A Basic Course in Real Analysis
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Lecture - 9
Type of Sets with Examples, Metric Space

In the previous lecture, we have introduced the concept of countable sets, and few examples we have seen, the countable sets or uncountable sets. Here, we will continue with the counts and countable sets with few more examples.

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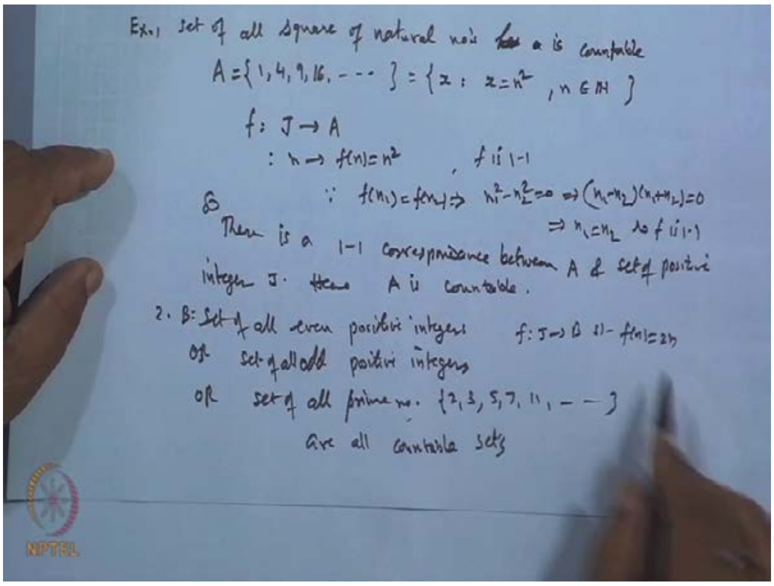

The first example is the set of all squares of natural numbers has a number is countable. Because the set of a squares of natural number means, it is 1, 4, 9, 16, and so on. That is a set of those elements x, where x is of the form n square and n is a natural number. So, this set has a 1 to 1 correspondence. If we define a mapping f from j to this set say, capital A which maps n to n square, that is f of n is n square then it is easy to show, that f is 1 1, and it has a 1 to 1 correspondence. Why 1 1, because f n 1 equal to f n 2 will implies n 1 square minus n 2 square is 0, which implies n 1 minus n 2 into n 1 plus n 2 is 0, but n 1 plus n 2 cannot be 0, this implies that n 1 is equal to n 2, so f is 1 1.

Therefore there is a 1 to 1 correspondence between the set A and set of positive integers J, 1 to n this one. Hence A is countable, similarly the other sets are also like set of all even positive integers or set of all odd positive integers or may be the set of all prime numbers, that is 2, 3, 5, 7 11, and so on. These are all countable sets, which can be easily shown by drawing a mapping from this set to the set of natural numbers, which is easy to say even in positive integer f of n is equal to a mapping can be defined from j to the set say B, such that f of n is equal to 2 n. Then, this is a 1 1 mapping (( )).

Similarly, here all integers 2 n plus 1 and like this, similarly for the primes also we can do for that. So, these are all the sets which are countable, and it is all subsets of a natural number. So, it means an infinite set is a set which has a 1 to 1 correspondence with its subsets also, that is why we define like that. Then another example, we just which we are doing the set of all rational numbers are countable.

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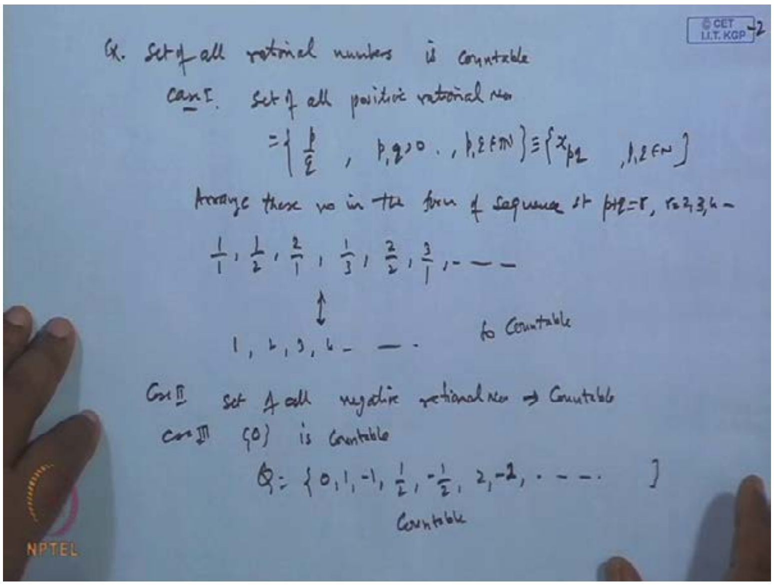

Suppose, I take set of all positive rational numbers that is it will be of the form p by q where p and q both are positive and are natural numbers. If you put it in the form of the sequence, it is of this type p q sequence, where p, q are natural numbers. Then we can arrange these numbers in the form of sequence, such that p plus q is r, where r is 2 3 4 and so on. It means, we can put it in this way 1 by 1 the first term, so that p plus 1 is equal to 2 then we can put it as another, say 1 by 2 2 by 1, and then we can go 1 by 3 2 by 2 then 3 by 1 and continue this. So, it has a 1 to 1 correspondence with the set of positive integer, which has a 1 to 1 correspondence with the set of positive integer 1 2 3 4 and so on. So, it is a countable set.

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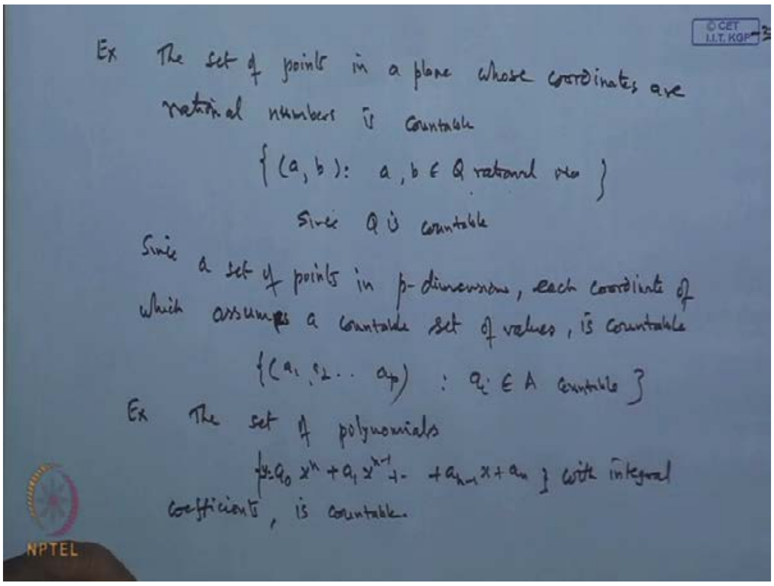

Then same case is with the set of all negative rational numbers. Then set 0 single term, say 0 is a countable set is a finite. So, if I take the odd union, then set of rational number is: set of all positive rational numbers, all negative rational numbers and including 0. So, if we put it in this form 0 1 minus 1 half minus half 2 minus 2 and continue, this plus minus and all these intra set is 1 to 1 correspondent with it and countable.

We were discussing about the points, whose coordinates are rational number. In another example we have seen, the set of points in a plane whose coordinates are rational numbers and is countable. So, set of points in a plane, whose coordinates are rational numbers, we can say a b, where a and b both are rational numbers, Q is the set of rational numbers, because any point in the plane will be ordered pair and we get this point.

Since Q is countable set, and this set is a ordered pair of whose elements are countable, then according to that result, which we have seen that, if s 1 s 2 s n are countable, then countable union of the countable set is countable. And accordingly we can say if there is n, each coordinates belongs to a set, which is countable then that collection of n will also be countable. So basically, double means coordinate and where these are rational this will be a countable set.

Actually, based on this result is that set of points in p dimension, this dimension is 2 in p dimension, each coordinate belongs to a set which assumes a countable set of values is countable. In fact, this was shown already. Suppose, a set of p dimension is there, we prove it by means of induction we have seen that, if the set a 1 a 2 a p where the a i's are in a set A which is countable, then this collection of set is also countable, and this we have shown when i is 1 then this coincide with the set A which is countable, then we assume for p say up to p minus 1 and for the p, when you write the point, it is of this form called ordered pair. Then when you fix up 1 value, other values keep on changing. So, it is a countable set.

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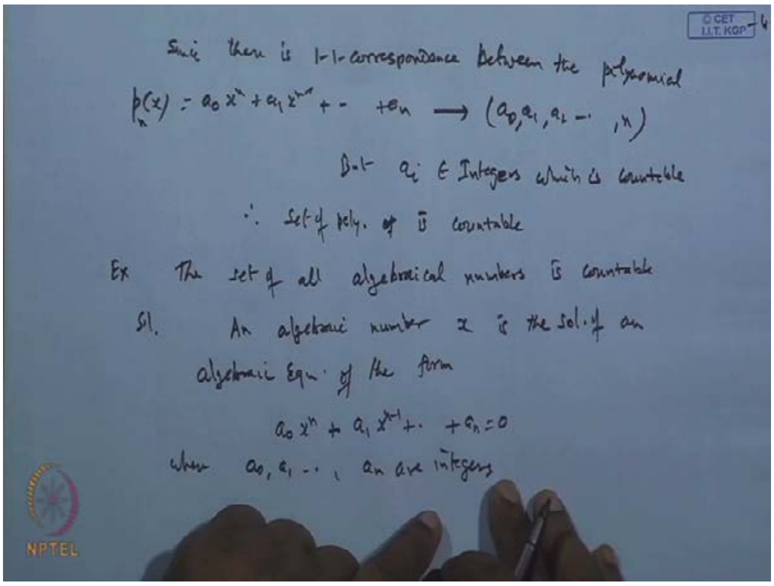

Now, based on the another result, which is the set of polynomials that is, a naught x to the power n plus a 1 x n minus 1 plus a n minus 1 x plus a n. this is the y, the set of all polynomials, where the coefficients are integral and is countable. It means, this set of polynomials where the coefficients are integers is countable. Now you see, that there is a 1 to 1 correspondence between the set of polynomials, and this set of p, since there is a 1 to 1 correspondence with the set of points between the polynomial a naught x to the power n a 1 x n minus 1 plus a n.

Let it be p x p x of the n to the tuples a naught a 1 a 2 a n plus 1 tuples. There is a 1 means corresponding to each polynomial we can get this tuple and if this is known, we can construct a polynomial of degree n. So, there is a 1 to 1 correspondence between these two, but what the coordinate are? These are the integral values and this integer is a countable set which is countable. So, this collection of tuples is a countable set. Because these integers and this collection of tuples are countable, there is a 1 to 1 correspondence between the elements of the set to this. So, this collection will also be countable. This shows that set is countable.

So, set of polynomials is countable of degree say n is countable. Then the set of all algebraic is countable. What is algebraic number? The algebraic number x is a solution of an algebraic equation of the form a naught x to the power n a 1 x n minus 1 plus a n equal to 0, where the coefficients a naught a 1 a 2 n these are all integers. So, this is the solution and number x satisfy this equation will be an algebraic number or corresponding to a solution of this equation will be an algebraic number.

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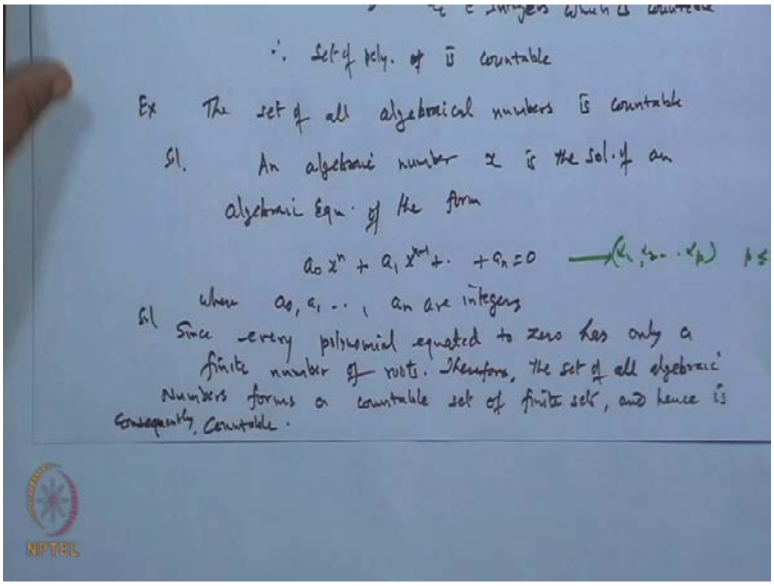

Now, if we remove this 0, just I take, then this is a polynomial of degree n and the coefficients are integers. According to the previous result, if the coefficients are integer, then collection of all such polynomials will be a countable set. Once the polynomial, when we put it equal to 0 you are getting a algebraic equation. The roots of this algebraic equation will be at the most n. These n roots which will satisfy this equation. Each equation will have the roots and since this collection of the polynomial is countable, therefore the corresponding roots will all set of all algebraic numbers which are the roots of this equation will be countable. Since every polynomial equate it to 0, has only a finite number of roots. Therefore, the set of all algebraic numbers forms a countable set of finite sets because these solutions are finite. So, it forms the finite set. When you find the roots of this polynomial, you are getting a finite roots, may be at most n. So, basically each equation corresponds to roots.

So, we can just say that this equation has the root say alpha 1 alpha 2 alpha n say alpha 1 alpha 2 alpha p where the p may be less than or equal to n. So, this equation will correspond to this. So, the collection of all such equation means collection of all such elements, but these are finite which is countable and then collection p a also is countable. So, this set will be a countable set. The algebraic numbers once you get, then all the transcendental numbers becomes the uncountable. So, let us see the few examples of an uncountable set.

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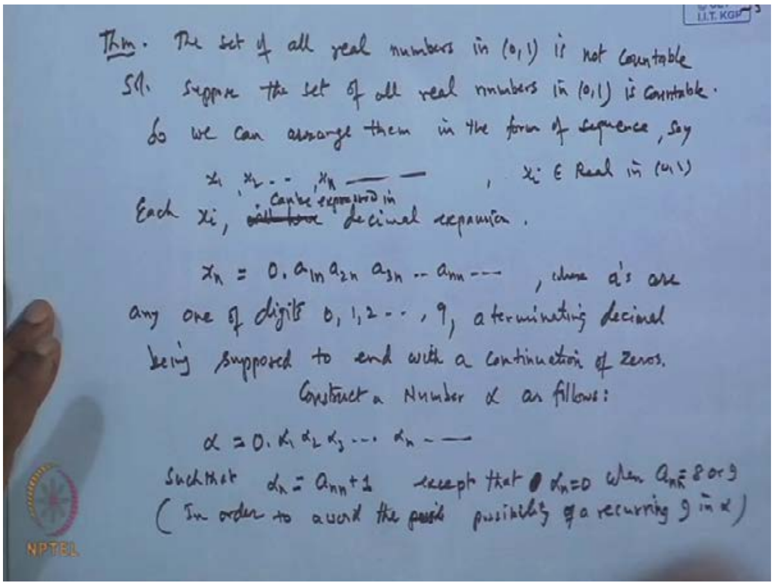

First example, which is very important is the real set of all real numbers in the open interval 0 1 is not countable. So, what the concept is that, all the infinite sets need not be countable. In this example, the set of all the real numbers, form the infinite sets, infinite real numbers lying between 0 1. But this set is not countable. The reason is like this. We assume that the set of all real numbers in the open interval 0 1 is countable. Once it is countable, we can arrange in the form of sequence say x 1 x 2 x n and so on. As they are infinite numbers, we get an infinite sequence.

Now, each x i these are real in the interval 0 1. We can write down the decimal expansion of x. So, each element x will have decimal or can be expressed in a decimal expansion. So, let us suppose the x n is having the decimal expansion as 0. Point a 1 n a

2 n a 3 n a n n and so on, where these coordinates are any one of the integer are any one of the digit 0 1 2 up to 9, we assume is a terminating decimal being supposed to end with a continuation of 0s. Suppose it terminate here then rest we will write 0 0 0 0 like that. So, it is a network, now with the help of this let us construct a new number.

Now, we construct a number alpha as follows: alpha a m writing the decimal expansion of alpha is 0 point alpha 1 alpha 2 alpha 3 alpha n and so on, where such that the alpha n is nothing but a n n plus 1 except that a n is 0, alpha n is 0, when a n n a n n is either 8 or 9 in order to avoid the possibility of recurring 9 in it, this is done in order to avoid the possibility of recurring 9 in alpha. This we are doing now.

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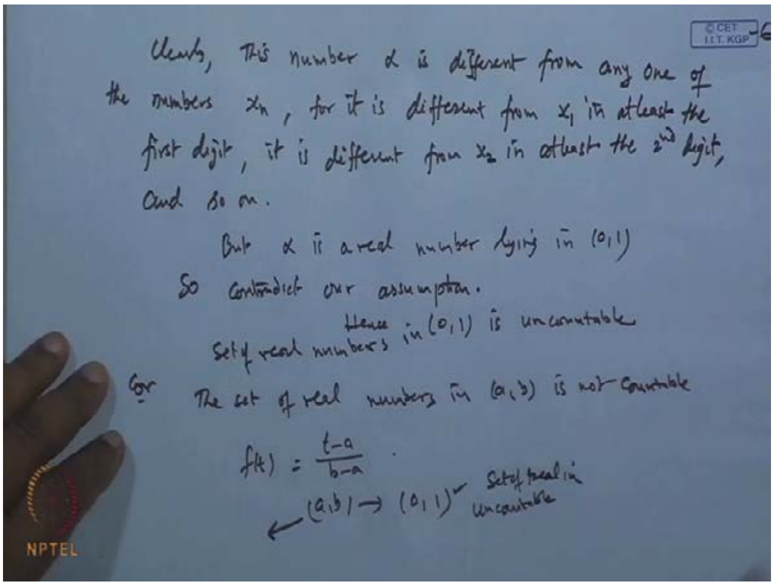

We claim that this point alpha, so constructed differs from each any term x 1 x 2 x n of the set of the points in the 0 1. Why at least at 1 place? Suppose I take x 1 then x 1 here is a 1 1 in the first place a 1 1. While in alpha, the first point is alpha 1. And alpha 1 is what a 1 1 plus 1. So, basically, whatever the first point is, first decimal place is there, we are replacing this by plus 1 next digit. The digit will be say, instead of this, we can write the 7 to 8 6 to 7 and so on. And whenever it is 8 or 9, then we can write this to be 0. Suppose this is 8 then we get 9. So, we put it alpha n 1 to be 0. Similarly when to x 2 the second decimal place in x 2 is a 2 n, but second decimal place of alpha is alpha 2 which is a 2 2 plus 1. So, again 1 is added here. Clearly we will see, this alpha is different from any one of the number x n, for it is different from x 1 in at least the first digit. It is different from x 2 from x 2 in at least the second digit and so on.

But, alpha is having a decimal expansion where the positive side is 0; mean all the terms are less than 1 lying between 0 and one. But alpha is a real number lying in the interval 0 1, and since we have assumed 0 1, the set of all real numbers 0 1 is countable. So, it can be arranged in the form of the sequence that is there, but this alpha does not fall in any one that does not coincide with any of these x n. It means, our assumption is wrong. If it is countable, then alpha must be 1 of the x n s, but this is not true. So, that shows that our assumption is wrong. So, contradict our assumption, hence 0 1 set of all points, all real numbers in the interval 0 1 is uncountable and is proved.

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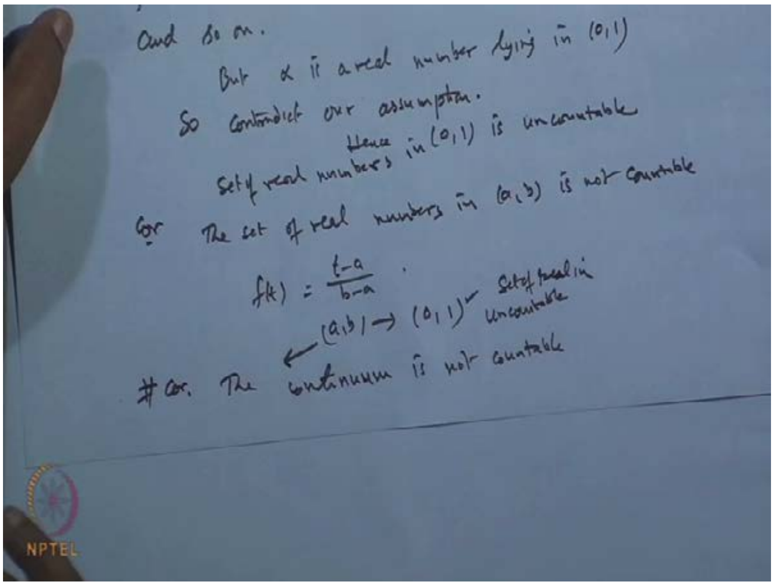

Now, if we take say any interval, the set of real numbers in any interval a b is not countable, because if I make the function t minus a over b minus a, then it will transfer the function, this mapping will transfer the interval a b into the interval 0 1 and this is a 1 1 transformation, it is a 1 1 mapping, and this interval is uncountable. Set of all points in this interval, is set of all real numbers in this interval is uncountable. So, in this also, set of all real numbers in this interval is uncountable and then if we extend it, then we say an important result is that is the set the continuum is not countable.

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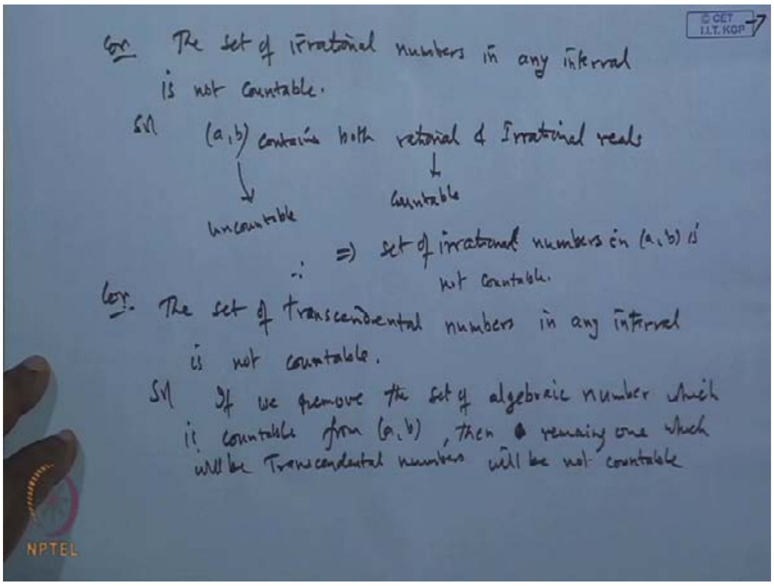

Continuum means, set of all real numbers over entire real line is not countable. That is, what is as if corollary of this we can prove one thing, the set of irrational numbers in any interval is not countable and solution is varying, because interval a b contains both rational and irrational points and irrational real numbers. So, the rational numbers are countable, this is uncountable.

Therefore, this implies that set of all irrational numbers in the interval a b is not countable and same way we can also write this corollary the set of transcendental numbers, which are not algebraic in any interval is not countable. Because the reason is, if we remove the set of all algebraic numbers from this, then we obtain an algebraic countable which is countable, we can get the transcendental left out.

Because the reason, if we remove the set of all algebraic numbers which is countable from any interval a b, then the compliment set of transcendental is unknown countable is uncountable, compliment then remaining 1 which is transcendental number will be not countable. So, there few examples we have seen now. Few more we will continue when we go for the concepts like dense set then perfect sets, etcetera. Then we will go for the few more examples where the countable uncountable nature of the set is considered.

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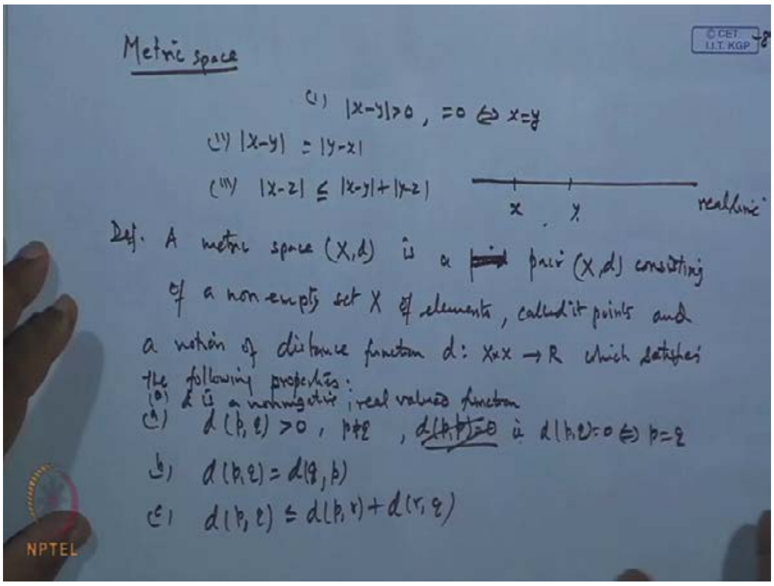

Now, let us come back to a 1 metric space, though this is the optional one, but we will discuss it here, as we have seen that, in case of the real line, when we pick up the 2 points x and y, then the distance between x and y is defined as mod of x minus y, this is the absolute distance between the 2 point on the real line and this distance will always be greater than 0 and will be 0 if and only if the x equal to y. Then if we measure the distance either from x to y or from y to x, both will be the same. Then third is when we take any point z either in between x and y or may be outside, then this will remain less than equal to x minus y plus y minus z.

So, these properties are satisfied by the absolute function or we say that the distance function, define on the real numbers on the real line, then we want it to extend it to an arbitrary metric set, because this is the set of real numbers. We are having this property, we want it to make it as an axiom for an arbitrary set and that list to the concept of the matrix space.

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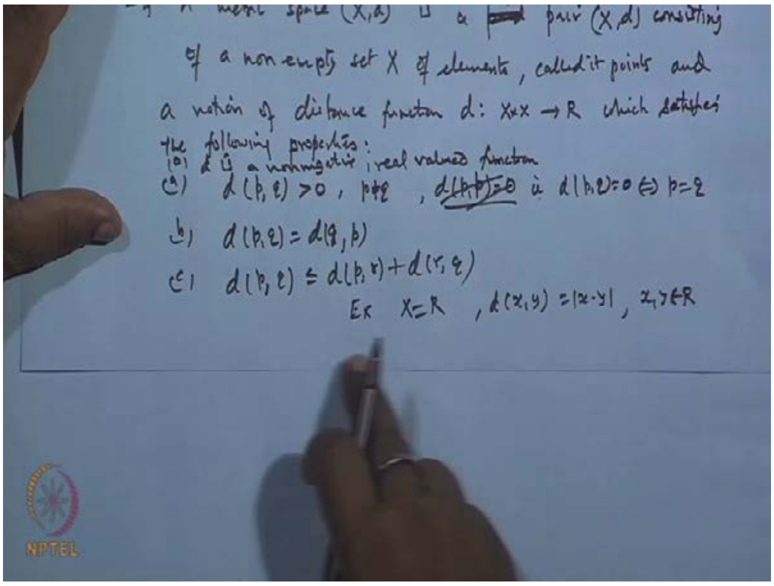

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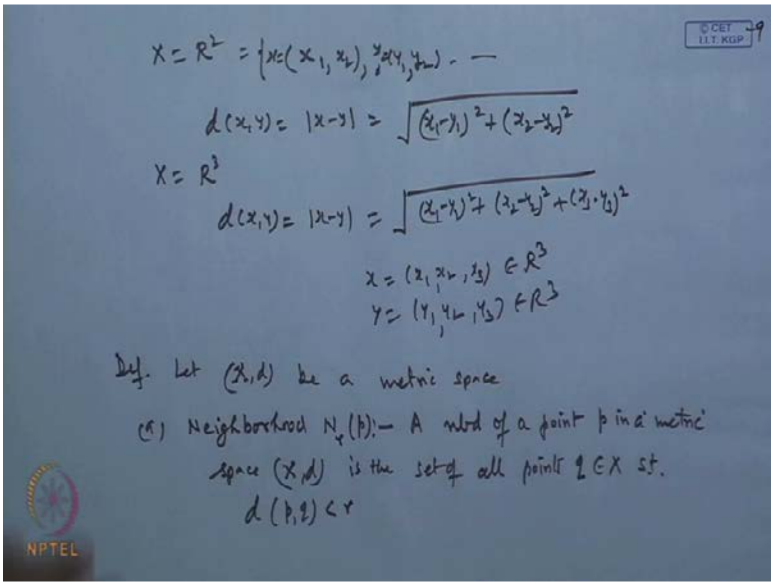

So, we define the metric space as a metric space x d is a pair x d consisting of a non empty set x of elements, we call it here as point, and a notion of distance function d define over x cross x to r, which satisfies the following properties. The first property is the distance between the 2 points p q is greater than 0, if p is different from q and distance between p q is 0.

In fact, other way you can say vice versa also. The second is distance d of p q is the same as d of q p then c is d of p q is less than equal to d of p r plus d of r q. So, a distance function satisfies the first. You can write the 0 property d is a non negative real valued function, and this property can also be written as d of p q is 0 if and only if p equal to q, we can replace it. In fact, this is, when p and q are identical both are equal to 0 and vice versa. So, if this properties are satisfied then we say this set x together with the notion of the distance forms a metric space and one of the example is our x is r set of real line d of x y is defined as x minus y where x and y are real then as usual, we have seen earlier, satisfy all the properties.

Then another example is, suppose x is say r 2 r 2 is the 2 dimensional plane where the elements is of the form say x 1 x 2 y 1 y 2 and like this so on. So, suppose this is x element, this another element and continue, then d of x y which is written as mod x minus y am writing just the same using the mod, but meaning of this is in 2 dimensional case it will be x 1 minus y 1 whole square plus x 2 minus y 2 whole square in case of 3 dimensional. The distance between x y is defined as we write the same way mod x minus y, but thing is the x 1 minus y 1 whole square x 2 minus y 2 whole square x 3 minus y 3 whole square this x 1 x 2 x 3 y 1 y 2 y 3 these are the coordinates belongs to a r 3 y y 1 y 2 y 3 belongs to r 3 and it can be shown that, these satisfy the condition of the metric. We are not going in detail; it is a part of the functional analysis. There it will be discussed what we are concerned is that, since, we have discussed it the concept of the neighbourhood open sets, close sets, open ball, closed ball in real line, with the help of the modulus functions.

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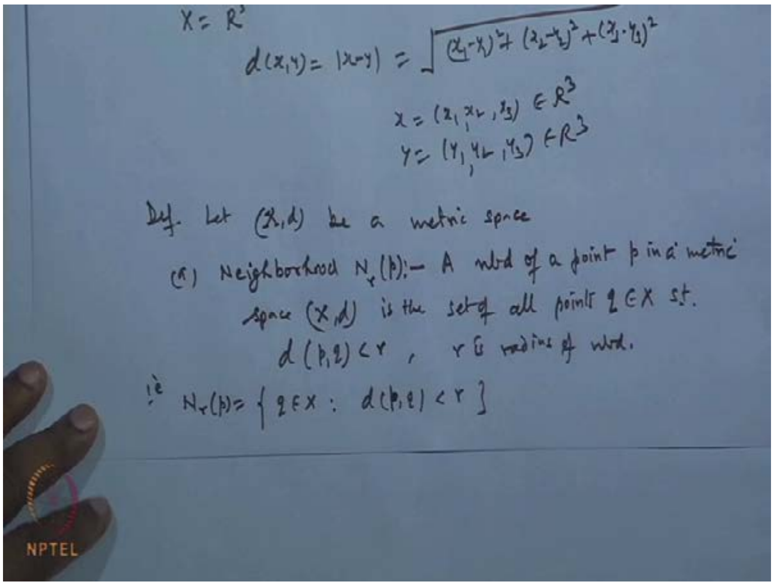

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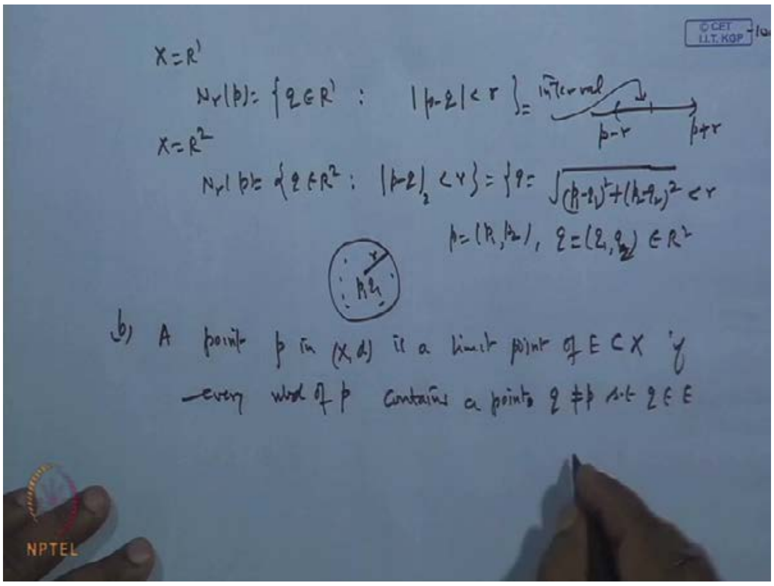

So, let us see the few concepts which are useful for further. Let x be a metric space let x d, am writing x d be a metric space and then we define like this. First definition is the neighbourhood, which we denoted by n r p, that is, neighbourhood of a point p with radius r. So, this we write as a neighbourhood, am writing n be the neighbourhood of a point p in a metric space x, d is the set of all points, q belongs to x such that the distance between p and q is strictly less than r then r is called the radius of this, then r is the radius of the neighbourhood it means our neighbourhood of a point in a metric space which we denote by n r p that is, it is the set of those point q belongs to x such that distance of p and q is strictly less than r.

So, this is enough for example, in case of real line if x is r 1 then this neighbourhood n r p is nothing but what q belongs to r 1 such that mod of p minus q is less than r that is, it is an interval lying between p minus r to p plus r and q will be a point somewhere here. So, this is the interval equivalent to the interval, when x is r 2 then n r p. Thus q belongs to r 2 such that mod of p minus q is a distance, am just writing the distance d 2 is less than this is the 2 distance 2 is less than r, that is it is the set of all q such that, under root p 2 points are there then p 1 minus q 1 whole square plus p 2 minus q 2 whole square is less than r that is where p is p 1 p 2 q is q 1 q 2 both belongs to r two. So, it is a ball centred at p 1 q 1 with a radius say r. So, all such points will be here. So, this is the neighbourhood of this, similarly, we can go for other then second definition is the limit point of the set a point p a point p in a metric space x d is a limit point of the set e which is a subset of x limit point of the set e if every neighbourhood of p if every neighbourhood of p contains a point q different from p such that q is in e. So, this is the point.

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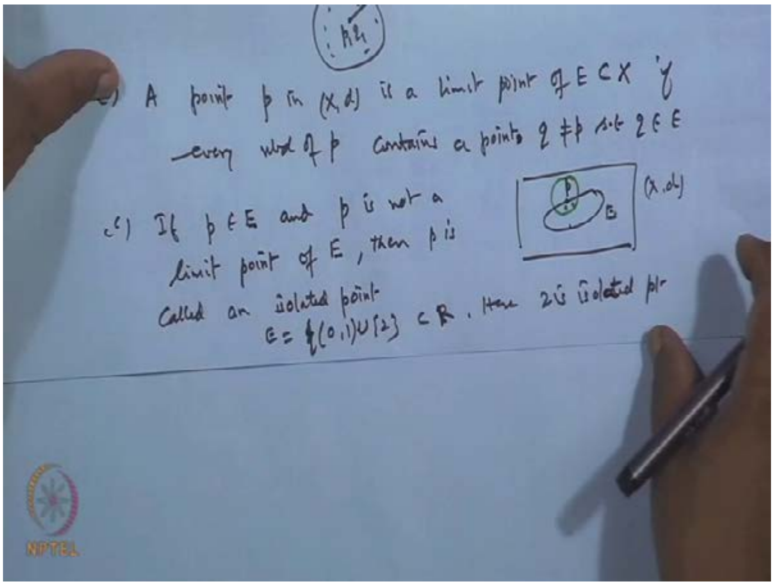

Newer limit point means, if this is a space metric x d and here is the set E, we say this p may or may not be belongs to E, but it will be a boundary point of E if it is a limit point then p is called the limit point of this, if every neighbourhood around the point if we draw neighbourhood means suppose in r 2 we get the neighbourhood in this way and like this. So, if every neighbourhood of the point p must includes the point q other than p which is in E, then we say p is the limit point of the set E, then the definition about the isolated point if p belongs to E and p is not a limit point of E, then p is called an isolated point. Suppose I consider the set E as the set or say 0 1 union 2, suppose I take this set 0 1 union 2 in r 2 which is a subset of r 2 in subsets of r. So, interval this 2 now, here 2 is an isolated point.

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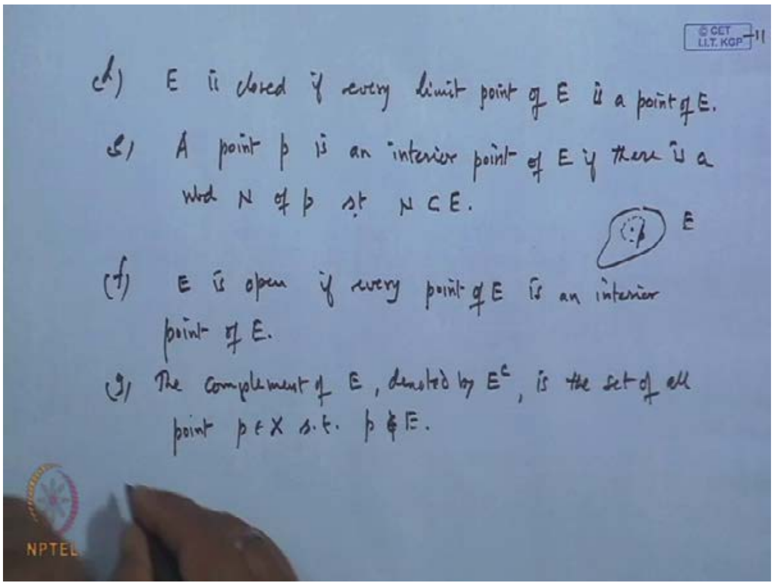

If we draw a neighbourhood around the 2, then it does not include any point of E. So, 2 cannot be a limit point of E, but 2 is a point of E. So, it is an isolated point p, is said to be closed E, if every limit point of E, then a point p is an interior point of E, if there exist or if there is a neighbourhood, n of p with a suitable radius, such that this neighbourhood is totally contained in E.

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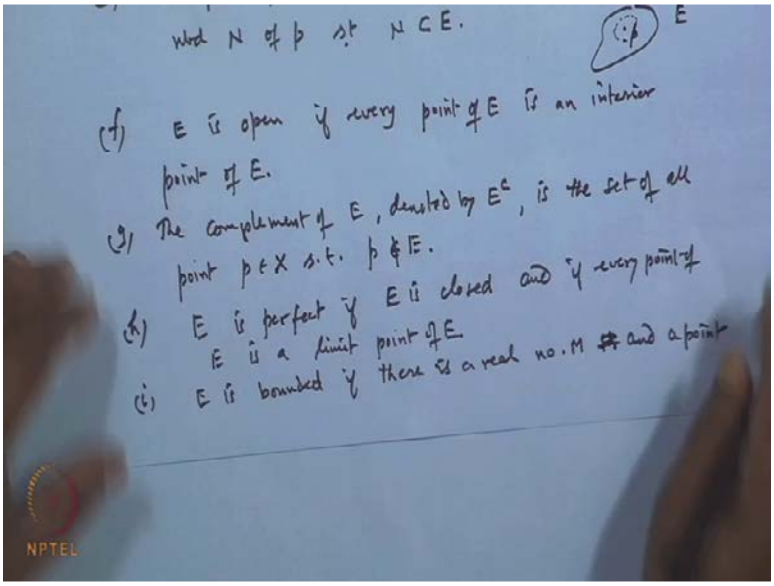

So, what he says is set E, we say this point p is an integer point if there exist a neighbourhood around the point p which is totally lies inside, then such a point p is called an integer points then e is said to be open set E in the metric space, is said to be open if every point of E, is an interior point of E, then this is called the g compliment, the compliment of E, denoted by E c is the set of all points p, belongs to x such that p is not in E and is said to be perfect in a metric space x d, if E is closed, and if every point of E is perfect, if every point of E is a limit point of E is a boundary set, we have already discussed.

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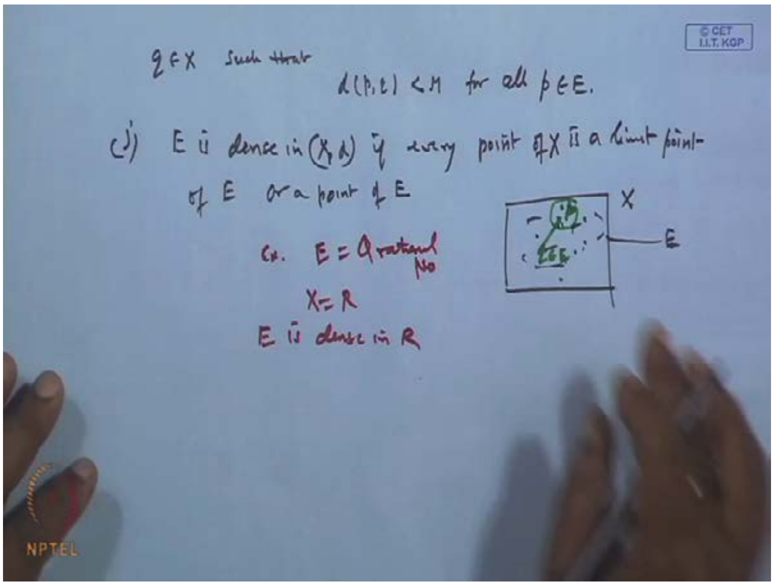

So, it is said to be bounded if there exist or there is a real number m such that, and a point p and a point q belongs to x such that such that the distance between p and q does not exceed by m for all p belongs to e. Lastly, we say J a set E is dense in a metric space x d if every point of x is a limit point of E or a point of E, it means what? Then we say is x a, set E, this is E is said to be dense in this, it means every point of x is a limit point of E or a point a.

So, if we take any point x say, here suppose I take any point x here, then this point will either be a point of E or if it not a point of E, then it must be a limit point of E. So, limit point of E, if we draw a neighbourhood around the point p, there must be some point q of E which is different from p. So, in this case the element of x and element E they are so close to each other, that you cannot separate out as soon as you take any x and draw a neighbourhood around the point x.

You will always find some point of E different from this or may be the point itself is a point E. So, such a case we say it is for example, the set of real line, for example, our E which is a set of rational number and x is r, then this E is dense, because any real number can be approximated by means of rational number. So, if we draw any real neighbourhood around the real number we get another real number is rational point which is different from this. So, this becomes the dense.