Probability and Statistics
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Lecture – 12
Problems in Probability-II

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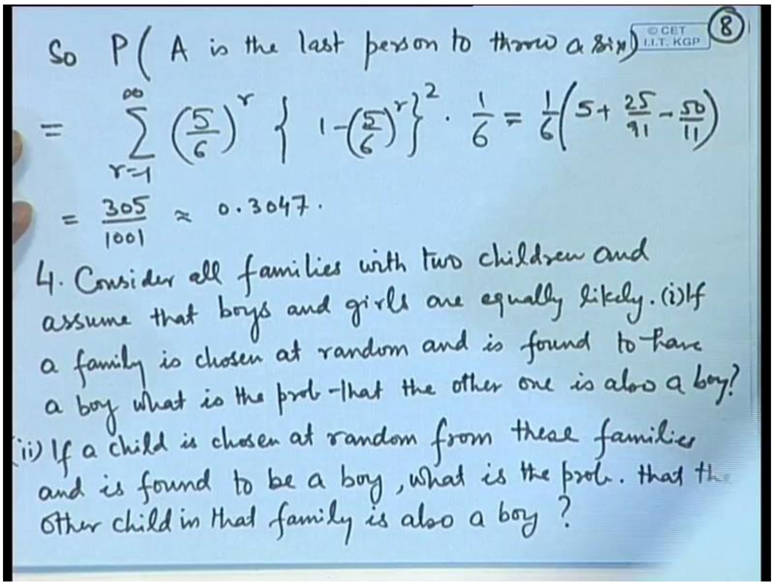

So, if we consider probability that A is the last person to throw a six; that will be equal to 1 minus 5 by 6 to the power r 2 times, so it is square of this. And then we consider the probability that A has no 6 up to its rth trail and the 1 by 6 on the r plus first trail, so that is 5 by 6 to the power r into 1 by 6. So, here we will get three infinite geometric sums and if we add; after simplification it turns out to be 305 by 1001. Once again you can see that this is less than the probability that A is the first to throw a six.

Let us look at some applications of the conditional probability now. Consider all families with two children and assume that boys and girls are equally likely. If a family is chosen at random and is found to have a boy, what is the probability that other one is also a boy, this is one part of the problem. In the second part we ask; if a child is chosen at random from these families and is found to be a boy, what is the probability that the other child in that family is also a boy. Notice here that the sampling scheme is different; in the first one the family is chosen, in the second one the child is chosen, you will see that representation of the sample space will be different in the both cases.

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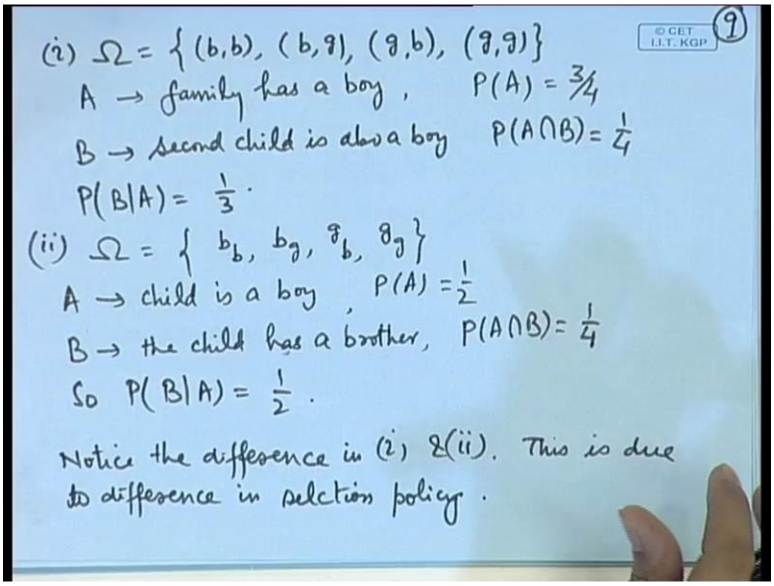

Let us take the first case: in the first case we are considering families with two children. So, we can put it as an order pair; both of them are boys, first is a boys, second is a girl; first is a girl, second is a boy or both are girls. So, A is a event that the family is a boy then probability of A will be 3 by 4 because out of this possibility if you see, because we are choosing a family so family means it could be this, this, this or this and therefore, one boy is appearing in three places; at least one boy.

So, the family has a boy a probability of that will be 3 by 4, so now, if I define the event that B that second child is also a boy then conditional probability of A given B is probability of A intersection B divided by probability of A. So, probability of A intersection B corresponds to the possibility that both the children are boys. So, this is only one possibility and therefore the probability of that is 1 by 4. So, if I take the ratio of probability of A intersection B with probability of A, we get probability of B given A as 1 by 3. That means, if in a randomly chosen family if a child is found to have a boy then the probability that the other one is also a boy is 1 by 3.

Let us look at the second part of this problem. Here the sampling scheme is different. So here from the collection of all the families, we choose a child at random; if a child is chosen at random from these families so; that means, the child can be a boy with a brother, the child can be a boy with a sister, a child can be a girl with a boy, the child can be a girl with a sister. So, here you can see the representation of the sample space is different; although here you may feel that we have written at in this way boy, boy, boy, girl etcetera. So, it can be also considered as a brother or sister relationship however it is not so, because here we are choosing randomly the family whereas, here we are choosing a child. So, the child may have a brother or sister. So the representation of the sample space is quite different.

So, what is the probability that the child is a boy, it will be simply half, because it could be a boy or it could be a girl; both are having two possibilities B; the child has a brother. So, if I look at probability of A intersection B; it means the child is a boy and it has a brother. So, it is this possibility that is 1 by 4; so probability of B given A becomes half.

Notice here, the answer in the beginning may look to be the same that if a family is chosen at random and is found to have a boy, what is the probability that other one is also a boy, we are getting the answer as 1 by 3; whereas in the second case, the child is chosen what is the probability that the other child in the family is also a boy, here it is half. So, it may look counter intuitive, but it is because the answers are coming different whereas, the event looks to be the same however it is not so, because the sampling scheme is different in both the cases; and therefore the representation of a sample phase itself is different in both the cases In the first case the sample space is described like this and in the second case it is described like this.

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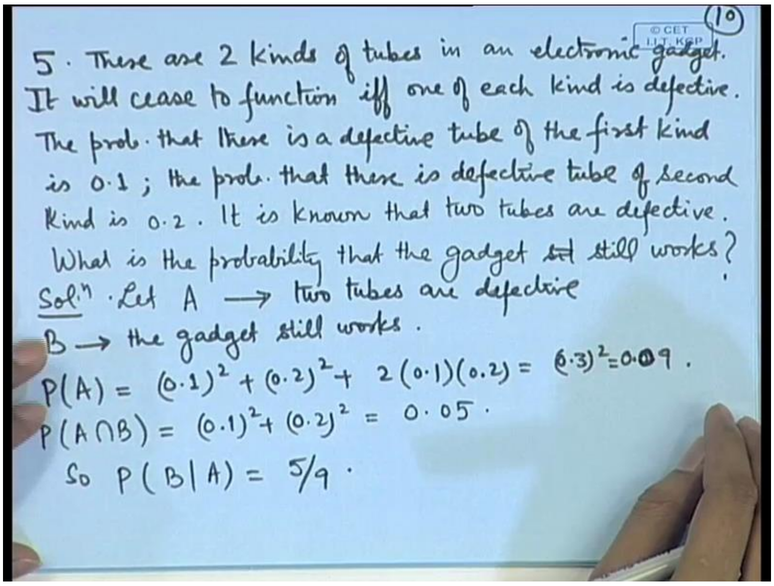

Let us look at some further applications of the conditional probability, so consider example; there are two kinds of tubes in an electronic gadget, it will cease to function if and only if one of each kind is defective. So, there are two kinds of tubes, so if both kinds of tubes so at least one of each kind is defective then the gadget will fail. The probability that there is a defective tube of the first kind is 0.1 and the probability that there is a defective tube of the second kind is 0.2. It is known that the two tubes are defective, now these two tubes could be any combination, both could be first kind, both could be second kind, one could be first time defective second time defective etcetera. So, what is the probability that the gadget is a skill working?

So, let us define the event A that the two tubes are defective and B is the event that the gadget is still works; that means, we are interested in finding out the conditional probability that B given A. So, we need to look at the probability of A intersection B and probability of A. So, probability of A that the two tubes are defective, so here we can have four different possibilities and here we will make use of the independence of the individual tubes to be working; that means, we assume that each tube fails or works independently of the other tubes. So, if both the tubes are defective of the first kind then the probability will be 0.1 into 0.1 that means 0.1 is square; both may be having defects of the second kind so it is 0.2 into 0.2 that is 0.2 is square. Or first one could have defect of the first kind and the second one could have defect of the second kind or vice versa. So, it will be 2 into 0.1 into 0.2, so this is equal to 0.09.

Now what is probability of A intersection B? A intersection B is the event that the gadget is still working and the two tubes are defective; that means, it ensures that we cannot have one tube to be defective of one kind and another tube to be defective of another kind because in that case the gadget will not be working. Therefore, both defects are either of the first kind or both are of the second kind, so here we have made use of that probability of union is equal to some of the probabilities of disjoint events and we have made use of the concept of the independence, so if we add this after simplification this turns out to be 0.05 and therefore, the conditional probability of B given A is equal to 5 by 9.

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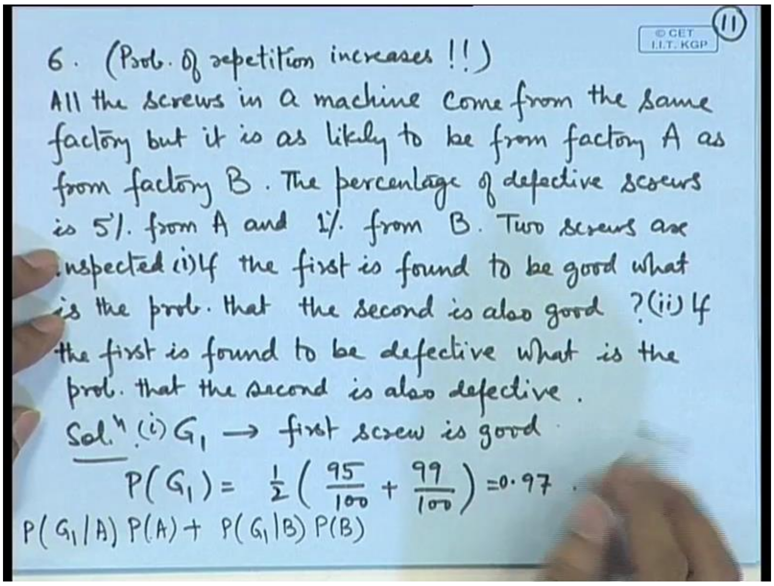

We consider some further applications of the conditional probability; here we will like to show one interesting phenomena, that is the probability of reputation of certain event increases; let me explain through the example. So, a machine has certain a screws fitted, so all the screws in a machine come from the same factory, but it is as likely to be from factory A as from factory b; that means, either all of the screws have been selected from factory A; that means, probability is half or all of them are taken from the factory B; the probabilities again half.

Now in both the factories some of the screws may be defective, so the percentage of defective screws is 5 percent from factory A and 1 percent from factory B. So, two screws are inspected 1 by 1, if the first is screw is found to be good; what is the probability that the second is also good. I am looking it in the reverse way also. If the first is screw which is inspected this found to be defective; what is the probability that the second is also defective. In order to evaluate this; let us define the events.

Let G 1 denote the event that the first inspected screw is good then what is the probability of G 1? Then probability of G 1 is actually by using the theorem of total probability that all the screws came from A into probability of A plus probability of G 1 given B into probability of B. Now probability of A and probability of B is half; what is the probability of good is screw coming from factory A. So, since 5 percent of the product coming from factory A is defective, so the probability that a good screw is there from factory A is 95 by 100.

In a similar way probability that the screw is good coming that it is from factory B, it will be 99 by 100 because 1 percent of the factory B products are defective. So, after simplification it turns out to be 0.97.

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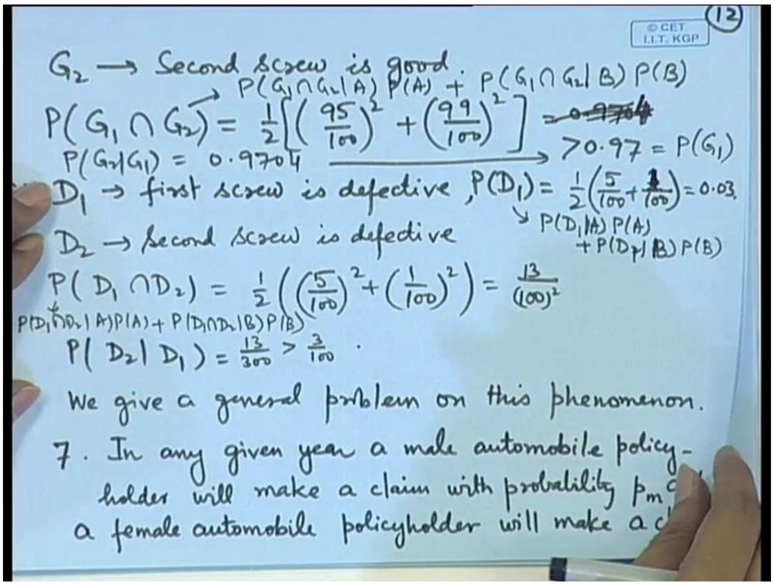

Now, we look at the event that the second screw is also good that is the one after another, the screws are inspected. So, first screw has been found to be good, what is the probability that the second is also good. So we look at the event G 1 intersection G 2, so first screw is good and second screw is good. Once again if we represent this event it becomes probability of G 1 intersection G 2 given A into probability of A plus probability of G 1 intersection G 2 given B into probability of B. So, here probability of A and probability of B both are half, here first a screw is good that is with probability 95 by 100.

Now, all the screws are coming independently from the factory A of factory B. So, at each inspection the probability of them being defective or good remains the same. So, it will be 95 by 100 into 95 by 100; if it is coming from factory A. In a similar way, if we are looking at from factory B it will be 99 by 100 into 99 by 100. So after some simplification probability of G 2 given G 1 that is probability of G 1 intersection G 2 divided by probability of G 1. So, after simplification it turns out to be 0.9704 which is clearly bigger then 0.97 that is the probability of G 1.

So the comment that I made in the beginning that the probability of some reputation of certain event increases; we are saying that if a screw is good then the second one is also good is probability is higher; that means if it is found to be good; that means, the supplier who has given we are taking from the good ones and therefore the second trail will have a higher probability of being good. So, probability of G 2 given G 1 becomes 0.9704, let us look at in the reverse way. Consider D 1 to be the event that the first inspected is screws defective, so probability of D 1 using the argument which in we considered earlier, so we can write it has probability of D 1 given A into probability of A plus probability of D 1 given B into probability of B.

So this is equal to half, the 5 percent from supplier factory A are defective that is 5 by 100 and 1 percent from the supplier B are defective, so it is 1 by 100; so it is 0.03. So, if I look at the event D 2; that the second is screw is also defective then once again we can use the same representation, this is equal to probability of D 1 intersection D 2 given A into probability of A plus probability of D 1 intersection D 2 given B into probability of B. So, by the logic which we used earlier it is half into 5 by 100 is square plus 1 by 100 is square that is 13 by 100 square. So if I look at probability of D 2 given D 1 that is probability of D 1 intersection D 2 divided by probability of D 1, so which after some simplification becomes 13 by 300 which is clearly bigger than 3 by 100.

Once again you can see that this probability has increased. What does it mean? It means that if the first one is the defective, it means that there is more likelihood that the supplier which is giving more defectives is the one which has actually given, and therefore the probability will increase that for the second one also to be defective. In fact, we can consider a more general problem here where we can replace this numbers by some abstract expressions, some numbers between 0 and 1 and is and similarly this probability of selection from each one in place of half; we can put some alpha and see that this phenomena is still holds.

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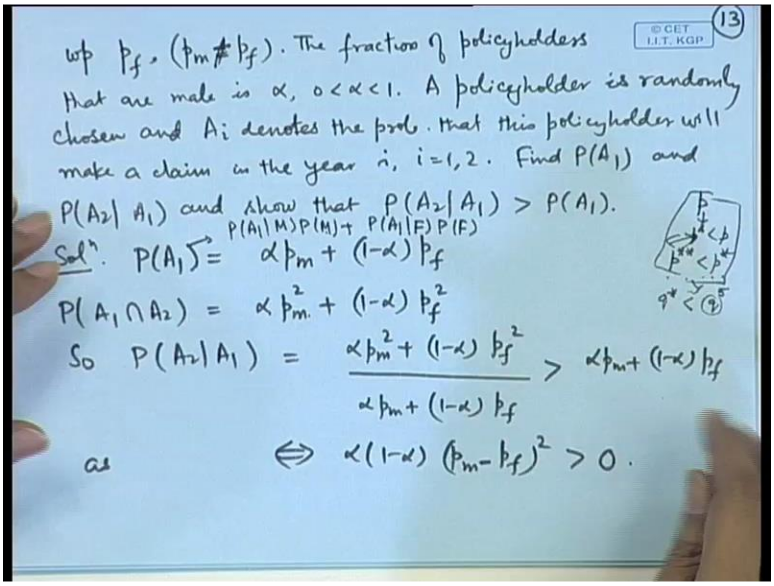

So, let us consider this problem; in any given year a male auto mobile policy holder will make a claim with probability p m and a female auto mobile policy holder will make a claim with probability p f; in general we assume p m is not equal to p f although it is not necessarily it may be equal also. The fraction of policy holders that are male is alpha where alpha is a number between 0 and 1. A policy holder is randomly chosen and a i denotes the probability that this policy holder will make a claim in the year i; for i is equal to 1, 2; what is probability of A 1, what is probability of A 2 given; A 1; in general show that probability of A 2 given A 1 is greater than probability of A 1.

So, once again it is a question of reputation; that means, if the person has made a claim in 1 year; then in the second year he will again make a claim the probability increases. Basically it means that if A is made a claim; that means, he is more accident proven person and therefore, it is more likely that in the next year also he will make a claim. If you look at the practical application of this, suppose you purchase a car and you take insurance, so you will have to pay certain amount say p. Now if you did not have any accident during the year, your premium gets reduce to some number p star less then p. If you in the second year again you do not have any accident, your premium will be further reduced in the next year and it will keep on reducing in the subsequent years.

Even if you purchase a new vehicle after say 5 years and the usual premium on the new vehicle is a q, you will be asked to pay only q is star which is less then q based on your past performance. However, if you commit an accident then and you make a claim then your premium will become much more. So, this phenomena that is the probability of the repeat event becoming more is used in practice by the insurance companies. So, let me give the solution to this problem, so using the same argument that probability of A 1; that means, the person makes a claim it is by using the theorem of total probability this is probability of A 1 given that he is a male plus probability of he makes a claim given that the person is a female. So, the probability of a male is alpha and probability of female is 1 minus alpha and probability of making claim for male is p m and probability of a female making the claim is p f.

So, it is alpha p m plus 1 minus alpha p f, by using the same argument probability of A 1 intersection A 2 becomes alpha p m square plus 1 minus alpha p f square and therefore, the conditional probability that A 2 given A 1 is equal to probability of A 2 intersection A 1 given divided by probability of A 1 is this. Now this conditional probability is greater than probability of A 1, you can actually write this expression and simplify it is reducing to alpha and 1 minus alpha, p m minus p f whole square. So, unless p m is equal to p f; this term is strictly greater than 0; so in the unlikely case where p m is equal to p f then this will be equal to 0; that means, the two probabilities will be same; however, p m is equal to p f simply denote that it does not make a difference that where from your choosing; that means, both have the same probabilities of say either defective or non defective or making a claim or not making the claim.

So, in that case the phenomena will not change the probability because here the effectives coming, because if we are saying that the person makes a claim so that means he is more accident (Refer Time: 20:21).

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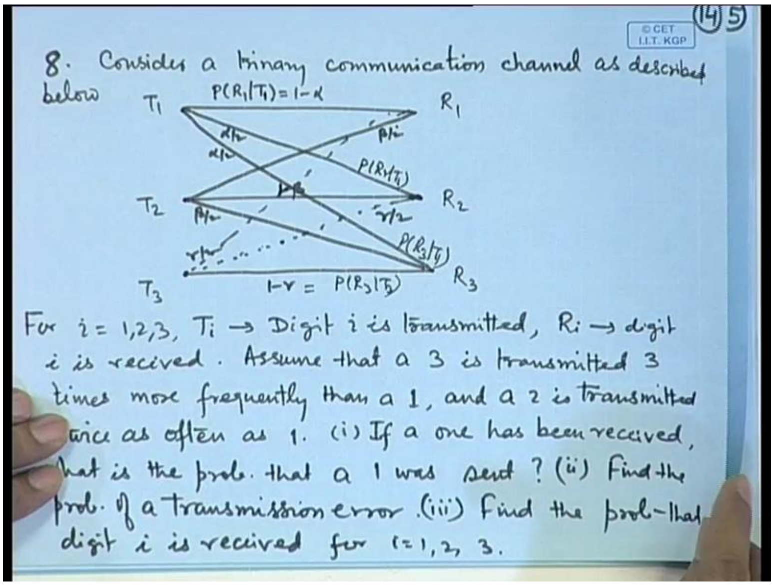

So, that is why he should have a higher probability in the next year. Consider a communication channel, so here again it is some further applications of the conditional probabilities. So, we have a trinary communication channel, so let me explain these things. So, T i mean that digit i is transmitted, so T 1 means that the digit 1 is transmitted, T 2 means digit 2 is transmitted, T 3 means digit 3 is transmitted. R 1 means digit 1 is received, R 2 means the digit 2 is received, R 3 means digit 3 is received. So, it is a trinary communication channel; now digits 1, 2, 3 are transmitted; however, due to noise in the channel; they may not be received as the same.

So, probability that a 1 is received given that it was sent is 1 minus alpha whereas, probability that it is received as 2 that is probability of R 2 given T 1 is alpha by 2 probability of R 3 given T 1 is alpha 2; that means, alpha by 2; that means, 3 is received given that 1 is sent; is alpha by 2; that means, with probability 1 minus alpha it is correctly sent and with probability alpha by 2, alpha by 2 each it is going as some wrong numbers. It is due to the noise in the channel, likewise probability that a 2 you received given that a 2 was sent is 1 minus beta and it is beta by 2 for the other two possibilities and similarly probability that a 3 is received given that a 3 sent is 1 minus gamma and gamma by 2 each is the probability that it is received as 1; R 2.

Further we assume that in this communication channel 3 is transmitted 3 times more frequently as a 1 and 2 is transmitted twice as often as 1, if a 1 has been received; what is the probability that 1 was sent, what is the probability that a transmission error has occurred, what is the probability that it is it i received for i is equal to 1, 2, 3. So, let us look at the probabilities of each of these, so firstly we look at the conditions of the problem 3 is transmitted 3 times more frequently than 1, 2 is transmitted 2 times as frequently as 1.

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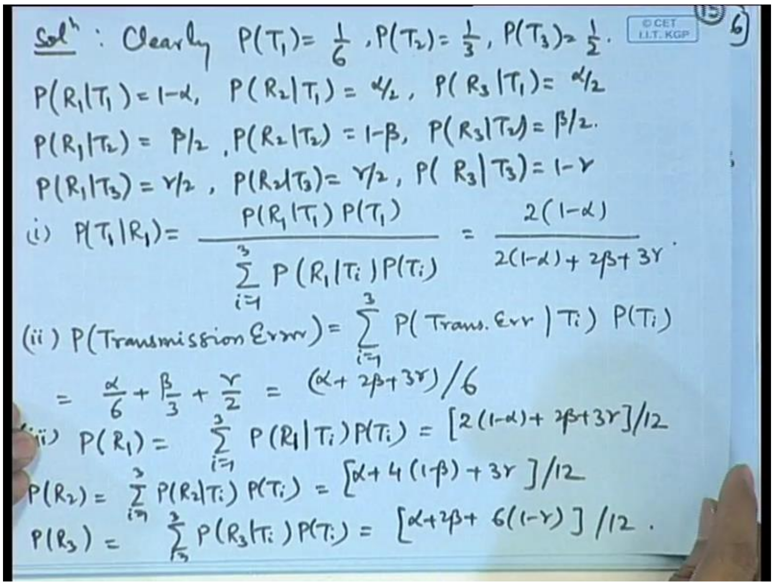

Therefore the probabilities of 1 being transmitted, 2 being transmitted and 3 being transmitted are as follows 1 by 6, 1 by 3 and 1 by 2. Further it is given that the conditional probabilities of R 1 given T 1, R 1 given T 2, R 1 given T 3 R 2 given T 1 and so on. Now if we look at probability of T 1 given R 1; that means the digit 1 is received; what is the probability that 1 was sent. So, it is a direct application of base theorem because T 1 is a priory event because the digit is sent before and it is received afterwards. Now in the light of the new information that what has happened afterwards, what is the probability of a prior event, this is what we call posterior probabilities and we will use base theorem year. So, probability of T 1 given R 1 is equal to probability of R 1 given T 1 into probability of T 1 divided by sigma probability of R 1 given T i probability of T i; i is equal to 1, 2, 3.

So, all the expressions are given here and we see substitute, so after simplification it turns out to be twice 1 minus alpha divided by; twice 1 minus alpha plus twice beta plus 3 times gamma. In fact, in a similar way we can calculate probability of T 1 given R 2, T

2 given R 1, T 2 given R 3 and so on. What is the probability of a transmission error; now transmission error is the post event; that means firstly something is sent, something is transmitted. Therefore, it is conditional upon what was actually sent, so there are three possibilities of sending the digits 1, 2 or 3. So, again by using theorem of total probability, probability of transmission error becomes transmission error given T i into probability of T i.

So, what is the probability of T 1 that is 1 by 6 and what is the probability of transmission error given that 1 was sent, it is alpha because 1 minus alpha of correctly sending, so it becomes alpha and 2; 1 by 6. In a similar way, if the digit 2 is sent then with probability beta, it is not received correctly with probability 1 minus beta it a received correctly and with probability 1 by 3; the digit 2 is sent, so the probability becomes beta by 3. In a similar way, the probability of transmission error if 3 is sent is gamma and half is the probability of sending the digit 3. So, the probability is evaluated here, what is the probability that digit 1 is received, what is the probability that the digit 2 is received, what is the probability that the digit 3 is received.

In each of these cases, the digit getting received is a consequence of digit beings sent. So, at each stage the theorem of total probabilities applicable and the expressions for the conditional probabilities are given here, we can utilize them to where the expressions for that the digit 1 is received or the digit 2 is received or the digit 3 is received.

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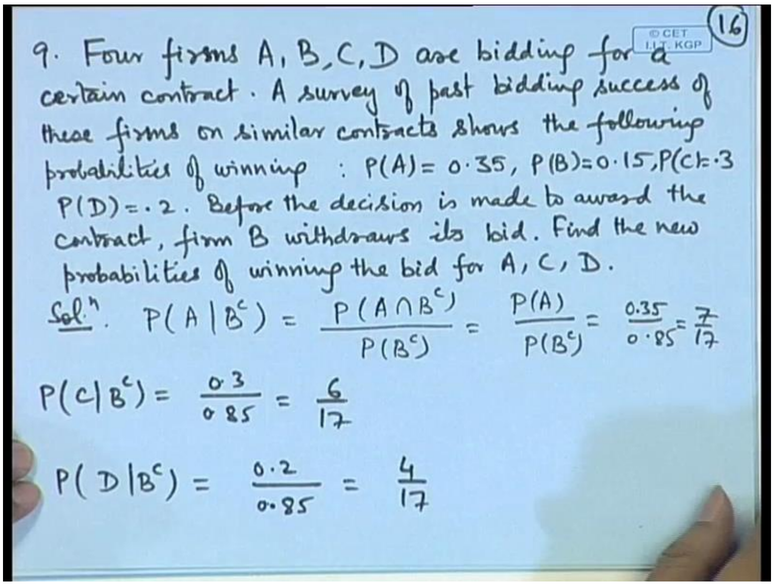

Let us look at some more applications of the conditional properties; four forms A, B, C and D; they are bidding for a certain contract. A survey of the past bidding success of these forms on similar contracts shows that the following probabilities of winning the contract are that is A will be in the contract with probability 0.35, B will be in the contract with probability 0.15, c will be in the contract with probability 0.3 and D will be in the contract with 0.2. Before the decision is made two award the contract, firm B withdraws its bid. Find the new probabilities of winning the bid for A, C and D. So, basically what does it mean, it means that if B has withdrawn; that means, B cannot win the bid. Therefore, probability of A winning is actually now the conditional probability of A given B compliment.

So, by using the definition of a conditional probability becomes probability of A intersection B compliment divided by probability of B compliment. Now here you notice that B compliment means that B does not win the bid, therefore A winning the bid is actually a subset of this. Therefore, A intersection B compliment is simply probability of A; so if we substitute the probabilities here we get it as 7 by 17. So, in a similar way probability of C given B compliment terms out be 6 by 17 and probability of D given B compliment turns out to be 0.2 divided by 0.85 that is 4 by 17. So if B has withdrawn; actually his shear of probabilities allocated to the other 3 bidders here and that is why the probabilities are getting modified in place of 0.35 it has become slightly more than 0.35 in place of 0.3 it has become slightly more than 0.3, in place of 0.2 it has become slightly more than 0.2.

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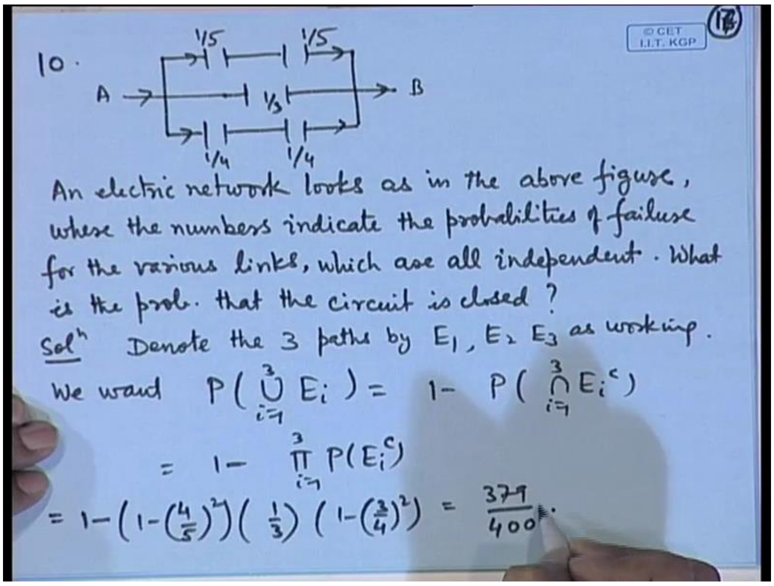

Let us consider a communication channel, this is an electric network the current flows from A to B; however, there are three independent paths here which we call say E 1, E 2, E 3. There are circuits here, so 1 by 5 denotes that the circuit in this failure of this link that is the probability of failure year; 1 by 5 is the failure of this link, 1 by 3 is the probability of failure of this link, and 1 by 4 etcetera are the probability of failure of these links. What is the probability that the current is actually flowing from A to B? So, if we denote the 3 paths by E 1, E 2, E 3 then it is probability of union of E i which is equal to 1 minus probability of intersection E i compliment.

Now, here each circuit is working independently, each path is working independently therefore probability of intersection becomes product of the probabilities. Now here it is 1 minus; now here probability of E 1 compliment, so E 1 compliment means that this circuit is not working, it is not working if either of this is failing are it is working if both are working; that means 4 by 5 is square. So, it becomes 1 minus 4 by 5 square; this will not work with probability 1 by 3; this will not work with probability 1 minus 3 by 4 square. So, after simplification it becomes 379 by 400 which is pretty high, so this is because of the redundancy in the system because if any of the path is working, the current will be flowing from A to B.

So, here in today’s lecture we have given various applications of the rules of the probability.

Thank you.