A Basic Course in Real Analysis
Prof. P. D. Srivastava
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 7
Equivalence of Dedekind & Cantor's theory

So today, finally we will see that Dedekind and Cantor's theory basically are equivalent concepts; both will give the same thing. Before going for this equivalence of Dedekind and Cantor's theory, we will see few more concepts, the things which can be represented with the help of the Cantor's theory like between any two real numbers there are infinite number of real numbers and so on and so forth.

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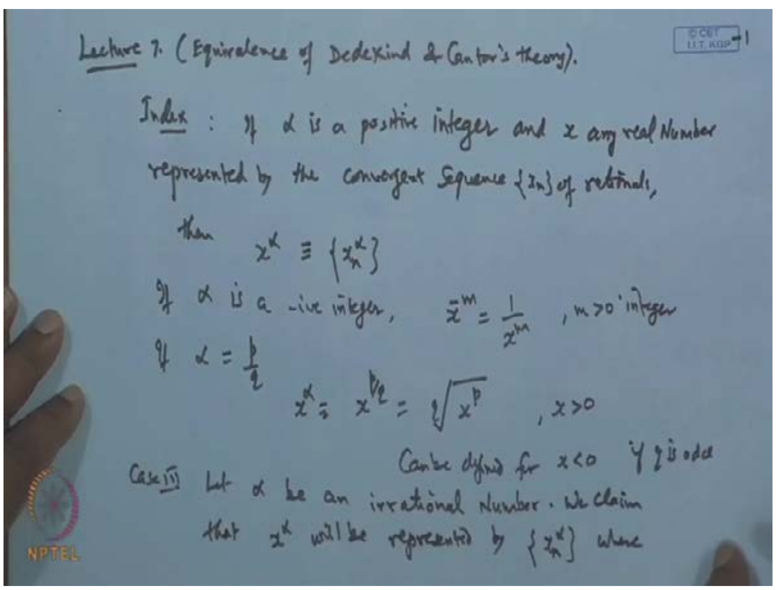

So, last time we were discussing about the index, is it not? And then in this case we have already seen that if n be a positive integer, if alpha is a positive integer and x is any real number represented by the convergent sequence x n, of course of rational convergence sequence rational then x to the power alpha will be represented by the sequence x n to the power alpha which also is a convergent form. This we will see in case of positive integer; if it is negative then we can just divide 1 by x n alpha if alpha is negative integer, then we can write then we can write x to the power minus, say m, this we can write as 1 by x to the power m where m is positive.

So, positive integer and also we have seen if alpha is fractional p by q where p and q are relational fraction, and in that case the x to the power alpha that is p by q this is the same as q th root of x to the power p where x is taking to be positive, or it is defined when q is odd and if q is odd then even this can be defined for x is less than 0 can be defined even for if q is odd number, okay. And the third case we are discussing was if alpha is a irrational number; let alpha be an irrational number we claim that x to the power alpha will be represented by a sequence x n to the power alpha.

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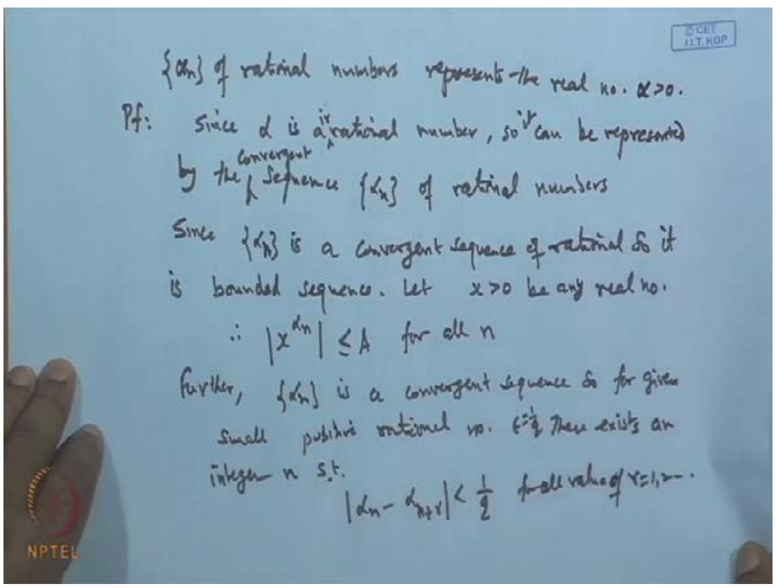

Where the sequence x n of rational numbers represents the real number x greater than 0; in fact this is given when x is greater. So, I will just revise it, but we have already discussed this part when alpha is positive integer and x be any real number which is represented by a convergent sequence x n of the rational. Then index we are talking about the index, x to the power alpha will be represented by x n to the power alpha, and if alpha is negative then we can write, say, alpha is equal to minus m then we can write it 1 by x m where alpha is equal to minus m. Then in that case x to the power minus m is 1 upon x to the power m, just m is integer. And if alpha is a fractional then rational number fractional then x to the power alpha is nothing but x to power p by q where this is nothing but x to the power p and q th route of this when x is positive.

But if x is negative it can be defined provided that q is an odd number. So, this for integer and for fractional we have already discussed. Now let us come to the irrational number. So, if alpha be an irrational number, and let us suppose x be a sequence x be any real number represented by a sequence x n of rational number x is positive real number, then we claim that x to the power alpha will also be represented by x n to the power alpha; that we have covered till then. So, let us see the proof of this, okay; so proof let us see. Now since sequence alpha is a irrational number so it can be represented by the sequence alpha n of rational numbers by Cantor's theory, is it not; any sequence convergent sequence of rational number represents either rational number or irrational number, anything.

So, if alpha is irrational number we can identify a sequence alpha n which is convergent sequence of alpha n which represents the alpha. So, it can be represented by sequence alpha n of rational numbers convergent sequence alpha n of rational number; that is important point, okay. Now alpha n is a convergent sequence. So, it is bounded; alpha n convergent is also bounded. Since alpha n is a convergent sequence of rational, so it is boundary sequence. It has been less than some positive number for all n; therefore, x is fixed. So, we get x to the power alpha n; let x greater than 0 be any real number, okay, x n be any sequence of real number, and so x to the power alpha n mod of this is less than equal to a, this just I wanted to, okay. Alpha be a irrational; we claim x to the power will be represented by, this you please make the changes, okay, by x to the power alpha n because we are choosing alpha n, is it not.

So, the sequence alpha n of rational number representing the number alpha, okay, just change please. What we are doing is x is any positive number, alpha is irrational. So, corresponding to alpha we are getting a sequence alpha n, x is fixed. So, x to the power alpha we claim that is represented by a sequence x to the power alpha n because x is not going to change, okay. Now here alpha n is a sequence of rational number, so it is bounded. Therefore x to the power alpha n will remain less than equal to A for all n; this is true for all n, okay, and A is any fixed number. Now sequence alpha n is a convergent sequence. So, for a given small positive rational number epsilon there exist some integer there exist an integer such that mod of a alpha n minus alpha n plus r; it means less than 1 by q for all n for all values of r 1 2 3. Here I am taking epsilon to be 1 by q, okay. So, for a given epsilon, say, 1 by q I am taking such that this is less than epsilon, is it okay. Now q is any positive number, okay.

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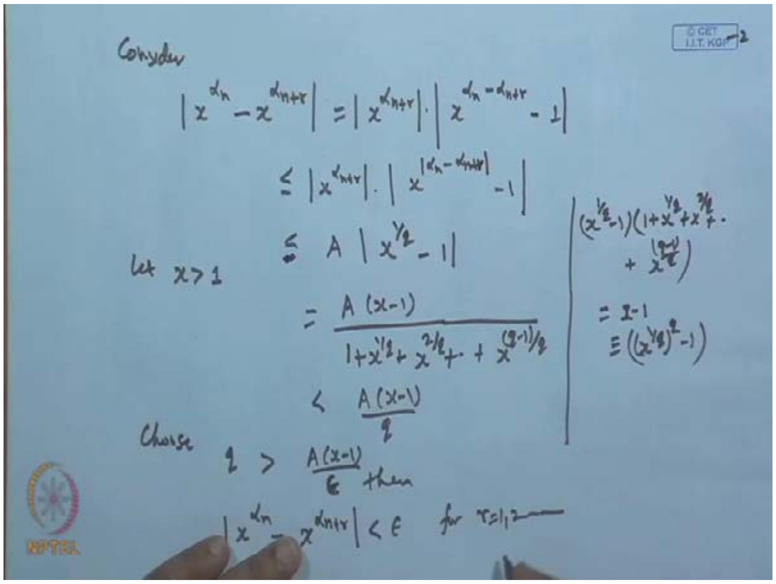

Now consider mod of x alpha n minus x alpha n plus r, consider this. So, take x to the power alpha n plus r outside. So, what inside we get x to the power alpha n minus alpha n plus r minus 1, is it not, okay. Now this may be positive may be negative, so it is further less than equal to mod of x alpha n plus r into mod x mod alpha n minus alpha n plus r minus 1, okay. Now this x to the power alpha n as we have seen from here this is this one x to the power alpha n is bounded by a for all n. So, we can take this thing less than equal to a into x and mod alpha n minus alpha n plus r is less than q, because this is convergent. So, it is less than 1 by q, so it is 1 by q minus 1, is it ok, and this is strictly so we can say this is strictly less than this, okay.

Now this can be written as I will write like this as x to the power 1 by q into this. If I put it this way A x minus 1 divide by, okay. So, let x is greater than 1, then we can write this modulus sign x to the power 1 by q is greater than 1; so modulus sign is removed. Now I am multiplying the denominator and numerator by this number x to the power 1 by q 1 plus x to the power 1 by q plus x to the power 2 by q up to x to the power q minus by q. So, when you multiply this we are getting basically this one, x to the power 1 by q minus 1. When you multiply this by 1 plus x to the power 1 by q plus x to the power 2 by q and so on plus x to the power q minus 1 by q, then this is the expression of x minus 1.

X is greater than 1, okay, because it can be written as x because this is equivalent to x to the power 1 by q q minus 1. So, x to the power n minus 1 you just apply the binary expansion you will get this thing. So, what I did is I am multiply the numerator and denominator by this term, x is chosen to be greater than 1. So, this sign modulus is removed, then no problem, and we get x minus 1 over this. Now since x is greater than 1, so each of this term in the denominator is greater than 1; x to the power 1 by q is greater than 1, x to the power 2 by q is also is greater than 1. So, total value is q. So, basically this is less than A x minus 1 by q, is it okay. Now if I choose epsilon; we want this whole thing to be less than epsilon, is it not.

To show this as a convergent sequence we wanted this to be less than epsilon. So, if I choose q greater than a x minus 1 by epsilon, then what happens? Then this entire thing mod of x alpha n minus x alpha n plus r this remains less than epsilon for all r 1 2 3 and so on, because you just see when you put it less than epsilon the q is greater than this number, okay. So, this one we wanted this to be there; so what they show? This shows that if x to the power alpha n is a sequence of real numbers this sequence, then it satisfy the condition of the convergence, because the difference between any two of the arbitrary term of the sequence remains less than epsilon. So, x to the power alpha n minus x to the power alpha n plus r is less than this.

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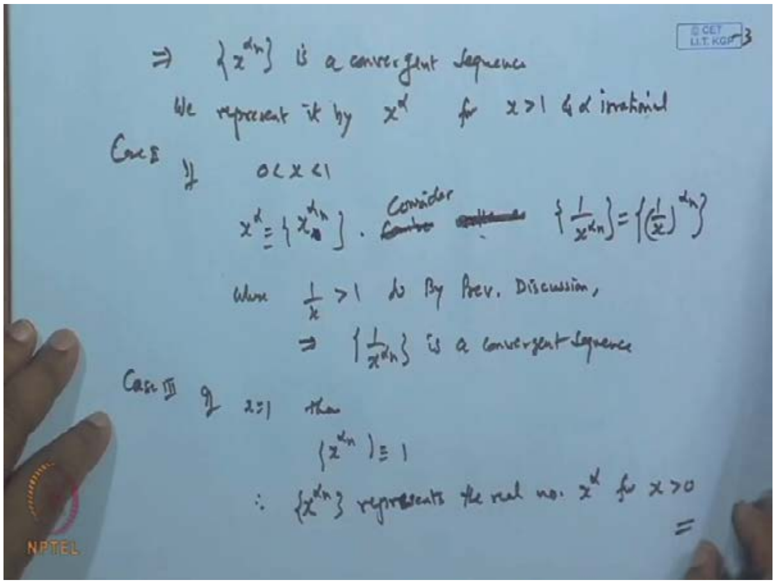

Now this shows this implies the sequence x to the power alpha n is a convergent sequence, okay. We represent this thing we represent it by x to the power alpha for x greater than 1 and alpha is an irrational point, is it okay. Now if alpha if x is lying between 0 and 1 then x n to the power alpha this can be written as or x to the power alpha n which is equivalent to x to the power, is it not? This can be written as 1 by x to the power alpha n if x is lying between 0 and 1. Then 1 by x greater than 1, is it not? So, if we consider this sequence then it can be written as this, no, no, x to the power alpha is this for x lying between. Now consider 1 by x to the power alpha n, okay; this is nothing but sequence 1 by x to the power alpha n where 1 by x is greater than 1.

So, as by the previous one; so by previous discussion this implies that sequence 1 by x to the power alpha n is a convergent sequence. Now if x is 1 then x to the power alpha n this sequence is equivalent to 1 only, okay. Therefore x to the power alpha n this sequence represents the number real number x to the power alpha for all x greater than 0, and this will be an irrational number, okay. So, that is all, is it okay; so this is the answer. So, index can also be justified with the help of Cauchy sequence.

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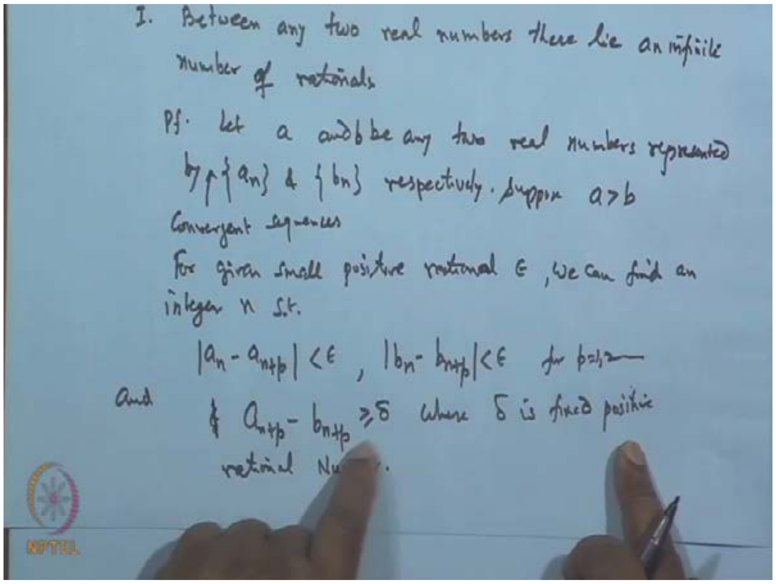

Now there are certain properties; in fact these properties are parallel to our property which we have proved in case of Dedekind's. The property says is between any two real numbers there lie an infinite number of rational. Okay, proof is let us support two real numbers; let a and b be any two real numbers represented by the sequence, say, a n and b n represented by convergent sequences a n and b n respectively, okay. And suppose a is greater than b, okay, so now what we have? We have a sequence a n, this is represented by convergent sequence a n; this means sequence a n is convergent, sequence b n is also convergent represent b, and a is greater than b. So now apply the criteria.

So, for a given epsilon greater than 0 for a given small positive rational number epsilon, we can find an integer n such that mod of a n minus a n plus b is less than epsilon mod of b n minus b n plus p is less than epsilon and for p is equal to 1 2 3 and mod of a n plus p because a is greater than 1. So, a n plus p minus b n plus p; this difference is greater than equal to some positive number delta where delta is a fixed positive rational number, okay. Now this is because a ns are convergent, this is because b n is convergent, and this because a is greater than b. So, a is greater than b means for any arbitrary terms if you picked from a and b the difference must be positive. So, there exist some delta some fixed positive integers such that all the terms of this difference will remain greater than equal to delta, because epsilon is very arbitrary small number. So, I can choose the epsilon less than delta.

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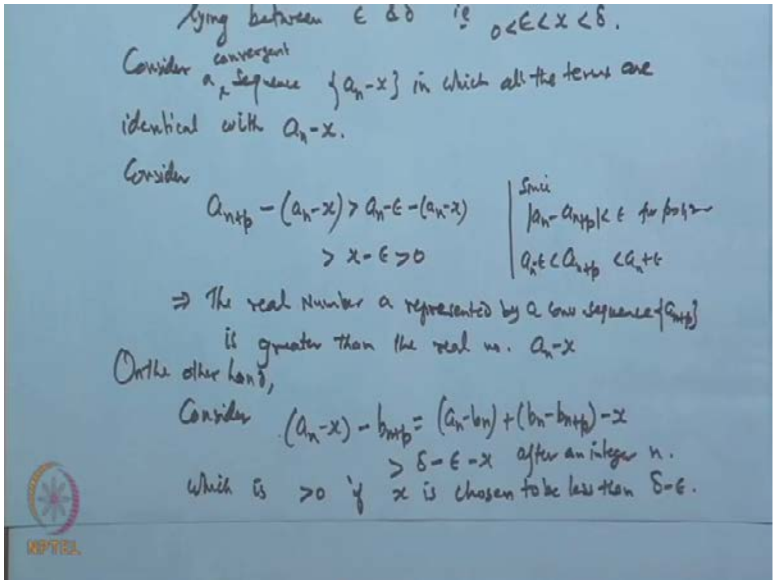

So, let epsilon is less than delta which is positive, okay; that is not a problem. Now let us picked up the rational number x, let x and pick up a rational number x lying between epsilon and delta; that is epsilon, this is greater than 0 less than x less than delta. Let x be a rational number lying between epsilon and delta, okay. Now consider a sequence convergent sequence a n minus x in which all the terms are identical with a n minus x. a n is a rational number; x is I am choosing to be also rational number. So, a n minus x is a rational number. Now we can identify a sequence or we can construct a sequence where each term is a n minus x. Suppose I take 1; 1 is rational number. I can construct a sequence 1 1 1 1 1; that is possible. So, a corresponding rational number we can identify this.

So, let us take a n minus x as a sequence with each term is identically equal to a n minus x, okay. Now this will be here. Now consider a n plus p minus a n minus x. Now from here if you look a n is convergent. So, this condition is satisfied a n minus a n plus p less than epsilon; it means a n plus p lies between a n minus epsilon and a n plus epsilon. So, from here we can say since mod of a n minus a n plus p is less than epsilon for p equal to

1 2 3. So, basically a n plus p lies between a n minus epsilon a n plus epsilon, is it not. So, if I want this number, say, this one then we can say a n plus p is greater than a n minus epsilon minus of a n minus x; just substitute here, is it not.

So, from here we get this is greater than x minus epsilon, but x lying between 0 and delta which is greater than epsilon. So, this is basically a positive. So, this implies that the number sequence a n p is a convergent sequence which represent the number a. So, this implies the real number a represented by a convergent sequence a n plus p or a n, it is same, is greater than the real number a n minus x, because this is real number which I am choosing a rational may be real, okay; each number greater than this. On the other hand consider the difference a n minus x minus b n plus p. Now this we can write it as a n minus b n plus b n minus b n plus p minus x.

Now a n minus b n and b n plus p. So, what is they say? This a n plus p minus a n plus p is greater than equal to delta. So, this will be basically greater than delta after a certain stage; this is after a certain stage this is greater than delta. This will be b n minus b n plus 1 lies between minus epsilon and plus epsilon. So, I can say b n, b n plus p is greater than minus epsilon that is greater than minus epsilon and then minus x, and then this is minus x, okay after some integer after certain integer n, suitable integer n can be obtained, okay.

Now this can be made greater than 0 which is greater than 0 if x is chosen to be less than delta minus epsilon, if I choose x to be less than delta minus epsilon, because what is this delta. So, delta minus epsilon is positive quantity. So, if I take an x any number which is less than delta minus epsilon. So, for that particular x this term will be greater than 0. Now this represents the real number a n minus x; this represents the real number b. So, the real number a n minus x is greater than b.

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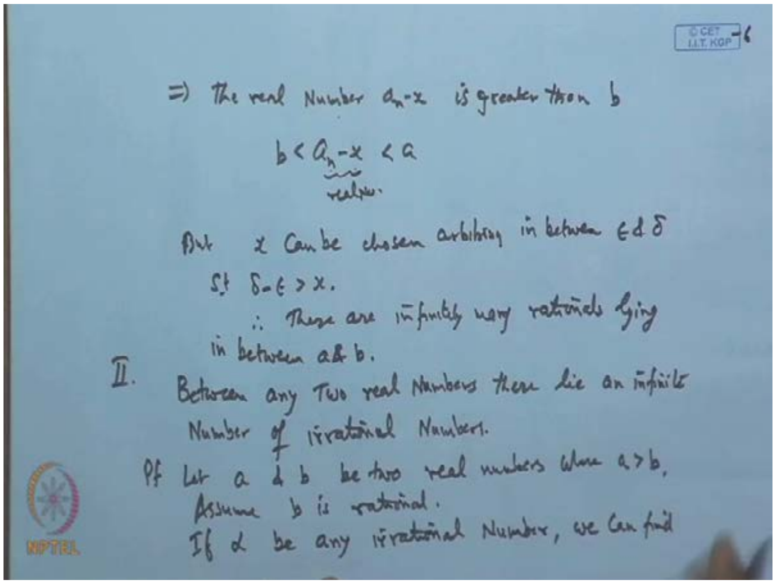

So this implies that real number a n minus x is greater than b. So, now what we conclude is that a n minus x, this is a real number which is greater than b but less than a, is it not. This is a real number this is lying between b and a, a is greater than b. But x is our choice, because one can choose infinite x which satisfy this condition between epsilon delta we can take infinite number of x which can satisfy this condition. So, as soon as you change this x there are infinite number of real point lying between a and b, and in fact these are all rational points. These are all rational points, because a n is also rational, x is also rational, and we are taking a sequence corresponding to a n minus x itself.

So, it is a rational number. But x can be chosen arbitrary in between epsilon and delta such that delta minus epsilon is greater than x, okay. Therefore there are infinitely many rational lying in between a and b; that is proven, clear. So, the second property is that in between any two real numbers there lie an infinite number of irrational numbers, okay, so there lies an infinite number. Proof is very simple; just suppose a and b are the two real numbers where a is greater than, say, b. Let us assume b is rational, okay. Now if alpha be an irrational number then we can find.

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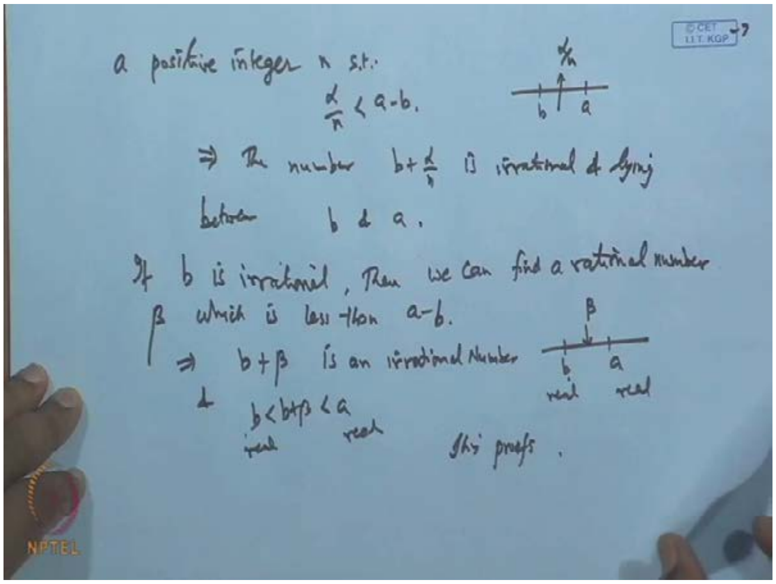

A positive integer n such that alpha by n is strictly less than a minus b. What happens is this is our number b, here is number a. Now if you picked up any alpha any irrational number alpha, so n can be made as large and we can choose as large as possible, so that alpha by n lies between here, alpha by n lies between this. It means this implies that the number b plus alpha by n; this number is irrational number, is it not, because alpha is a rational, n is integer of course and lying between b and a, because it is greater than a b; b plus alpha n is greater than b but is less than a because of this reason. So, this number is an irrational number lying between b and a. It means between any two real number we can find out an irrational number. Now if b is an irrational number because we have assumed one to be rational; if b is irrational then we can find a rational number beta which is less than a minus b.

This is our b, this is a. So, if b is an irrational number then we can find a rational number between b and a, because a and b are the two real numbers. These are real; this is real. Between any two real number there are infinite many irrational points, so between a and b one can introduce a irrational number, one can identify a irrational number beta, because between any two real there are infinitely many rational points are there. So, there exists we can find beta which is less than a minus b rational; beta is rational which is lying between this. So, this implies that b plus beta, b is what? B is irrational. So, b plus beta is an irrational number, and b plus beta is greater than b but less than a, is it not, because it is less than a minus b. So, b plus beta is less than a. So, in between two real's, this is real, this is real; we can identify a irrational point at end. So, therefore, this proves the result, okay; this proves the whole thing, is it okay. Now before going for the limit, etcetera, we will see the main thing that is the equivalence definition of Dedekind's and Cantor's, okay.

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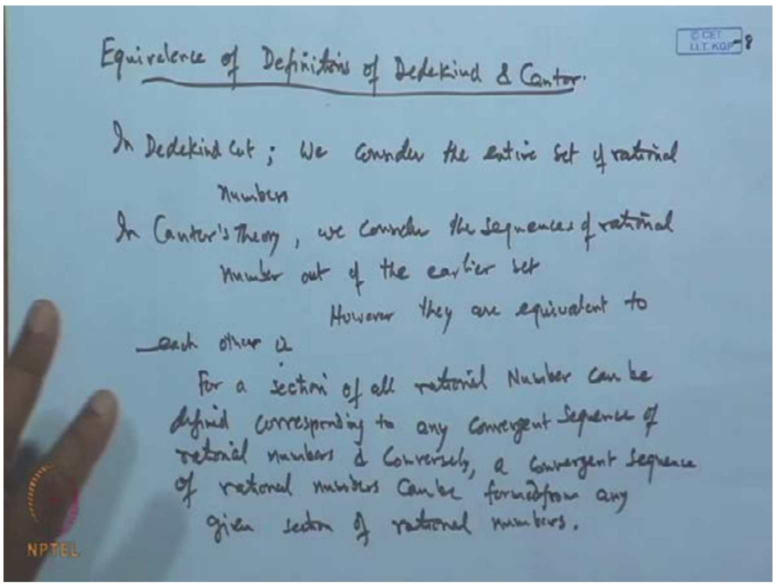

Let us see equivalence of the definitions of Dedekind and Cantor; these are the main means we have developed the concept of the real numbers with the help of Dedekind cut as well as the concept of the convergent sequence of rational points, and this is given by Cantor's, and the cuts given by the Dedekind, and what we see that these two theory basically are equivalent. They give the same set of the real numbers; do not give it, okay. So, what is the difference between them? If we take the Dedekind's theory Dedekind 's cut here what we do is here in Dedekind's cuts we consider the entire set of real numbers consider the entire set of rational numbers, and on this set of rational numbers we are introducing the cut, is it not. Then we say a lower class and upper class and that things any rational number which belongs to either lower class or upper class and like this.

So, in Dedekind's theory or Dedekind's cuts we consider the entire set of rational number together at a time, okay, while in Cantor's theory the sequences we consider the sequences of rational numbers out of the earlier set, is it not, the set of rational point. We picked up the sequence of the rational number and then we said this converges to a real, converges to a rational, converges to an irrational point.

So, this in Dedekind's cut we consider at a time all the rational points and put it in the form of the sections cuts while in case of this we bring that sequence of the rational numbers and then we assign the limits to this sequence convergent sequence of rational numbers sometimes rational and irrational. So, this is the main idea which they have done; however, fundamentally they are true.

However they are equivalent to each other; that is for a section of all rational numbers can be defined corresponding to any convergent sequence of rational numbers and conversely a convergent sequence of rational number can be formed from given section of rational number given section of rational numbers, okay. This is the one. So, we can introduce that and we can define. Now let us see the proof of this, how these two theories are equivalent. So, we need the following theorems. First result is to justify it we have the following results, oaky.

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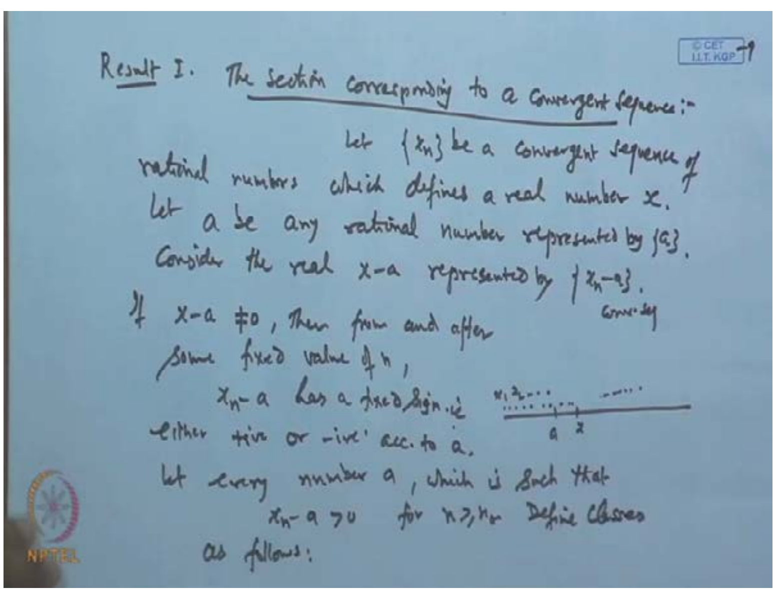

The first result is the section corresponding to a convergent sequence, okay; let us see. So, let x n be a convergent sequence of rational numbers which defines the real number x, and let a be any rational number and represented by a sequence a where all the terms are the same a and consider x minus a. Now consider the real number x minus a represented by a convergent sequence x n minus a, let us see this, okay. So what we have? We have this concept; suppose we have this x here and a is any rational number here, okay, the x n if a converges in case of rational number defines the number x, so x 1, x 2, x n can be obtained here. This is x 1, x 2 and so on or it may come from here also x 1, x 2, x n; a is a number which is the a itself; so we do not care for it, we do not bother about much,

Then x n minus a; this is main important thing represented by x minus a. Now if this number if x minus a is different from 0, now since x minus a represented by x n minus a which is different from 0. So, either it means what? That after a certain stage all the terms of the sequence will have one sign either it will have a positive or it will have a negative, any number if x minus a is different from 0 because this we have seen. If x minus a is different from 0 number then either it will be a positive number or it will be a negative number, is it not. So, the corresponding sequence x n minus a will have only one sign; either it will be a positive or it will be a negative after a certain stage is obtained, okay. Obviously because x n minus a represent the number x minus a, which is a convergent; this is a convergent sequence.

So, it is then from and after some fixed values of n x n minus a has a fixed sign, is it not, this is clear, because of this has fixed sign; that is either positive or negative. This sign will be there either positive or negative according to the value of a; depends on a whether according to a, if a is greater than this negative otherwise it will be positive. So, let us suppose let every number a which is such that x n minus a is positive for n greater than equal to say n naught after certain stage it is greater than n naught, okay. Now let us define classes as follows.

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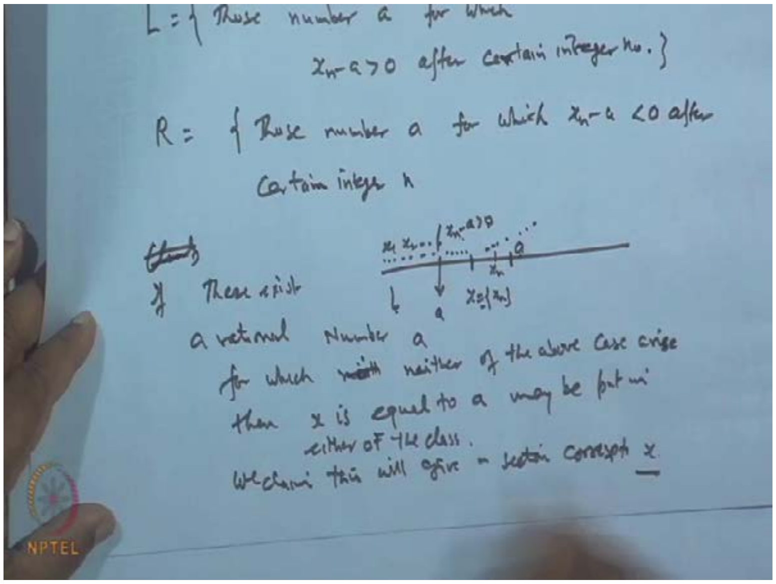

What we take L is the class of those number a for which x n minus a is positive after certain integer n naught. Let us put it in the class L after a certain integer and r where every number a for which is negative and those numbers a for which x n minus a is negative after certain integer n, let it be placed in the class r, okay. So, those numbers which are greater than 0 is in the lower class which are less than 0 in the upper class of certain integer of this, negative from after certain integer. If there exists an integer for which neither of them, okay, and third case is let us see. So, this is our what we have given the x equal to this sequence convergent, so I am taking x 1, x 2, x n like this. I am putting those rational number those number a in the lower class such that after a certain stage the difference between x n minus a is positive.

So, these points will come, is it not. So, those real number for which this is greater than 0 will be put in the lower class while the upper class means a is this. Suppose I take this number like this, so a is this number, x 1, x 2, x n, will be this. So, x n minus is a will be negative after a certain stage. So, those numbers you are puttying to be in the upper class. Now obviously this forms a section A; this section clearly (( )) in third case what about the A? Third case is now if there exist a rational number a for which neither of these cases rise, neither of the above cases arise, then may be neither of the above cases arise, then x is equal to a and may be put up in either of the class, okay. Now we claim this way that section x this will give the corresponding to the number x, okay. Now this is we claim this will give a section corresponding to x. So, we will it next time the detail for this, ok but it is almost completed just a few minutes left.

Thank you.