A Basic Course in Real Analysis

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Lecture - 8
Finite, Infinite, Countable and Uncountable Sets of Real Numbers

Today, we will discuss the basic topology on the set of Real Numbers.

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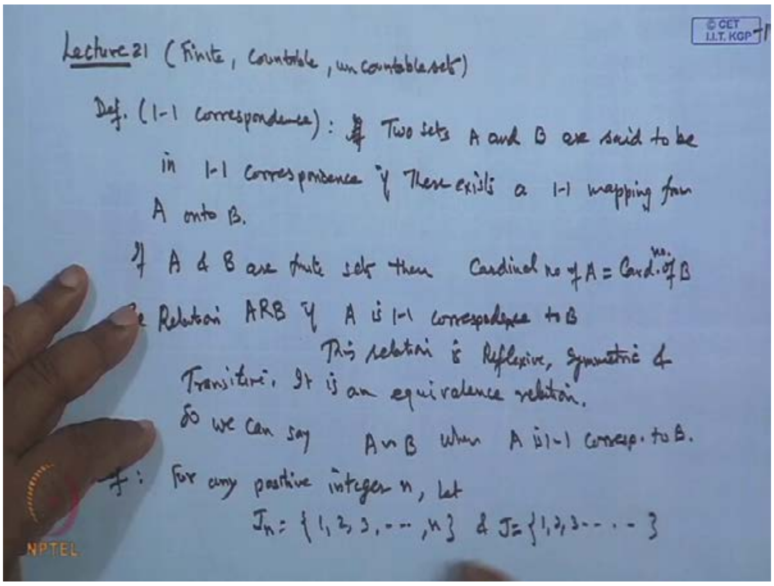

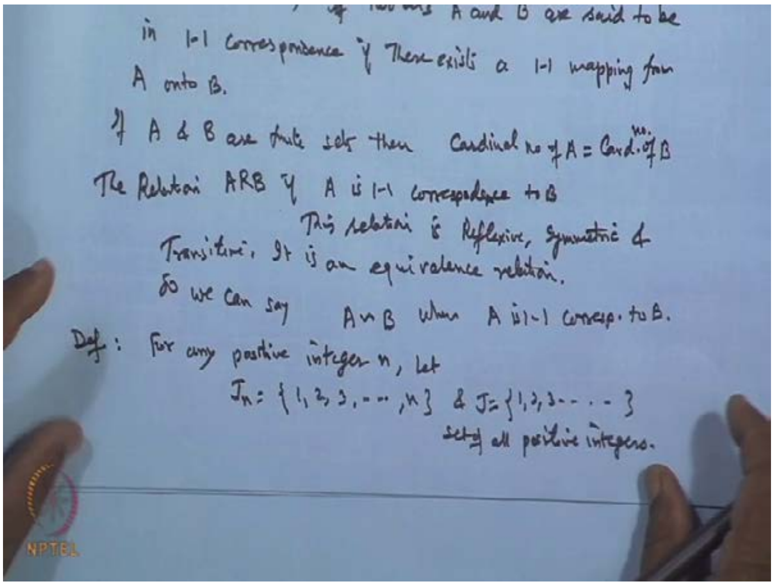

And first we will discuss what is 1 1 correspondence then, we will go for the countable and uncountable concept of the countable and uncountable sets. The 2 sets A and B are said to have 1 to 1 are said to be in 1 to 1 correspondence in 1 to 1 correspondence, if there exists if there exists a 1 1 mapping 1 one mapping from A onto B onto B.

And then, if A and B are finite then, we say the cardinality of A and cardinality B is the same, if A and B are finite sets then the cardinality or cardinal number of A is the same as the cardinal number of B number of b. But, if A is infinite then, there is no sense of talking the number of elements in the set both. So, in that case when A is and B are infinite set, then instead of saying the cardinality is the same, we said they have a 1 to 1 correspondence that is more meaning full than saying the numbers are same.

So, this is one and the relation, which we get if we put the relation, suppose A is related to B a is related to B, if A is a 1 to 1 correspondence a is 1 to 1 corresponds to B, then this relation the relation this relation is this relation is obviously, is reflexive symmetric and transitive and transitive. So, it is a reflex, it is a equivalence relation, so it is an equal, so it is an equivalence relation.

So, we also say, so, we can say we can say that A is equivalent to B a is equivalent to B when they have, when A is 1 to 1 correspondence to B, so that is the way we define. Now, using the concept of the 1 to 1 correspondence, we can now define the finite set infinite set countable and uncountable set the definitions. For any positive integer for any positive integer positive integer say n, let Jn represent the sets having the element 1 to 3 say up to n, the first n in natural number of positive integers and let J is the set of all positive integers 1 2 3 and so on, this is the set of all positive integer positive integers.

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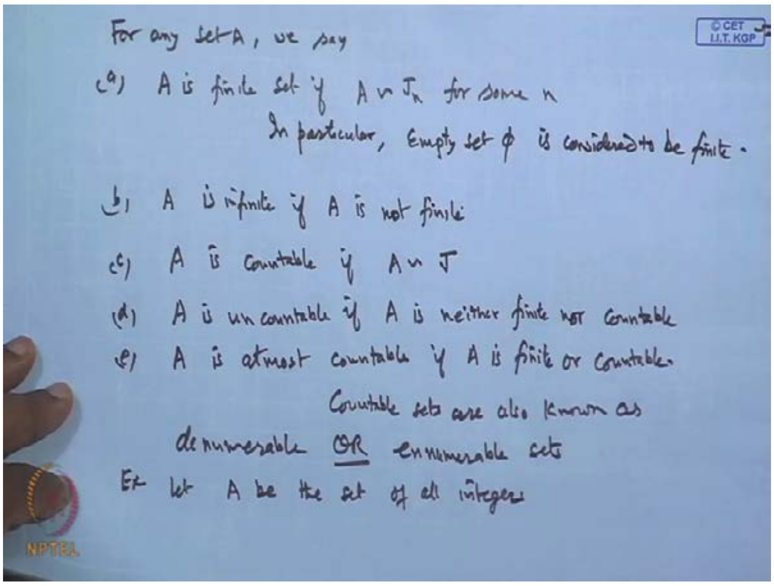

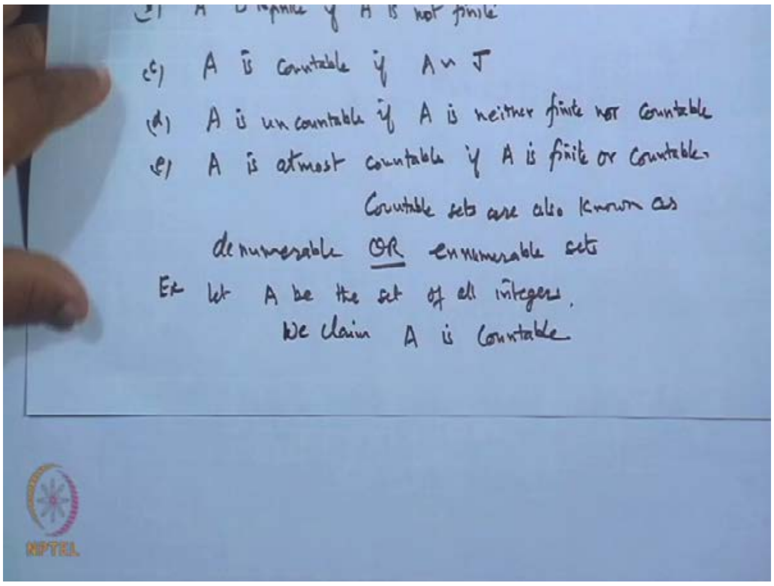

Then for any set A for any set a for any set a, we define, we say A is finite a is a finite set a is finite set, if A is equivalent to J n for some n for some n. Obviously, once it is equivalent to some n then n is fixed, so J n is finite, the number of the terms n only, so a will also be finite, 1 to 1 correspondence and set will be a finite. Empty set in particular consider to way to finite set, so in particular empty set phi is considered to be is considered to be finite finite.

Then A set B set A is said to be infinite, if A is not finite, in fact, this definition we can further modified and we get a better way of defining the infinite set s, in the next top next part when we discuss about count ability.

So, A a is finite means, if it is not infinite, A is infinite means, if it is not finite and A is countable, if if A is equivalent to J, that is there is A 1 to 1 correspondence between the elements of A and that J, all we can define A mapping from set of positive integer to A, which is 1 1 then such a set A is said to be A countable set. So, this J, we have though started from 1 to n 1 to infinity, we can also take from 0 1 to n, we starting the point x 0 0, corresponding the point x 0 x 1 corresponding to point x 1 and so on. So, we can also consider then positive a non negative integers, positive integers been including 0.

D A is uncountable, A is said to be uncountable, if A is neither, neither finite, nor countable nor countable and E is say A is at most countable, A is at most countable, if A is a is finite or countable. Finite set is also countable set,, but if the set is finite as well is and also count means finite, that is it will be considered countable countable set itself a countable.

So, we say a is almost countable means either A is finite or may be a countable set that is about countable. The countable set also known as countable sets are also known as as denum denumerable set denumerable or ennumerable sets or ennumerable sets. Let us see the some examples, we are we say, let A be the set of set of all integers a be the set of all integers, then set of all integers, we claim that this set of integer is countable, we claim A is countable is countable, it means we are able to define a 1 to 1 correspondence between the sets of positive integer and the set of the elements A.

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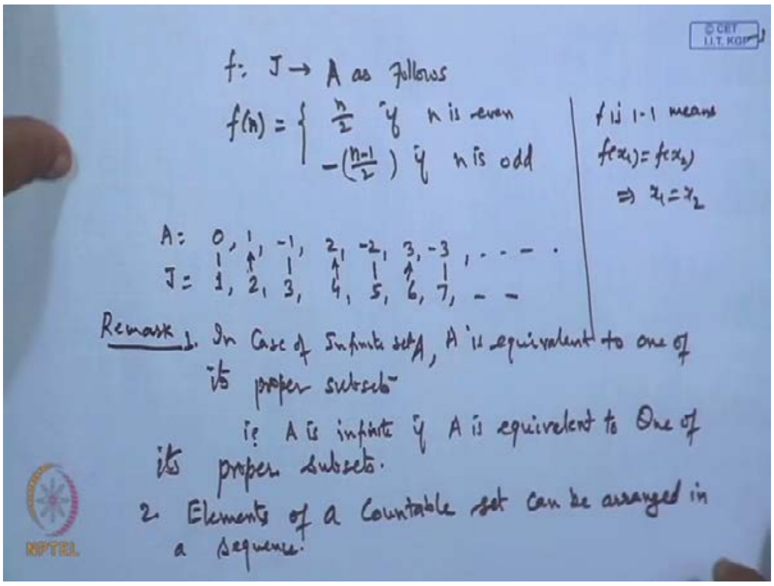

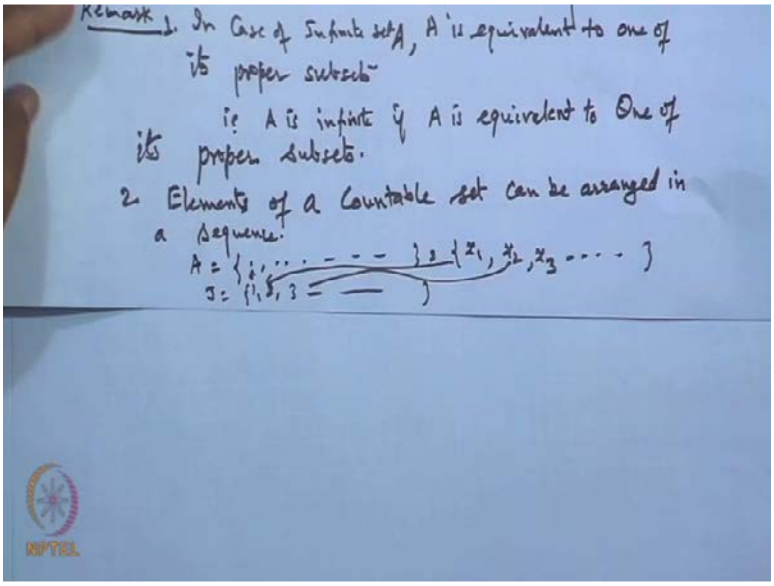

So, let us define a mapping f from the set of positive integer J to A, A as follows, if I take the image of any n under f is say n by 2, if n is even integer, even positive integer and otherwise when we say n minus 1 by 2, if n is odd positive integer. So, what we see here is, that if we take A, which is the set like 0 1 minus 1 2 minus 2 3 minus 3 and so on and J, this is the set of positive integer 1 2 3 4 5 6 7 and so on.

So, what is says is as soon as n is even, it will be n by 2 this is related to here, then 4 will go to here, 6 will go like this and when n is odd when n is odd then your getting this thing, this thing, this thing and of course,, 1 will go to 0. So, there is 1 to 1 correspondence between the elements of the set A and set J, so obviously, this n this mapping is 1 1 mapping, 1 1 mapping we can just check it, because f is 1 1 means f of x 1 equal to f of x 2 should implies x 1 equal to x 2.

So, here if we say n is even then obviously, when you can take f n 1 equal to f n 2, obviously, n 1 comes out to n 2, similarly here, when n is odd, we are getting same, so obviously, f is 1 1 here, so that is why this type set of positive in that is of positive integers is a countable set [FL]. One more thing which we can see here is a remark, what we have seen is that, this A is set of integer, but J is a set of positive integer, J is a proper subset of A.

So, but they are having 1 to 1 correspondence, so what we can say is what we conclude or we observe here is in the case of the infinite, said we can say A is infinite, then A is equivalent to 1 of it is proper subset, in case of the infinite set a proper sub set may be equivalent to the set itself. So, incase of infinite set a proper subset A may be is equivalent to in case of finite set say A a is equivalent equivalent to 1 of its proper subset.

And this is also a way to define a infinite set, we say a set is infinite, if A is equivalent to 1 of its proper subset. So, we say in that is that is A is infinite a is infinite set if infinite set, if A is equivalent to equivalent to 1 of its proper subsets, that is rule. Now, another remark we can put it here, that every elements of any countable set can be arrange in the form of sequence, elements of a countable set of a countable set elements of a countable set may be arranged can be arranged arranged in a sequence.

Because basically, what we we have a 1 to 1 correspondence with set of positive integers, so corresponding to 1, we have getting x 1, corresponding to 2, we are getting x 2, corresponding to 3, we are getting. So, this form basically sequence, because is like this, if A is any set having the elements see here, these are the elements for this set a then J is what j is 1 2 3 and so on, it has 1 to 1 correspondence. So, corresponding to 1 your getting x 1, corresponding to 2 your getting x 2, corresponding to 3 your getting x 3 and so on, so this has 1 to 1 correspondence with this is a not like this.

So, we get the 1 to 1 correspondence between the set of positive integer and the elements of the set, so element of a countable set can be arranged in the form of sequence. So, this is also remark, we will use it.

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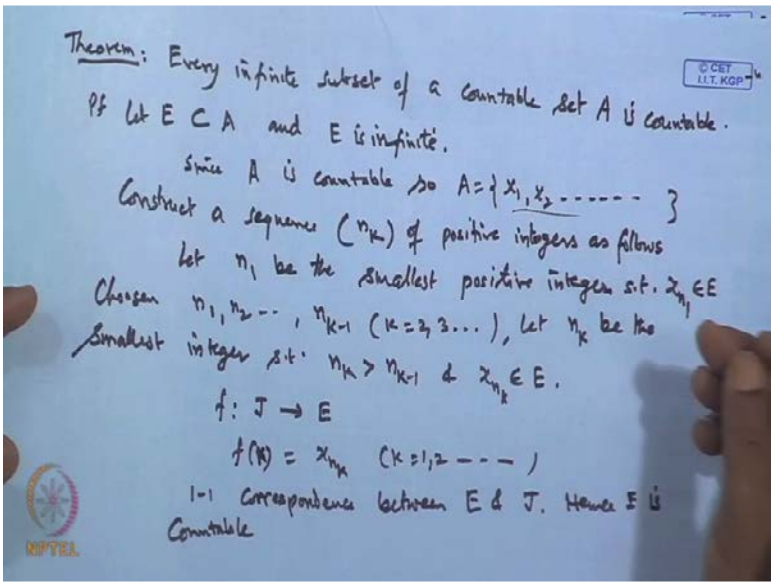

Now, this is interesting result and result says, every infinite subset subset of a countable set of a countable set A is countable, so proof is let E be an infinites, let E be a subset of A and E is infinite is infinite. Now, since A is since a is countable, so we can arrange the element a in the form of sequence, so A will have the sequence like x 1 x 2 and so on all the elements of set can be arranged in the form of the sequence.

Now, let us construct a sequence construct a sequence n k of positive integers as for as follows, suppose n 1 be the smallest positive integer be the smallest positive integer, such that x n k x n 1 is an element of E is from this 1 2 3 and so on, suppose I am taking n 1, n 1 is the smallest integer. So, that first x n 1 corresponding to this x 1 is an means out of this the first element, which your getting is x n 1 belongs to E, then assume that n 1 chosen n 1 n 2 say n k minus 1, where k is 2 3 4 and so on. These are the after choosing in such a way, when n 2 is greater than n 1 and such that x n 2 belongs to E and so on.

Let us take n k now be the smallest integer be the smallest integer with the smallest integer, such that be the smallest integer such that n k is greater than n k minus 1 and the corresponding terms of the sequence x n k belongs to E. So, now, let us introduce the function f from J to E, so if we take f of n as x of n k f of k let us take f of k as x of n k, where k is 1 2 3 and so on, then what we see here, there is 1 to 1 correspondence between J and E, because for k is 1 x 1 x n 1 is E k is 2 x n 2 in E x n k in E and like this.

So, this is a 1 to 1 correspondence between between E and J, hence E is countable hence e is countable, so this shows that every infinite subset of countable set is countable clear is it, so this one. Now, here, as we have seen that if A is countable, we can put in the form of a sequence x 1 x 2 and all the elements, we can arrange in the form of sequence x 1 x 2 x n, if I generalized it, say why because 1 to n is basically set of positive integer. So, instead of this we can take the collection family of set also.

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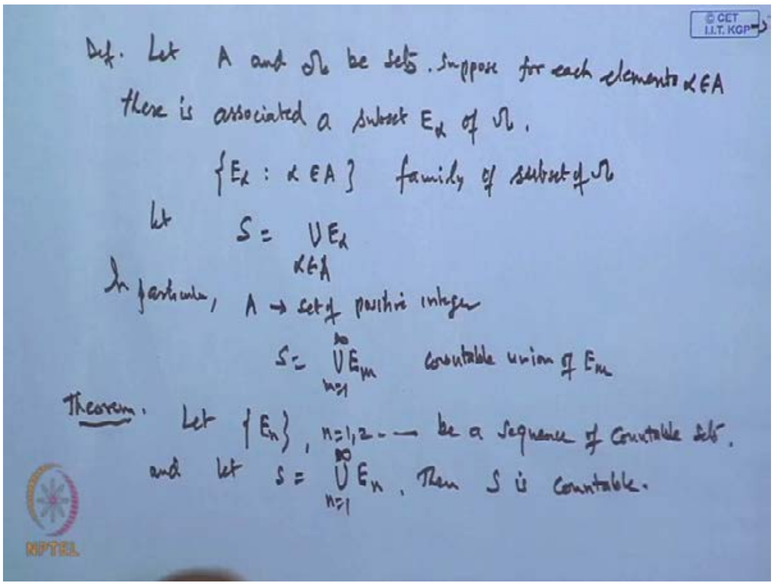

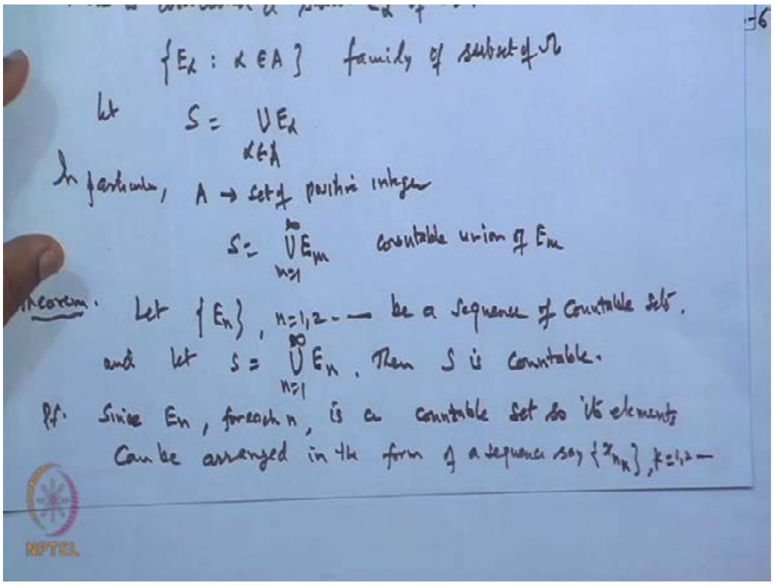

So, we define like this let A and omega be sets and suppose with each element alpha of A, the associated set of omega, which is known as suppose for each for each element element alpha belongs to A alpha A, there is there is associated a subset E alpha of omega subset say n, subset denoted E alpha of omega. Then the collection of these E alpha, then this collection E alpha here, alpha belongs to A this collection alpha is the collection of the sets is the family of the sets family of sets or subset of q subsets of omega, family of subsets of omega.

Now, if we take s, s the union of E alpha when alpha belongs to A, then this for at least 1 we used, then any element belongs to this means it will be in 1 of the alpha E alpha like this for 1 A, at least we used it. And in particular in particular, when A is an positive integer, A is the set of positive integer, then s becomes union E m, m is 1 to infinity and this we say it is countable union of E ms like this. Similarly, for the inter section also, we can enter this will be needed, we can justify.

Now, let n of the results in the sequence of the countable sets, let us suppose E n e n and n is 1 to no and so on, be a sequence of sequence of countable sets countable sets and put and let s is the countable union of ens s is a countable union of ens n is 1 to infinity, then what we claim is then s is countable, then s is countable.

So, countable union of a countable set is countable, that is what this result says, now what is En, Ens are giving to be countable, so, it can be arranged in the form of sequence. So, proof is since En for each and since E n for each n is a countable set countable set. So, its elements can be arranged in a form of the sequence in the form of sequence in the form of the sequence say x n k where, k is 1 2 3 and so on, k is 1 2 3.

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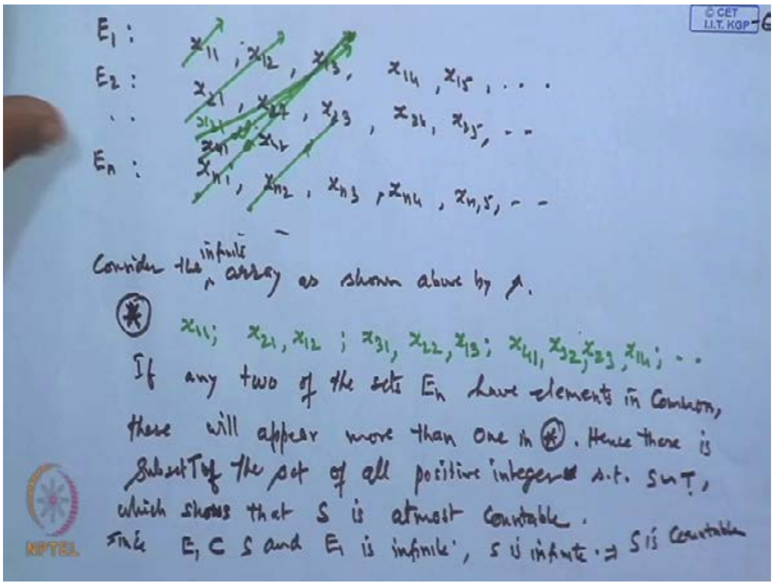

It means, that is E 1 set E 1, we can arrange the form for the set E 1, we can arrange the elements like this x 1 1 x 1 2 x 1 3 x 1 4 x 1 5 x 1 6 and so on. For E 2 the elements suppose, we are arranging x 2 1 x 2 2 x 2 3 x 2 4 x 2 5 and so on like this and Ens suppose, we arranging x n 1 x n 2 x n 3 x n 4 x n 5 and so on like this, so continue this.

Now, let us consider the foring array consider consider the array array as mention above infinite array as shown above by arrow, what is this, suppose I take this arrow first, let us take this another pen then, suppose I take this as first element this way, then I take choose this 1 then I take this arrow, then I take this one, x n 1 x n 2 x 3 1 oh sorry, this is like this x 3 1 x 3 2. So, basically this will be the x 4 1, this is x 4 is wrong, so this will be x 4 1 x 4 2. So, this way will go this way will go not this like this. So, if we continue this way that is, what we are doing is, we are taking up this way like this.

The first element, we are choosing x 1 1, the first array then second 1, I am taking as x 2 1 x 2 2 x 1 2 then, third array, we are taking as x 3 1 x 2 2 x 2 2 x 1 3 x 1 3 and 4th 1 is say x 4 1 x 3 2 and then x 2 3 and then x 1 4 like this continue this. So, if we arrange this in the form of the sequence, then what we get is, we are getting first element second element third element and so on like this. So, in this way, we are getting 1 to 1 correspondence between the elements of the set and set of positive integer.

It may so, happen the some of the elements of this sequence may be repeated, then what we can do is, we can get the subset of this set, since it is infinite subset. So, subset of this we can find out a integers, a subset of T integer J, which is also countable, so with that it will be a 1 to 1 correspondence, we drop this common element and make the correspondence with the set of positive integer.

So, if if any 2 of them, if any 2 of the any 2 of the sets, E n have elements in common common, then these will appear these will appear more than once, in this arrangement in this arrangement say is star in this arrangement is star.

Then what we share, hence they are exist, hence they are is a subset of subset T of the set of all positive integers T of the set of set of all positive integers all positive integers, such that such that s and T, which s is equivalent to T, which shows that, s is at most countable, that s is at most countable. Let us see what is the meaning of this, once we have arranged this union of this sn even E to n, this each E n, we are arranging in the form of sequence.

So, now, take the arrow like this, so choose the element first element 1 1, then second element x 2 1, third element x x 1 2, fourth element may be the x 3 1, then x here, x 3 1, this will be the x 3 1, then this array, here the sum is 4, here the sum is 4. So, this one and continue, so this has a 1 to 1 correspondence with a set of positive integer, in case if suppose some elements are repeated, then what we do, be just drop that element take only once.

So, that will be have a 1 to 1 correspondence with the set subset of J, that is there will exist a set T of a set of positive integer, which has 1 to 1 correspondence with this set of elements s, so s becomes countable. Now, s may be finite or may be infinite, so but, s cannot be finite, so we can take to be the let us by because since E 1 since e 1 e 1, which is contain in s, because s is the countable union of these Ens.

So, E 1 is contain in s an E 1 is giving to be infinite and E 1 is infinite, because this is already given that sequence of countable sets, Ens is the sequence of the countable sets, which are infinite of each element, then E 1 is infinite then s will be infinite. So, this shows s is countable this implies s is countable s is countable and that is proved the result, so this is one.

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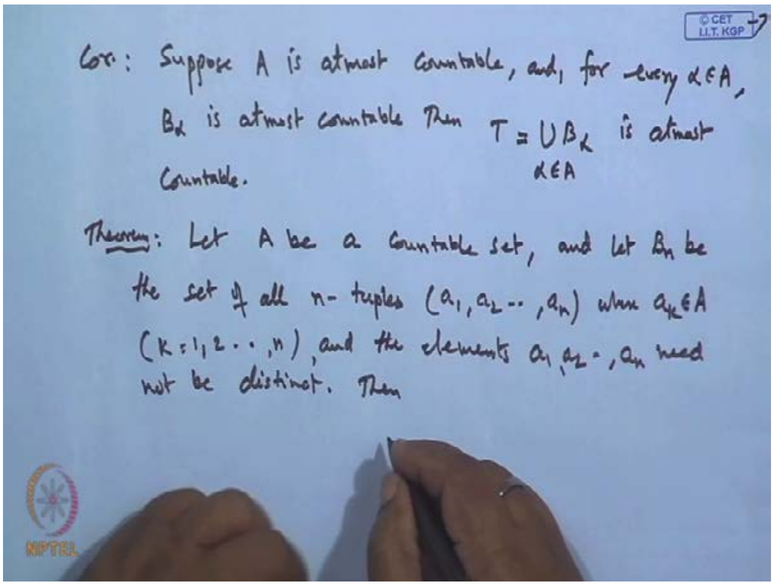

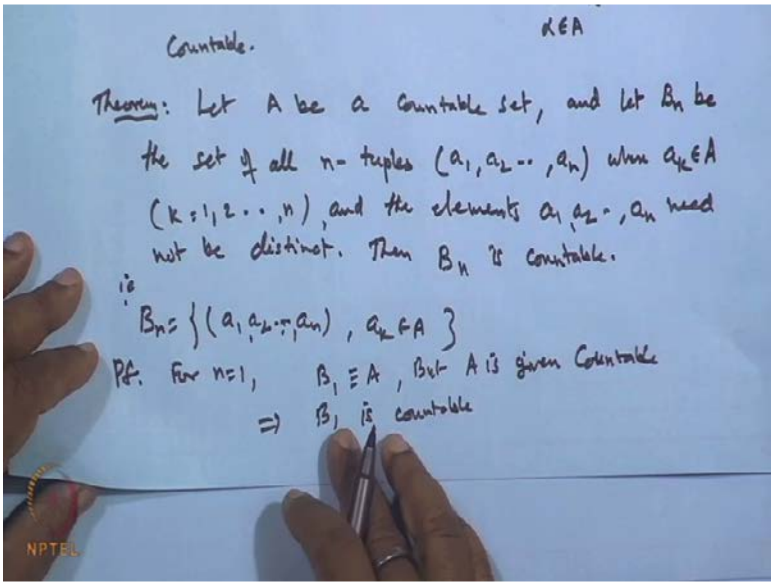

Same result, we can generalize it, so s is coronary, we can say suppose A is at most countable at most countable and let for every alpha belongs to A, B alpha B alpha is at most countable with a subset omega of course, at most countable. Then then T, which is the count union of B alpha, when alpha belongs to A is count is at most countable at most countable means, either it may be finite or infinite any countable all infinitely countable.

So, that is with another results, which we in this you since, we have let a be the countable set be a countable set and let B and let B alpha B n be the set of all n tuples set of all n tuples of the say a 1 a 2 a n, we are Aks, this element, they are the elements of A k is 1 2 say up to n. And the elements a 1 a 2 will not be, in the elements elements a 1 a 2 a n need not be distinct distinct, then B n is countable is countable.

So, what this theorem says is that, if we construct a set B or B n, that is B n is the set of all n tuples a 1 a 2 a n where, a k is all in A, if this quadrants of this n tupeles belongs to a countable set then this collection of all n tuples will also be a countable set, that is what he said.

So, in particular, when you take n equal to 2, the order set of order appear where a 1 a 2 belongs to a set, which is countable then this set of for n is 2, becomes countable an this will be give leads the proof for the rational number to be countable. So, let us see the proof of this first this be proved by induction, sup what is our B 1, for n is equal to 1, B 1 is basically is only single element E 1, it means B 1 coincide with A, but A is giving countable A is giving countable. So, once it is countable, so this implies the B 1 countable B 1 is countable, so let us assume that up to B minus 1 is countable.

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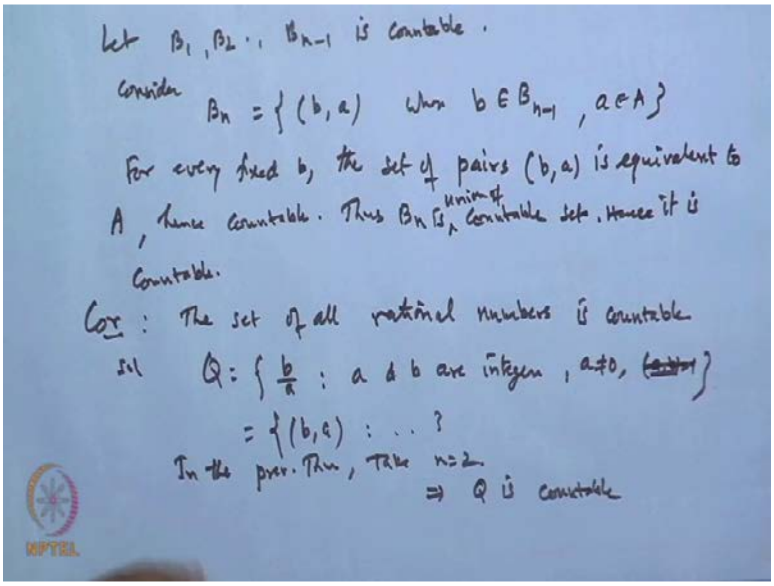

So, let us assume let let B 1 B 2 B n minus 1 is countable set, now we will proof for n B n, so consider the B n, B n we can put it B n in the form of order pear b a where, what b where b belongs to B n minus 1 a tuples here, B is B 1 B 2 B n and n minus 1 and then comma any terms is a, so B 1 in a belongs to A clear. Now, if I fix b then once you fix up b, it means each element of a, we are combining with b that is all.

So, basically your getting a itself is a not, so that is nothing but, so not for every fix b, the set pairs the set of pairs b a is equivalent to equivalent to equivalent to set of B a equivalent to A, but A is given to countable, hence hence countable.

So, thus B n is countable, thus B n is countable countable union of B n is the is union of union of a countable set union of countable set of countable set is a union of countable set, because a is countable and this your fixing B and minis 1, which is also countable. So, basically B n is the union of countable sets B n minus 1 in A, so it is con hence it is hence it is countable by the previous result therefore, B n is countable, so this proved the theorem.

Now, as a particular case I told then when you fix up the n is 2 then, we get the very important result the all set of rational numbers is countable, the set of set of all rational numbers rational numbers is countable why, what is our rational number. The set of rational number q is basically of the form say b by a B by a where a and b are integer integer a is not equal to 0 and the deviser of this one is the do not have a common factors in it this is one. Even if it is not 1, even we do not put it say 2 by 4 1 by 2, we can take also that one, so this is all general in this form.

Now, what is this, this is basically an ordered pair say b a where, the order is b by a, so a is an integer, b is also an integer and both are countable set, so basically it is union of the countable sets. So, the in the previous theorem in the previous theorem, if we take n is equal to 2 then this shows that q is countable, because b is in I which is countable a is in I which is also countable therefore, it is countable, so set of all.

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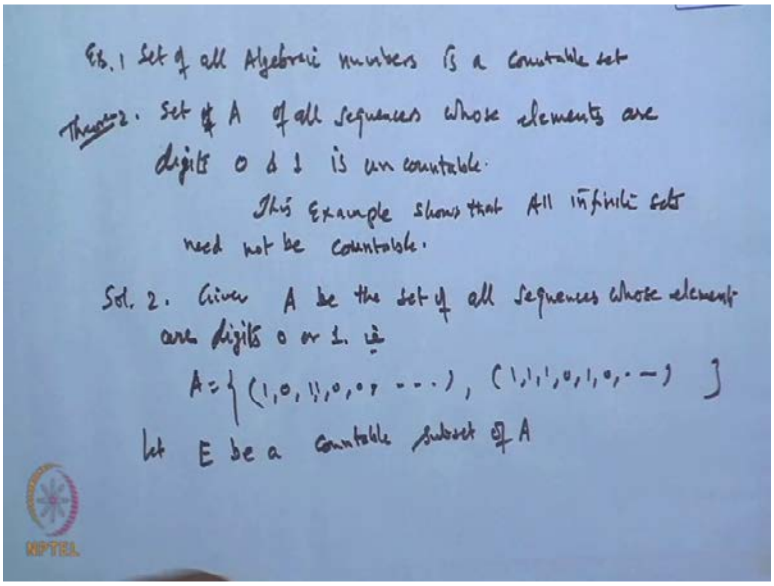

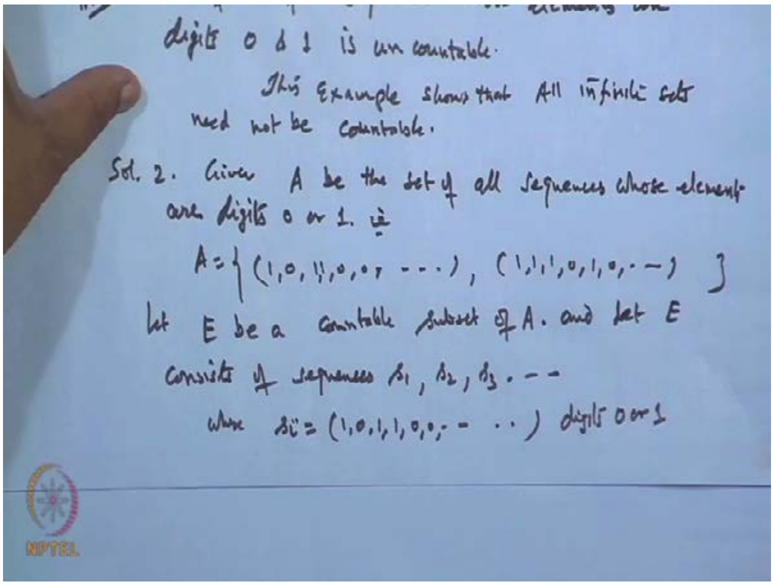

Now, this we will just given example,, but we show it later exercise, which is we will prove it, now set of all algebraic number algebraic numbers set of all algebraic numbers is a countable set. Second we can show and we will prove that, I am giving example, but otherwise next, we will show it also when it is set of all, that is all this infinites, real numbers set of real number, set A set a of all sequences, set a of all sequences all sequences whose elements whose elements or all digits 0 and 1 0 and 1 all this 0 1 is not uncountable set.

So, In fact, this result shows the second example shows, so this this example shows that all these infinite set need not be countable, that all infinite set sets need not be countable. In fact, we will show that set of real number is not countable set of rational are countable set of irrational becomes uncountable set. So, let us see the results the theorem, which is fall in the theorem proof of solution for 2 for this you can also say in the form of theorem as A.

Let us suppose A E be the countable subset of these let E, let given A be the set of set of all the sequences all sequences whose elements are elements are say whose elements are like this all this infinite sequences. The sequences set of all sequences whose elements are digits 0 or 1 that is a will be a set of this type of sequence say 1 0 1 comma 1 0 comma 0 like this or may be 1 1 1 1 0 like this means all these sequences.

In fact, the digits are 0 or 1, this claim that this set will be uncountable set. So, let let us suppose E be a proof let let E be a countable subset of A, let us suppose let e be a countable countable subset of a let us take this, it means every E can be arranged in the form of sequence. So, let E consists of the sequences E consist of sequences say s 1 s 2 s n s 3 and so on, because this is countables, we can arrange the element in the form of this now is s 1 s 2 and s n where, s I is this may be few 1 0 1 1 0 0 0 and so on, means an digits are are either 0 or 1, digits 0 or 1.

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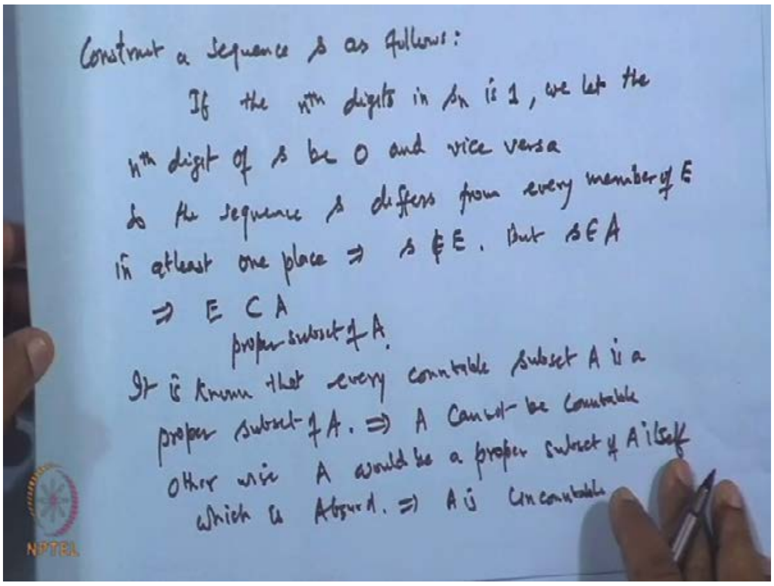

Now, from this let us constructed new sequences, so construct a sequence construct a sequence as as follows, what is the way we are constructed in sequence is that, if the nth digit in s n, if the nth digit digit in the nth digit in s n is 1, then we let the nth digit nth digit of s be 0 and the vice versa means. What how we are choosing is suppose s 1 s 2 s n is giving sequence, so I am taking the constructed sequence s, such that if the nth digit is s n is 1 then, we replace 1 by 0.

It means the first first term first term of this term, we will look that s 1, if the first term in s 1 is 1, we will get a 0, if first term in the s n s 1 is 0, we will take it 1. Similarly, for a second term of s 2, we look, the s the second term of s, we look the sequence s 2 and see, what is that digit, whatever the digit, there we will take that just opposite of means, if it is 1, we will take 0, if it is 0, it is 1 it means the sequence s, so tend will differ from all the terms of the this set e.

So, obviously, so what we see here is, that the sequence s sequence s differs from every member of E, every member of e in at least at least 1 place, because, we are constructing in such a way const, so once it is differing from each element and E is already countable, we are arranged the system in formal sequence s 1 s 2 s n. And since s is not co siding with in any one of the term, so this implies that, s is not an element of E, it means what, but s is what, but s is is in the element of what a because, a is collection of all the sub sequences with digits either 0 or 1.

So, E is a proper subset of this this implies, E is a proper subset of a subset of a is a proper subset an once, we have already shown and it is known or it is shown that it is shown that every countable subset every countable subset of A is a proper subset of a is a proper subset of a every countable set of a is a proper set of a, this we have shown already every countable subset of A is A proper subset of A. So, A cannot be countable, because s 1 is a countable, then this subset, which we are getting must be proper.

So, this shows this implies that, A is a cannot be countable, because if a is countable then here itself, we are getting is it not because otherwise a would be proper subset of itself otherwise a would be a proper subset of a itself, would be a proper subset of A itself which is absolute which is absolute. Because, a set cannot be proper a set will be proper subset of itself, it cannot be a proper subset and here if you assume A to B, a countable then it must be have a proper subset.

So, this is every proper subset, but here we are getting that this cannot be, so this shows that A uncountable, this shows a is uncountable a is uncountable. So, this a thing now one more example, which we will deal is set of number countable and other.

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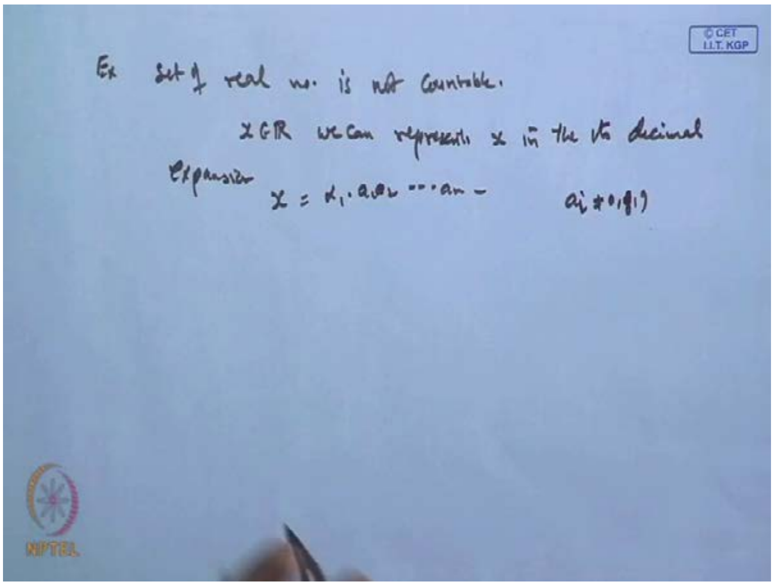

Lets take the few more that is this example as we have seen the interval 0 1 set of real numbers is not countable. Now, this we will prove it by using the decimal expansion suppose, I take x any element of R, we can arrange this in the form of the sequence say x is in R, then we can write it, we can represent x represent x in the form of in its decimal expansion decimal, we can write the decimal expansion for this.

So, x is something alpha 1 point say a 1 a 2 a n and so on then this a 1 a to n and these are numbers different from 0 8 and 9. So, in order to repeat the whole 9 and then, what we do is we can construct the another numbers say first place a 1, we can replace it by number, which is not available here.

Just changing just like a previous think, we did it s B n construct a sequence, where we are replace the first element of the sequences by looking the s 1, if it is 1, we will take 0, if it is 0, it will be 1. So, similarly here also be looked at and then replace it by number, which is not out of this. So, that at least each element will differ from the constructed 1, so we will say, I will complete this thing next.

Thank you very much.