two sets and alternative way of telling it is that it is equal to chi E plus chi F, where the sum is taken modulo 2 and additional exercise in this direction can be that if we consider A, B, C and d.

Then A delta B is equal to C delta D; if and only if A delta C is equal to B delta D. In order to prove this statement 5; we proceed as follows. We show that either equality is equivalent to the statement that every point of x is in 0, 2 or 4 of the sets A, B, C, D. So, let us consider x to be any point in this space omega then there are five possibilities. Let us consider this possibilities; 1 possibility is that x belongs to none of the sets. If x does not belong to any of the sets then with respect to this point A delta B and C delta D they must be equal because x is none of them.

If we consider x belongs to exactly one of sets say x belongs to A; in that case A delta B will consist of the point x and C delta D will not include this point.

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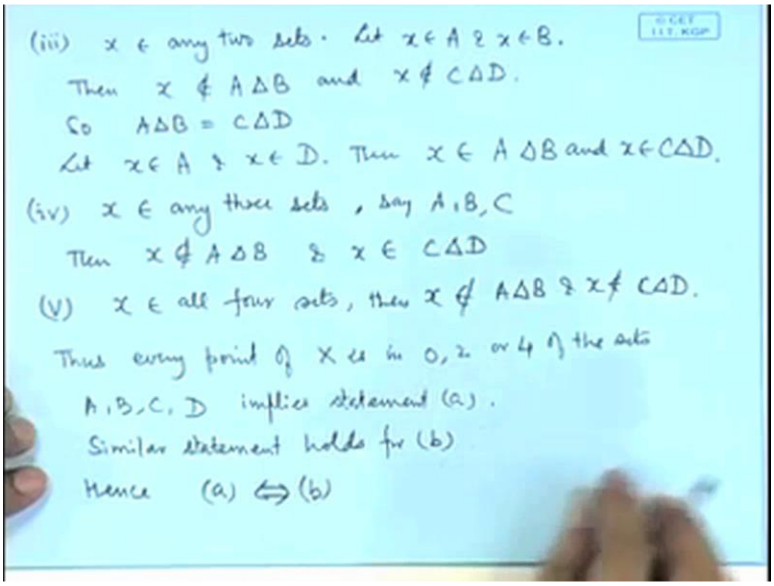

If we consider x belonging to any two sets; let us take x belongs to A and x belongs to B then clearly x does not belong to A delta B because the points which are common to both are excluded from the symmetric difference and x does not belong to C delta d. So, A delta B will be equal to C delta D, let x belongs to a and x belongs to D then x will belong to A delta B and x will belong to C delta D. Let us consider the possibilities that x belong to any three sets; say A, B and C then x will not belong to A delta B and x will belong to C delta D. If we consider x belongs to all four sets then clearly x does not belong to A delta B and x does not belong to C delta D. Thus we have proved that every point of x is in 0, 2 or 4 of the sets; A, B, C, D implies this statement one, similar statement holds for B, hence A and B must be equivalent.

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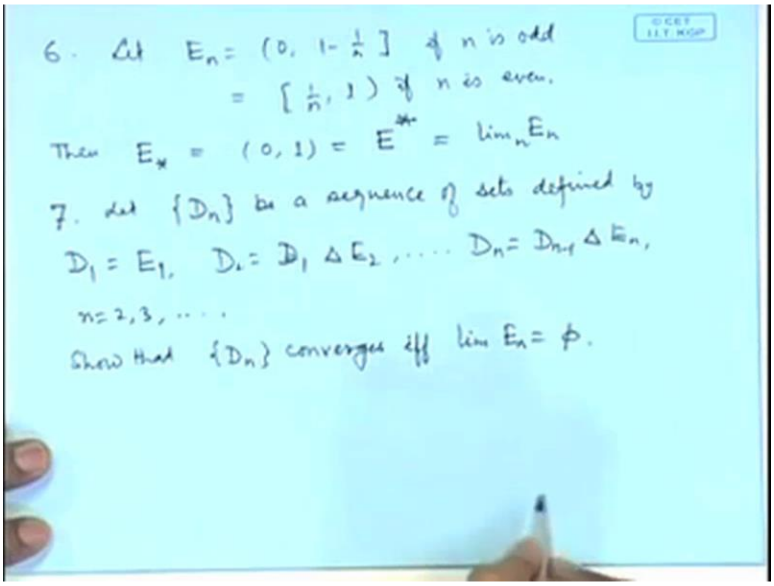

To end this class we look at one or two more examples of the (Refer Time: 25:28) infimum limit inferior and let E n be equal to 0 to 1 minus 1 by n semi open interval if n is odd and it is equal to 1 by n to 1; if n is even. Then if we look at limit inferior, it is the open interval 0 to 1. If we consider any real number between 0 to 1 then we can always find a capital N such that the point a will belong to both 0 to 1 minus 1 by n and 1 by n to 1 for all n greater than or equal to capital N, therefore the point will certainly belong to the limit inferior set. Since it will belong to limit inferior it will also belong to the limit superior and limit superior cannot be bigger than the interval 0, 1.

Therefore, it will also equal to limit superior, and therefore the limit of the sequence exist and it is the open interval 0 to 1. Let me complete today's lecture by giving the final exercise; let us consider a sequence of defined by that D 1 is equal to E 1, D 2 is equal to D 1; delta E 2. In general D n is equal to D n minus 1 delta E n for n is equal to 2, 3 and so on, so that the limit of the sequence D n exists if and only if limit of the sequence E n is equal to.

In next class we will introduce the concepts of certain algebraic structures such as rings; sigma rings, fields and sigma fields which are eventually going to be used for the definition of a probability function.

Thank you.