A Basic Course in Real Analysis
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Lecture - 6
Cantor's Theory of Irrational Numbers (Contd.)

So, in the last lecture we have started the Cantor's theory of irrational numbers, and in fact this is another way of introducing the concepts of the real number which is entirely different than the Dedekind's set on it. So, here with the help of the convergence sequencing of irrational number, Cantor's has introduced the concept of the rational and irrational points, and then he let on show that basically these two approach are basically equivalent; they are not different.

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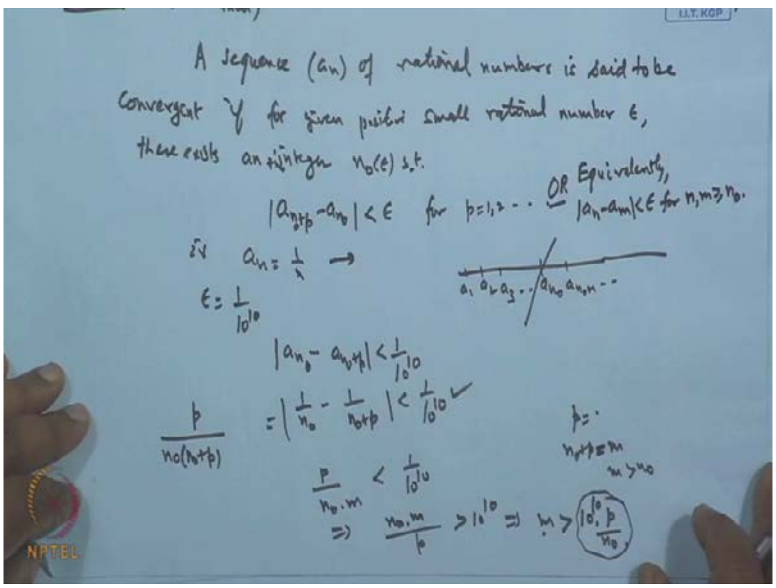

So, yesterday we discussed about the convergence part of the rational number convergence sequence of the rational number and various property. So, let us just recollect a sequence a n of rational numbers is said to be convergent if for a given positive small rational number epsilon any positive small rational number epsilon there exist, for a given positive rational number there exist an integer n naught positive integer, such that the difference between n plus p minus a n naught a n naught plus p minus a n naught remains less than epsilon for all p is 1, 2, 3, and so on. Then such a sequence a n of rational number is set to be a convergence sequence.

It means that after a certain stage if these are the points a 1, a 2, a 3, a n naught, a n naught plus 1 and so on, this is sequence of rational number. The sequence will be convergent, if the sign is some position number given any position number epsilon, corresponding to this epsilon one can identify a naught. It means that a naught will depend on this epsilon, such that if I truncate this series then after this all the terms of this sequence will satisfy this condition. So, difference from a naught to any other term can be made as small as we please. For example suppose I take a n is to be 1 by n a sequence of rational number we say it converges to 0. It is a convergence sequence, because take any epsilon say epsilon is 10 to the power 10, then corresponding to this epsilon one can find a naught such that mod of a n naught minus a n naught plus p less than epsilon means 1 by 10 to the power 10 a naught, and that will be equal to what? This is 1 by n naught minus 1 by n naught plus p.

This remains less than 1 by 10 to the power 10, ok; if I simplify this part then what you get basically? This is nothing but what, n naught n naught plus p and then p over this part is less than this, ok; it is also positive. So, this is less than. Now this entire thing p over a naught plus p we can already find from here; suppose I take p equal to, say, 1 or 2 anything then from here we can write 1 by n naught let it be n naught plus p to be, say, n. So, m is greater than n naught when p is less. So, m is greater than n naught. Then this becomes n naught m and then p you can just write p is 1 2 3. So, we can say p is less than 1 by 10 to the power 10; therefore, n naught into m divide by p is greater than 10 to the power 10. So, m can be obtained. So, from here m will be greater than 10 to the power 10 into p by n naught, is it not?

So, corresponding to this n naught this number if I choose this number all the term which are greater than this number will satisfy this condition is it not; it will satisfy this condition. It means you can easily find out and truncate the series where this happens, ok. So, we can get it this way like this. This can also be said equivalently this part when we say such than this is a or equivalently here also or equivalently we can say that for a given epsilon greater than 0 one can find a integer n naught such that mod of a n minus a n less than epsilon for all n m greater than equal to n naught. Both are equivalent. So, either we can use this or we can use this, so that I will do first. Then we have already seen that if a sequence is a convergence sequence of rational number then it must be a bounded sequence, and the terms of the sequence we can find out the a n plus p which can lie between a n naught minus epsilon and a n naught plus epsilon.

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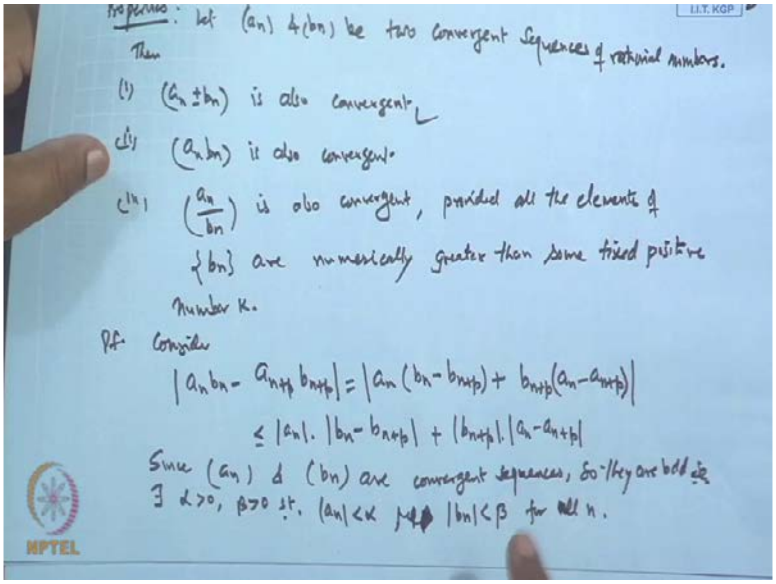

Then the resultant properties of it; if suppose there are two sequences various properties the addition, multiplication and subtraction. Let an and b n be two convergent sequences of rational numbers, ok, then one, a n plus minus b n this new sequence where addition of the two sequence or subtraction of the sequence is also convergent. Product of the two sequences a n into b n is also convergent sequence, but division of the two sequences a n divided by b n is also convergent provided the sequence b n, provided all by elements of the sequence b n are numerically greater than some fixed positive number k, ok. We have already checked the first one; this already proved with epsilon delta. The proof of this let us see the others. This one already done yesterday, so no point of repeating the whole thing; let us see here.

We wanted to show if a n, b n are the two convergence sequence of rational number then their product will also be a convergent. It means we wanted to check basically this condition a n naught minus a n naught plus p is less than epsilon for all this, for this. So, instead of this a n naught we will say a n naught, b n naught, a n naught plus p, b n naught plus p. If this remains less than epsilon for given then we can say it is ok. So, construct a sequence. Consider the difference mod of a n b n minus a n plus p b n plus p. Consider this sequence, ok, and for p is equal to 1 2 3 and so on. Now this can be written as a n b n minus b n plus p plus b n plus p into a n minus a n plus p; just adding and subtracting this term a n into b n plus p and get. Now this is less than equal to mod of a n into mod of b n minus b n plus p plus mod of b n plus p into mod of a n minus a n plus p, is it not.

This modular sign become mod of x 1 plus x 2 is less than mod x 1 plus mod x 2. Now since a n is a convergence sequence, so according to that every convergence sequence is a bounded sequence; similarly b n is a convergence sequence. So, this has to be bounded. So, since a n and b n are convergence sequences, so they are bounded sequences; that is there exists alpha beta there exists some alpha greater than 0, beta greater than 0, such that all the terms of the sequence mod a n is less than alpha and mod b n is less than beta for n all n for all n; every convergence sequence is bounded sequence, so this happens. So, we can find alpha and beta such that this. Now further b n and a n are convergence sequence, so we will make the criteria from here, ok.

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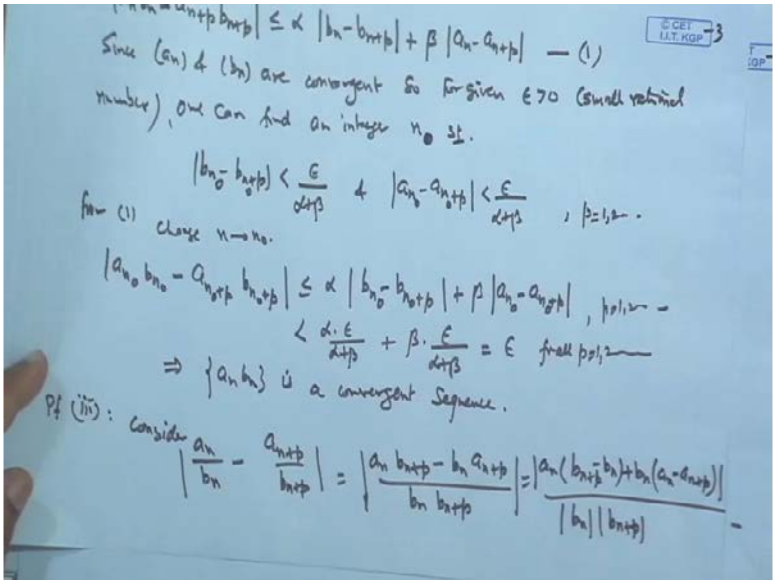

So we can get from here is therefore mod of a n b n minus a n plus p b n plus p. This is less than equal to alpha times mod b n minus b n p plus beta times mod a n minus a n plus p, ok, this is ok. Now since a n b n are convergent so for any given arbitrary small positive number epsilon one can find a common n, so that this can be made as small as this, and this can also be made as small as. So, since a n and b n are convergent so for given epsilon greater than 0 a small positive rational number small rational number one can find an integer n naught such that mod of b n minus b n plus p remains less than epsilon over say alpha plus beta, and mod of a n minus a n plus p remains less than epsilon by alpha plus beta for all a n, ok.

So, this is true; a naught I have to write for all n this can be, ok, for n we can find an n such that this holds, sorry instead of n naught we can just say n, ok. Let n naught, n naught, n naught where the p is 1 2 3 and so on, ok, by definition, clear. Now again pick up from one; this is one. So, from one what we get a n naught b n naught minus a n naught plus p b n naught plus p is less than equal to alpha times b n naught minus b n naught plus p plus beta times mod a n naught minus a n naught plus p. Change n to n naught so that it is true for p 1 2 3 and so on. Now this is less than this. So, will remain less than alpha into epsilon y alpha plus beta; this is also less than beta into epsilon over alpha plus beta which is equal to epsilon. So, this is true for all p starting from 1 2 and so on.

So, this implies that sequence a n, b n is a convergent sequence, ok. So, product of the two sequence convergence sequence is convergent, clear. So this is the proof for the part two; now proof for part three. We will saw that division is also if the two convergent sequence are there then one can divide a n by b n provided then b n is greater than 0 or greater than all the terms are greater than some positive constant k; then the a n over b n is also a convergence sequence. So, let us start with this division. So, what we do is again in a similar way we start with this. So, consider say mod of or a n by b n minus a n plus p over b n plus p; consider this. So, what is the mod of this? Now this we can write it as a n b n plus p minus b n a n plus p over b n b n plus p mod of this, and now again you break up in the form. So what we get is this is basically equal to a n b n plus p minus b n and then plus b n a n plus a n minus an plus p divided by mod of b n mod of b n plus p, is it not; that what it is. Now further this is less than equal to mod of this, mod of this and [FL].

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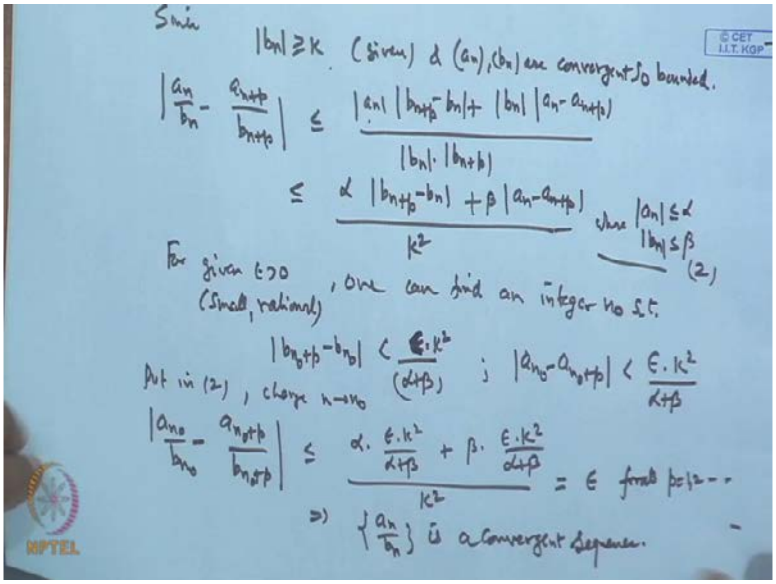

Now it is given that b n, since each term of the sequence b n is greater than numerically greater than or equal to k; this is given, ok. So, we can say that mod of a n over b n minus a n plus p over b n plus p is less than equal to mod of a n into mod of b n plus p minus b n plus mod of b n into mod of a n minus a n plus p divide by mod of b n mod of b n plus p, ok. Now this is given and sequence a n and b n are convergent, so bounded. Therefore there will exist alpha and beta so that mod a n is less than alpha, mod b n is less than alpha. So, this is less than equal to alpha times b n plus p minus b n plus beta times mod a n minus a n plus p, and since b n is greater than mod b n so 1 by b n is less than equal to 1 by k. So, this is k square where mod of a n is less than equal to alpha, mod of b n is less than equal to beta where alpha and beta both are positive because of the boundaries, ok, so bounded.

Now pick up the epsilon. So, for given epsilon greater than 0 let it be the equation two, ok. For given epsilon greater than 0 which is a small positive rational number one can find an integer n naught such that this thing is less than suitably less than epsilon or something, and this is also a small quantity. So, we can say that mod of b n naught plus p minus b n naught is less than alpha over alpha plus beta and then k square we want to remove. So, let it be, sorry this is epsilon not alpha; this is epsilon. So, epsilon into k square by alpha plus beta and mod of a n naught minus a n naught plus p is also less than epsilon into k square divided by alpha plus beta, because basically this is a small quantity. Once you multiply the epsilon the whole thing become very small.

So, n naught can be showed in such a way so that this entire thing is less than this, similarly this is less than this. Substitute this in two; so put in two. So, what we get is a n naught b n naught minus a n naught plus p over b n naught plus p. Change n to n naught, then we get this less than equal to alpha. When you write this thing you are getting epsilon by k square epsilon into k square by alpha plus beta. This is also beta epsilon k square by alpha plus beta and divided by k square. So, finally what you get? K square get cancel, alpha plus beta get cancel, and finally you are getting what? Epsilon only, and this is true for all p equal to 1 2 3. Therefore the sequence a n by b n is a convergence sequence of rational number. This is what we get, is it ok. Now after developing this thing Cantor's was able to introduce the concept of real numbers.

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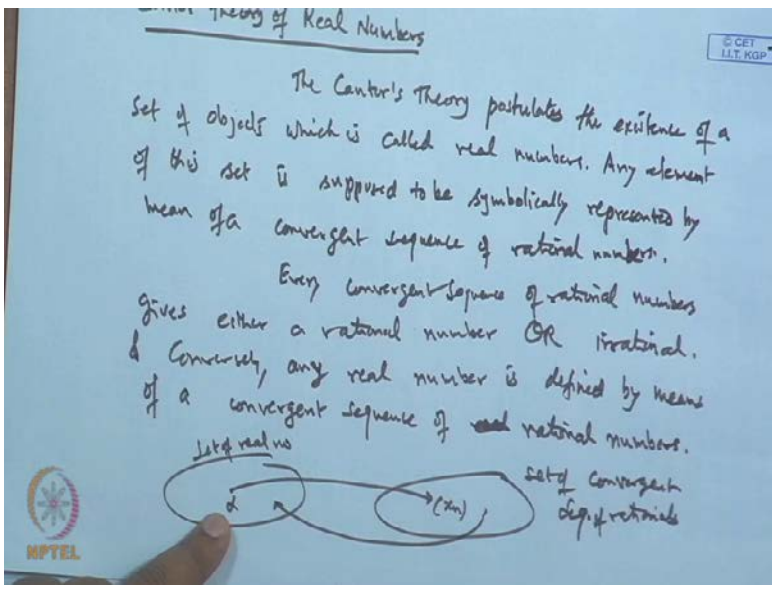

So, what is the Cantor's theory of real numbers? Yes, you wanted something, ok; but why it is greater? I think it is drawn. So, Cantor theory of real numbers; ok what is the Cantor theory? The Cantor theory postulates the existence of a set infinite set of object which we call it as a real number. So, Cantor's theory postulates the existence of a set of object infinite set of object of course set of object which is called real numbers, and every object of this set is associated with a convergence sequence of the rational number, and any element of this set is supposed to be symbolically represented by means of a convergent sequence of rational numbers. It means what is it? The meaning of this is that what the Cantor's say that it we pick up any convergence sequence any sequence of the rational number which is convergent, then this sequence will converge either to a rational number or to a number to which is different from rational; that they call it irrational.

So, every convergence sequence of rational numbers will give either a rational number or irrational number which are not rational we call it as irrational number, convergent point means every sequence convergent sequence of the real number there will be every convergent sequence of rational number defines a real number which is rational or irrational, and converse is also true; conversely any real number is defined by means of a convergence sequence of rational number. So, this is the theory given by Cantor's, and this is known as the Cantor's theory of real numbers. So, what this postulates is what it says is that if we have a class of the objects which we call it as a real number. Then any real number if you picked up then corresponding to real number we can identify a sequence of the rational number which is converse to that point. If it is rational then the limit point is called the rational real; if it is not rational then we call it as irrational, but the convergence sequence we can identify.

And conversely if we are having a convergence sequence of the real number or set of the real number then every convergence sequence of the rational number will give a number which is either rational or irrational. So, there is correspondence between them set of real number and set of the convergence sequence of rational. This is the set of real number and here is the set of convergence sequence of rational. Pick up any sequence alpha corresponding to this we can identify a sequence x n which will converse to this and vice versa, ok. Now if you pick up any sequence we can find out here the number this, so like this, ok. This theory is given by Cantor's but this is a very simple way he has introduced. Dedekind's had introduced in a very complicated way. What he did he has considered the objects of the points and then introduced the sections and then lower class upper class and then finally he got that rational numbers and irrational numbers like that.

So, he had developed the three types of the classes and then introduced, but here simply in the form of the convergence sequence he has introduced the concept of rational and irrational numbers, but what is the drawback? Main drawback in this is that a real number if I picked up a real number then there are many sequences of rational point which will converse to the same real number.

Student: If we are taking many sequences converging we can find.

Many sequence infinitely many we can find out. This is the drawback.

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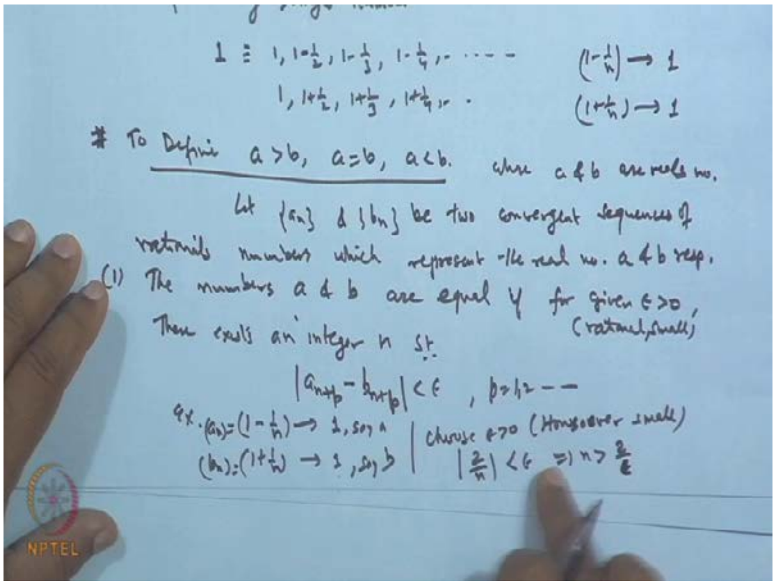

The drawback of this theory is that it does not explicitly express any single number. For example suppose I take the sequence, say, one I choose the number one then one will correspond to this sequence, say, 1, 1 minus half, 1 minus 1 by 3, 1 minus 1 by 4 and so on; that is 1 minus 1 by n this sequence will go to 1. But on the other hand this will also bellow 1 plus half, 1 plus one-third, 1 plus 1 by 4 and so on. This will also 1 plus 1 by n; this sequence will also go to 1. So, there are many Fibonacci sequence available which can go to the one in fact any real number is picked out. Similarly for irrational also; it is not hold only for one sequence. There are many sequences of rational which can be generated whose limit point is the rational number given. So, this is the main drawback. But if we looked at finally in spite of this drawback both the theory are equivalent; both the theory gives that equivalent concept, and both are equivalent basically.

They are not generating an extra element which is not available in the first text means there also we are getting a continuum the entire real line, with the help of this also we are getting a continuum, and just like in previous case when we apply the Dedekind's theory section cuts over the set of the real number we are not getting a further extension. Similarly here also when we applies the Cantor's theory of sequences over that set of real numbers set of sequences of real numbers we are not getting a further generalized extension. So, basically we are not getting a new thing; both are equivalent, ok. So, that is what the theory says; is it clear. Now there are certain relations like how can define the two sequences, two elements are equal, one element is greater than other or less than other in the form of the in this Cantor's way.

So let us see. To define a is greater than b, a is equal to b, a is less than b where a and b are real numbers. Suppose we have this; how to define it? Suppose let sequence a n and b n with the two convergent sequences of rational numbers which represent the number the real number a and b respectively. Suppose we have two sequences convergence sequence of the rational number which represent the real number a and b. Then we say number one; we say a is equal to b. We say the number a is equal to b; the numbers a and b are equal if for a given epsilon greater than 0 rational number is small, ok. For a given epsilon greater than 0 there exist a positive integer n such that mod of a n plus p minus b n plus p is less than epsilon for p equal to 1 2 3 and so on, ok; for p 0 obviously 2 there no problem for 1 2 3 then we say that two sequences two real number are equal means the number represented by this sequence, number represented by this sequence are basically equal number for a given.

So, for example if I take these two sequence for example if I take the sequence 1 minus 1 by n this sequence and the sequence 1 plus 1 by n, ok. This is our sequence a n, this is the sequence b n. Now they represent the same number one and this is also one. So, basically this is, say, a, this is say, b. We say a is equal to b, if the difference between the terms of the sequence after a certain stage can we made as small as we please. So, if I take any epsilon greater than 0 choose any epsilon greater than 0 whosoever is small may be, howsoever small, then what is this difference of this will be a n plus b minus p means it is basically what difference? The difference will come out to be 2 by n only. The difference of this will come out 2 by n. It can be made less than epsilon.

So, n must be greater than what? n by 2 by epsilon. So, after this stage we get all the terms of the sequence are basically very, very small very close to each other very close. So, though that sequence looks like a different sequence; we are not saying these two sequence are identically equal we are not saying. We are not saying that sequence a n b n are equal sequence we are not saying this. What we are saying is the corresponding limits are equal corresponding limits because when the two sequence are said to be equal then they have corresponding elements are equal.

So, that we are not saying, why? Because two sequences are equal obviously they will represent the same point. No, but since we are in Cantor's theory a point can be represented by many convergence sequence. So, how to define that two limit point to be equal? We say they are equal only when that these two sequence which are different but their behavior after certain stage is within our limit; that is the difference is very, very negligible very almost equal to 0 when n is sufficiently large. Then we say these two points these limits points are equal, is it ok; that is fine. In fact if you go to 0 when n is sufficiently large difference will go to basically 0.

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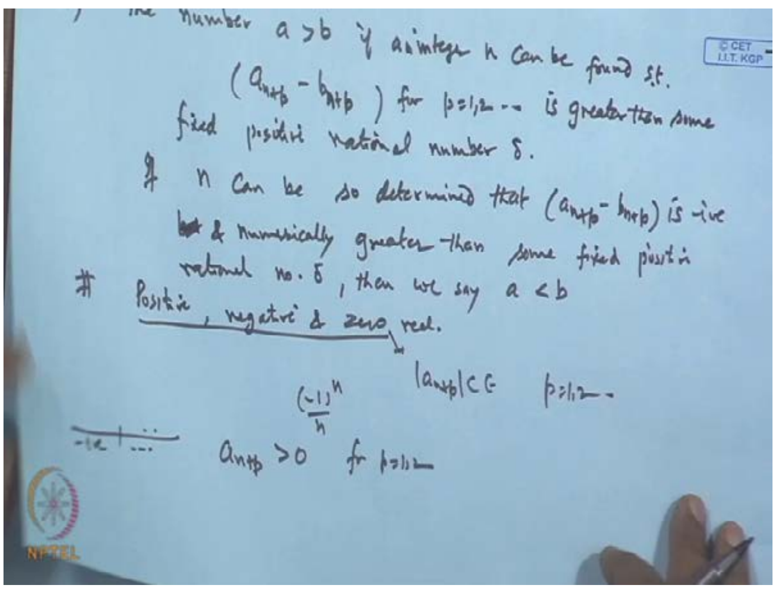

Now similarly we say that is second, ok. Similarly we say second part the number a is greater than b if a value of n can be found if an integer n can be found such that a n plus p minus b n plus p; this difference for p is 1 2 3 and so on is greater than some fixed positive rational number delta, ok, means whenever this a n plus p minus a n is greater than some positive rational number delta then we say the sequence the number a is greater than b. Now if n can be so determined that the difference is negative that the difference a n plus p minus b n plus p is negative, but numerically greater than some fixed positive numbers and numerically greater than means without sign numerically greater than some fixed positive rational number delta then we say a number a is less than b and when equal we have already discussed.

So, given any sequence two convergence sequence which represents the number a and b then one can also identify the ordering between them by seeing this thing. So, this way he has introduced the ordering between the two real numbers, ok. Now if we take this two sequences a n and b n then whether this a n and b n they are positive, negative and this positive, negative and zero number zero real numbers; positive real number in a similar way we can define zero number when mod a n plus p except 0 number when the mod of a n plus p is less than epsilon for certain n greater than a n. For given epsilon we can find a n such than this true for all p then we say the sequence of this a n are basically going to zero. Why mod is there because there is a chance of getting the 0 from the negative side; it means the elements some elements may be positive some may be negative, it may go alternatively positive negative, but it is opposed to 0.

For example if I take minus 1 to the power of n then what happen? It goes to plus 1 minus 1, plus 1 minus 1 but the limit of this is tending to 0. So, in this case when the terms corresponding to 0 or when it tend into 0 the sign of the a n plus p may vary, but when you take the positive or negativeness then what happen is when we say the positive; it means after the certain stage a n plus p must be greater than 0 for p 1 2 3. So, after a certain stage the terms may be negative. They are negative here but after certain stage all the terms should be positive. Then only we say that the number which is represented by the sequence will be a positive number. When the sequence is given some terms are negative and some are positive, but when n is sufficiently large the terms are going to be positive. So, the limited point will be a positive content will represent a positive real number. But otherwise if you take the few terms are positive but rest are negative, and when n is sufficiently large the terms are coming to be negative the limiting point will be a negative. So, in that case we say the real number is negative, ok. So, that is what we are saying.

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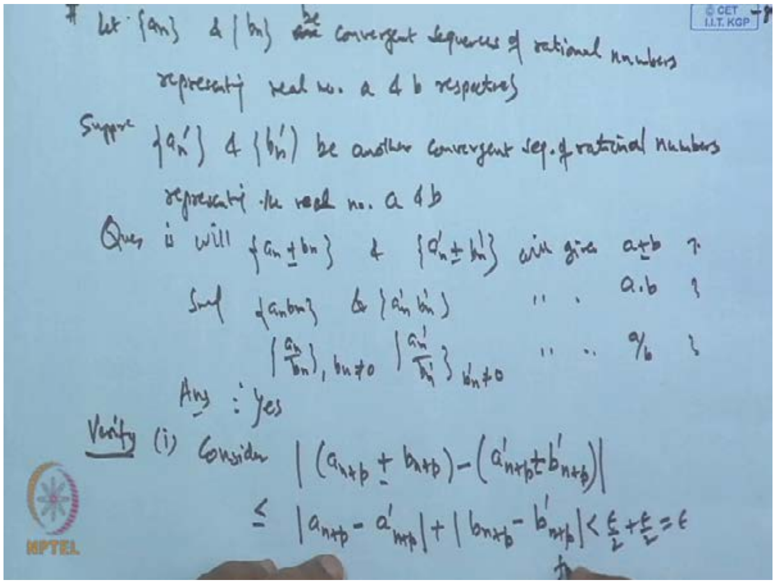

Next is there are some difference and the product is defined like this a n b n so that is noted. Now as we have seen that if a n and b n are the two sequences are convergent sequences of rational numbers, let a n b n be convergent sequences or convergence sequence of rational numbers representing suppose representing real number a and b respectively, ok, clear. Now if I take since these are not only the representation but suppose a n days and b n days be another sequence another convergent sequence of rational numbers representing the real numbers a and b. Now question is will sequence a n plus b n, ok, or a n and minus b n and a n dash plus minus b n dash will give a plus b or a minus b. This is a one set of the sequence, a n converges to a, b n converges to b. Then we are finding a n plus b n; limiting behavior of this will be a plus b. If we take minus then it will be a minus b.

Now what I am claiming is in place of a n b n I am choosing a n days and b n days which are the sequences also converging to a and b. Then instead of a n plus b n if I take a n days and b n days weather this sequence convergent sequence will represent the same sum as the sum of this sequence. Similarly if the product a n and b n and product a n days b n days will give the same number a n to b; that is what is there. Otherwise if you give a different then the entire theory will be flopped. There is no point of developing the theory in such a way. So, the answer is yes. Similarly a n by b n where the b n is not equal to 0 of course and a n days over b n days where the b n days is not equal to 0 will give a by b. So, these are the answers for question; the answer is yes.

Let us verify one and then we will see, ok. So, suppose I take one, ok. So, let us consider mod of a n plus p plus b n plus p or minus also minus a dash n plus p minus plus b dash n plus p or minus sign; both I am taking at a time plus and minus. Now this will be less than equal to mod of a n plus p minus a dash n plus p, ok, plus b n plus p minus b dash n plus p because minus will not affect much, ok, when minus minus plus and we get. Now what is given? a n and a n dash represent the same number a. So, by definition the way he has introduced the number a if it is represented by two different sequences then basically after a certain stage the difference can be made as small as we please.

So, since these sequences represent the same number so this difference can be made as small as we please, say, epsilon by 2. Similarly this difference can be made as small as we please epsilon by 2. So, total is epsilon for all p for p equal to 1 2 3. So, what it shows? That this difference is less than epsilon; it means this sequence a n plus b n and a dash n plus p b dash n plus p will give the same number, and that number is nothing but a plus b. Similarly for the difference, is it ok, and the same trick same case you can use for the verification for the product.

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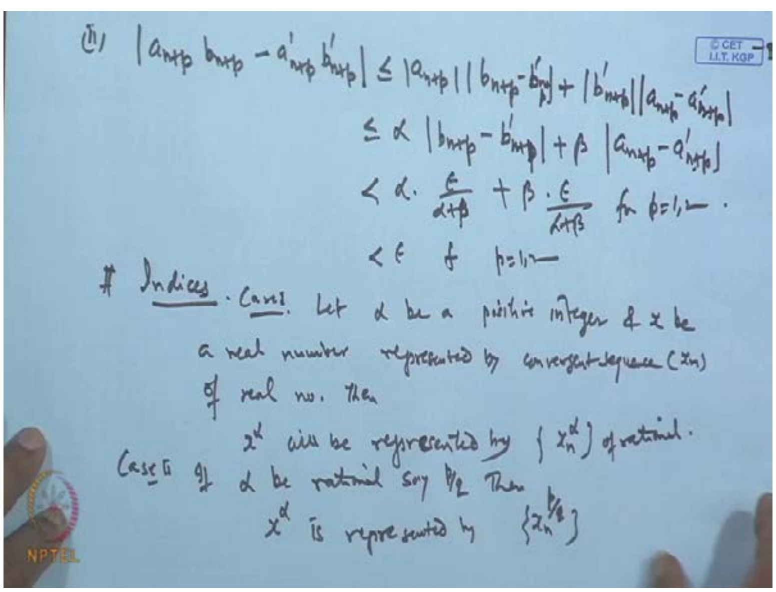

Product of these two in the similar way we can write for the product if we write a n plus p b n plus p minus a dash n plus p b dash n plus p. Then this is less than equal to a n plus p mod of this mod of b n plus p minus b n and then b n dash or dash b n dash plus p, sorry. This is plus p, ok; then plus b dash n plus p mod of a n plus p minus a dash n plus p, just making adding. Now again these are bounded because these are convergent sequence. So, this is less than equal to alpha into mod b n plus p minus b dash n plus p, and this is beta mod of a n plus p minus a dash n plus p, ok. Now again this b n and b dash n represent the same number. So, it can be made as small as we please. So, I can make this thing as epsilon by alpha plus beta, is it not and this also because it is also same sequence so after certain stage we can made it to be epsilon by alpha plus beta for all p 1 2 3, but this is epsilon.

So, this is true for all p 1 2 3. Therefore these two sequence basically give the same number that is there. Similarly also we can justify. So, what we say that these properties are satisfied with this. Similarly associative property, commutative property, etcetera can be justified with the help of the Cantor's theory; that is as the limit of the sequence of convergence theorem. Now here one more thing; the indices in this can also be generated defined in terms of sequence of rational numbers in terms of the sequence. So, this is our indices. Suppose I take x be a case one. Let a be a alpha is a positive integer; let alpha be a positive integer and x be a real number represented by a convergent sequence x n of real numbers, ok. Then x to the power alpha this number this is also real number will be represented by sequence x n to the power alpha. This is the way he has introduced.

Suppose alpha be any positive integer and x be a real number which is represented by x n. So, x to the power alpha will be represented by a sequence x n to the power alpha of rational points. These are the rational numbers, is it not where x n is also any integral power also will be rational. Similarly, if alpha be a fraction be a rational number, say, p by q then x to the power alpha is represented by x n to the power p by q this sequence x to the power alpha is represented by x n to the power p by q and where x n sequence converges to x. Now only here the difference is since q this square root of this here note it the x to the power 1 by q q th root is valid only when x is positive or q is x or if q is odd and x is negative. Alpha is integer, and here it is rational. So, p is integer but 1 by q I am talking; 1 by q because this is not this is defined for even the x n are, say, negatives, but here x n cannot be negative unless if x n is negative q has to be a odd numbers, then only it is defined. So, for irrational case we will see later on.

Thank you very much. Next class we will see.