A Basic Course in Real Analysis
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Lecture No - 5
Cantor's Theory of Irrational Numbers

So, today we will discuss the another theory, which developed the real number system through which first we have seen already Dedekind's theory. How the real number has been developed with the help of the curves, rational curves and the Cantor's theory is another way of getting the real number system. In this theory we are instead taking the curves, we will defeat, we will take the sequence concept, concept to the sequence is taken consideration and then every real number will be a limit point of the sequence of the rational number. So, this way the theory has been developed and though there is some problem in this Cantor's theory, but later on we will see that basically these two theory. One is given by Dedekind's other one is given by the Cantor's, basically they are equivalent theories.

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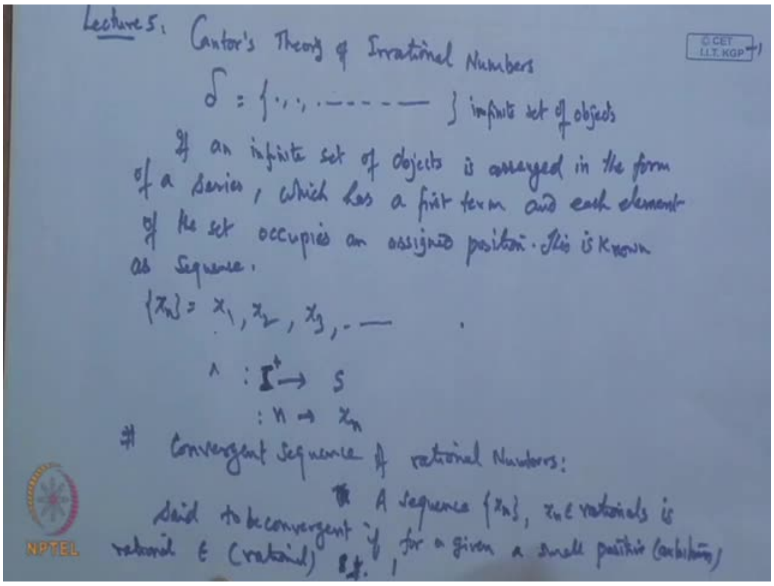

So, let us see the Cantor's theory of irrational number. Suppose we have a set S, it is a infinite set of objects, infinite set of objects, some objects are there (( )). Now, when we arrange the elements of this set in the form of the series, such that it has a first step and each element of this set occupies a definite position, then this arrangement is known as the sequence.

So, basically where this we can see, if an infinite if an infinite set of objects objects is arranged in the form of in the form of a series. Here series we means the sequence way, a series which has a first term and each element and each element of the set, each element of the set occupies an assigned position assigned position or place. So, in other words we can also say that this and this arrangement this is known as and this arrangement, this is known as sequence.

Suppose we take the first element from here, put it as x 1 then comma. Let pick up any element and put it at the second place, x 2. Then pick up an another element put it at third place and so on. So, each element will occupy a definite position and there is starting from the first place. So, this sequence, this way we denote as a x n and is a sequence. If these are rational number then this is a sequence of the rational number and if it is integer, sequence of the integer and say irrational number and like this.

So, basically what we are getting is there is a one to one correspondence; a sequence is a mapping which transfer S to the one set of positive integer, a mapping from the set of positive integer to the element S that is corresponding to a positive integer n. We have a element x n of S, so this mapping gives a sequence and we arrange in the form of the this here, separated out the point then we call the sequence. So, this is what we are getting either x 1, x 2 or a 1, a 2 whatever may be.

Now a convergence sequence of rational number, convergent sequence of rational numbers, how to define the rational numbers, convergence sequence of a rational numbers. The sequence or a sequence x n, where the x n's are rational are rational is said to be convergent if for a given if for a given, for a given arbitrary small number epsilon, given a small positive number, arbitrary, any positive, very small positive number epsilon which is a rational number, a small positive rational number epsilon greater than S. such that a sequence x n is said to be convergent for a given positive number epsilon there exist, sorry there exist there exist a positive integer n naught such that if there exist a positive integer in n naught such that the difference between the term a n naught minus a n naught plus p remains less then epsilon for p equal to 1, 2, 3, then such a sequence is said to be a convey.

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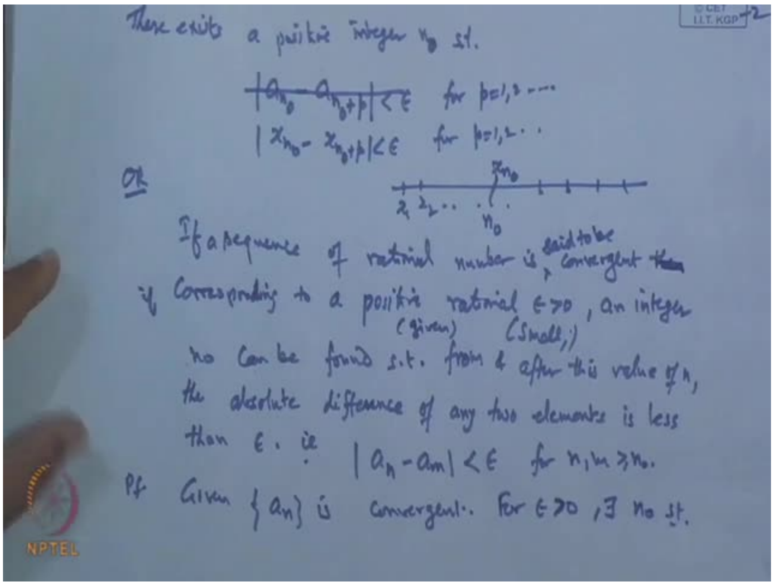

So, basically where this is a terms of the sequence x 1, x 2 sorry, here it is x n. This is this is what we are getting a positive x n minus x n naught plus p, this is less than epsilon, for p is equal to 1, 2, 3. So, these are the terms of the sequence, we say the sequence of the rational number is convergent if for a given epsilon a rational number greater than 0, there exist an n naught such that from this term onward if I pick up any arbitrary point here, the difference between x n naught and that point will remain less than epsilon.

Then only we say the sequence is a convergent sequence or in other words you can say, or in other words if a sequence is convergent, if a sequence of rational number is convergent then then the difference between, then the… Then corresponding to corresponding to corresponding to a positive rational number epsilon greater than 0 which is a small and one can choose a small number, corresponding to epsilon greater than 0 one can find an integer n naught can be found such that from n, after this value after this value of n the absolute difference the absolute difference of any two elements two elements is less than epsilon.

So, that is in other words you can say, sequential rational numbers is said to be convergent is said to be convergent if if corresponding to a positive rational number greater than 0, this is a given number. For a given epsilon greater than 0 there exist an integer n naught such that from, and after this value, the absolute difference of any two elements is less than epsilon. That is the difference between a n minus a m can be made less than epsilon for all n, n greater than or equal to n naught. That is the meaning of it, so the proof of this just depends on this definition.

How, suppose we have a sequence is given to be a convergent, given sequence a n sequence a n is convergent, a sequence a n is convergent. Then for a given epsilon greater than 0 there exist an n naught such that modules of a n naught minus a n naught plus p is less than epsilon by 2 for n is greater than n naught or that is for p equal to 1, 2, 3 and so on, clear.

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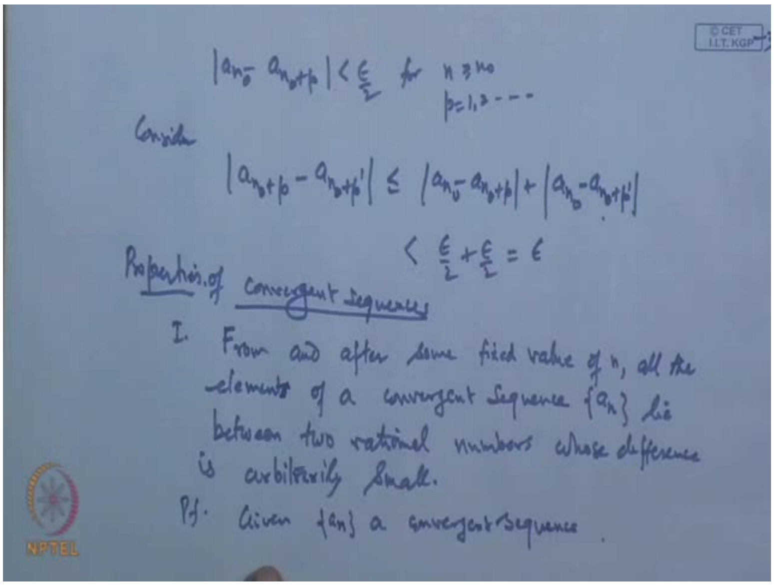

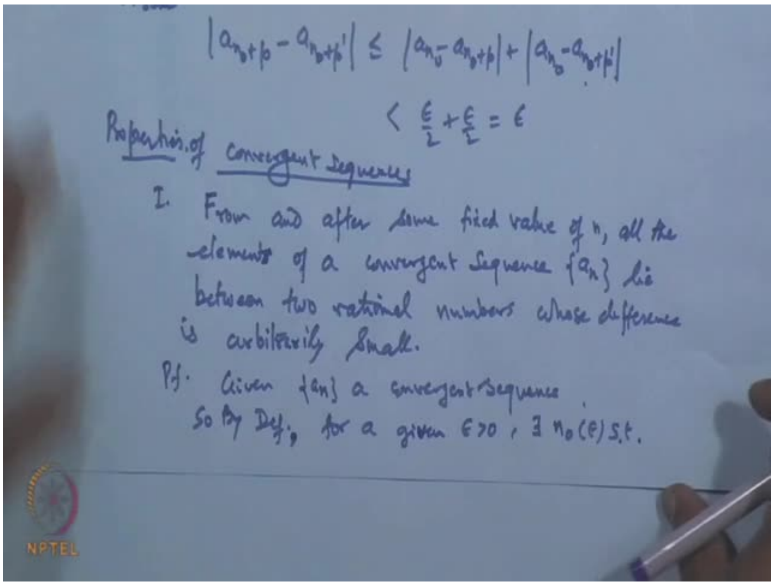

Now consider these point modules of a n naught plus p minus a naught plus p dash. This will be less than equal to mod of a n naught minus a n naught plus p plus mod a n naught minus a n naught plus p dash. Just subtract and adding a naught in them. Now, if this is less than epsilon by 2 for n is greater than n naught, then we can say this is less than epsilon by 2. Further we can choose in a similar way n naught is the maximum value among these, then we can also take this to be less than epsilon by 2. So, total is less than epsilon, is it not. So, difference between arbitrary term will always be less than epsilon after a certain stage, that is this is become n, this becomes m. So, this will be follows from and in fact this m can be obtained, (( )).

Now they have another properties of these convergent sequence properties of convergent sequence sequence. The first property is that from and after some fixed value of n from and after from and after some fixed value fixed value of n from and after some fixed value of n all elements, all the elements of a convergent sequence, say a n. Convergence sequence a n, lie between lie between two rational numbers rational numbers, whose difference is whose difference is arbitrarily small arbitrarily small. What is the meaning of this thing, from an after some fixed value of n all the element of a convergent sequence lies between two rational numbers whose difference is arbitrarily small.

So given is a n, be a convergent sequence, given sequence a n given sequence a n a convergent sequence, a convergent, given sequence a n a convergent sequence.

So, by definition for a given epsilon, by definition.

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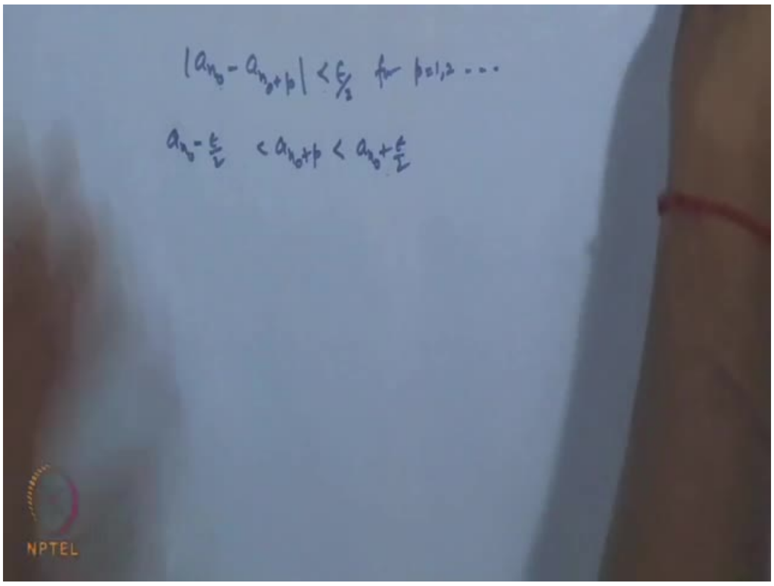

For a given epsilon greater than 0 there exist an n naught, depends on epsilon, of course such that such that the difference between a n naught minus a n naught plus p is less than epsilon for p 1, 2, 3 and so on by definition. Now, in place of this I am taking epsilon by 2 because this epsilon by 2 another number epsilon that is all. Now, that two points when you exchange remove this moder sign if a n naught is greater than a n naught plus p or a n naught may be less than a naught plus p.

So, basically the a n naught plus p will lie between these two bounds, a n naught plus epsilon by 2 and a n naught minus epsilon by 2, is it not. Because if suppose a n naught is greater than a n naught plus p than what happens, a n naught minus a n naught plus p without moder sign, is less than epsilon by 2. So, epsilon by 2, you take it this side so a n naught minus epsilon by 2 is less than this number. Now, if this is greater than this number, a n naught then because of the naught, we can write a naught plus p minus a naught, a naught is positive quantity and we get this number.

So, what we get that, given any sequence, that is what this result says that from and after if a sequence is a convergent sequence. Then from and after the value from after some value fixed value of n all the elements of convergent lies between two rational number whose difference is arbitrarily small. Since a n naught is epsilon and rational epsilon is also rational. So, we can get these two are rational so here it is rational, given rational number.

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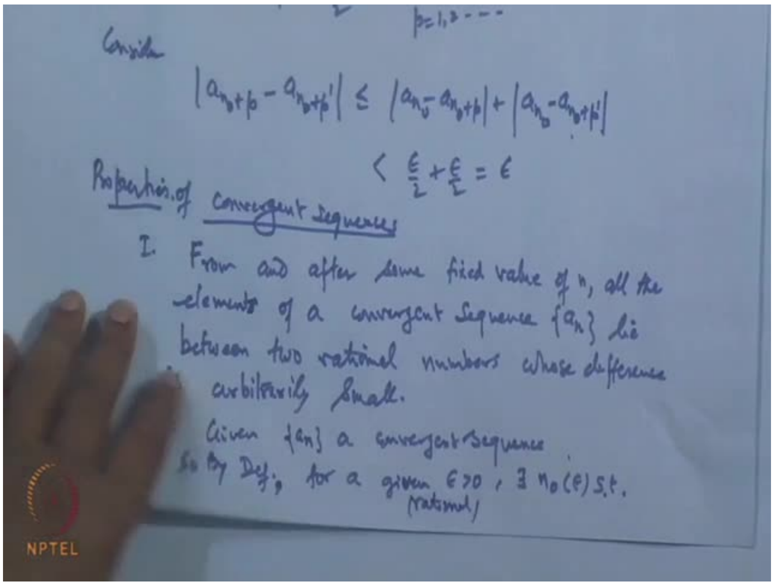

Now, if I take the difference between these two, the difference of these is epsilon so this number lies between two rational number, whose difference is arbitrarily small, that is what I said.

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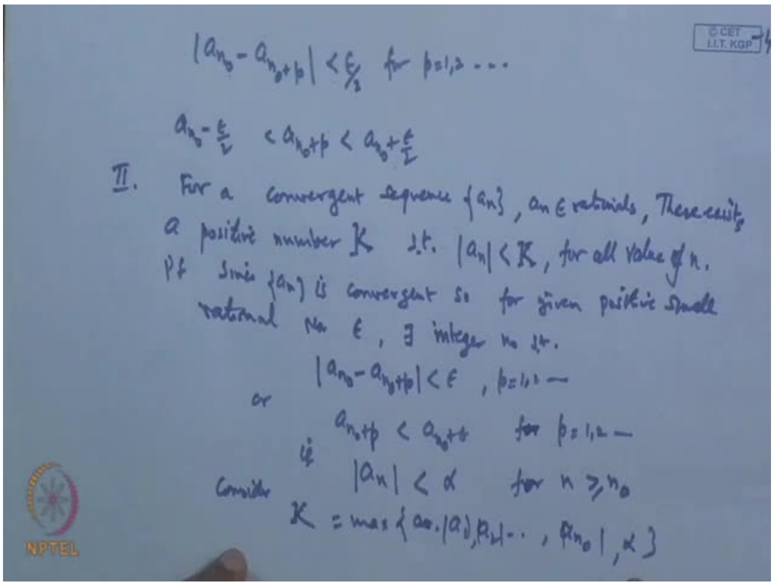

Now second property is of this for a convergence sequence, a n of course, you are taking a n’s are rationales rationales there exist a positive number K, exist a positive number K there exist a positive number K, such that mod of a n is strictly less than K or dominate or bounded by K, for all values of n for all value of n. So, this is not only true for, any real number of course, we will whole system.

So, what it says that for a convergence you can say a n and epsilon for a convergent sequence a n, there exist a positive number for a convergent sequence, a n there exist. Why this a n belongs to rational, the a n are rational then they are exist a positive rational number K such that mod a n less than equal to k. So, every convergent sequence is a bounded sequence basically this is the (( )).

The reason is very simple a n is given to be convergent so since sequence a n is convergent. So, for a given positive a small rational number epsilon there exist an integer n naught such that mod of a n naught minus a n naught plus p is less than epsilon, for p is 1, 2, 3 or you can say a n naught plus p will remain less than a n naught plus epsilon, for all p 1, 2, 3 for all p 1, 2, 3. It means the a n's are less than this number because a n naught is fixed.

Now, so a n naught plus epsilon is fixed, so this imply that is the, all terms, a n this term will remain less than less than n number, alpha remain less than a number alpha, for all n greater than or equal to n naught. Is it not where all these term are like this. So what all the left now, consider K is the maximum of maximum of a naught, a 1 a 1, a 2 and a n naught take the positive way, absolute values and then alpha also, if I take the K is the maximum then obviously this will give you.

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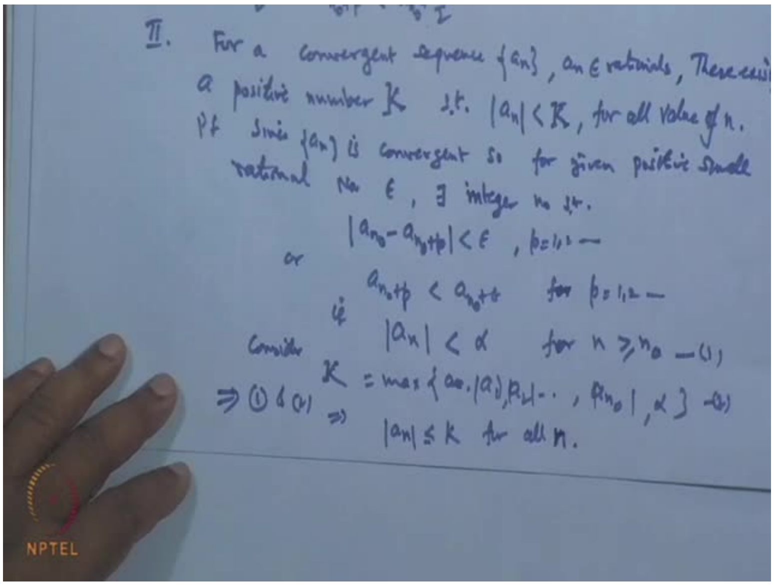

So, this will implies, using the first and second we can say a n will remain less than equal to K, for all K for all n. That is mod a n is less than K for all k. So, this proves the design this is a bounded sequence. Every convergence you can of rational is bounded, in fact it is for real's also the result is true so that is what we get it. Now, there are certain conditions for this sequence. Now, one more small result is there of course, this result is third.

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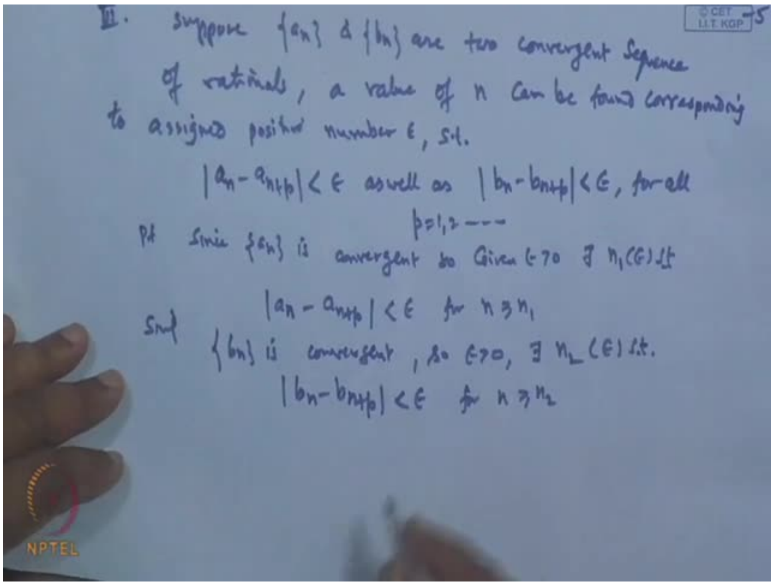

What this result says is suppose a n, b n are the two convergent sequence of rational suppose a n and b n are two convergent sequence of rational's, number, of rational's, two convergent sequence of rational's number. Then a value of a, then a value of n can be found, can be found corresponding to corresponding to corresponding to an assign, an arbitrarily small positive number epsilon, corresponding to assign positive assign positive number epsilon.

Such that both these difference, such that mod of a n minus a n plus p is less than epsilon as well as mod of b n minus b n plus p is less than epsilon, b n and b n plus n for all positive integer for all p 1, 2, 3 and so on. All positive integer value p, we say common of n can be obtained and the reason is very simple. When a n, b n are the two so a n is convergence. So for given epsilon greater than 0 there exist an n 1 depends on epsilon, such that mod of a n minus a n plus p is less than epsilon, for all n greater than equal to n 1, is it not. For all n greater than equal to n 1 onward, that is n 1 is fixed. So, basically this we are getting n 1 all n greater similarly. Sequence b n is convergence so for the same epsilon greater than 0 there exist an n 2 depends on epsilon. Such that modulus of b n minus b n plus p remains less than epsilon, for all n greater than n 2 and p of course, 1, 2, 3.

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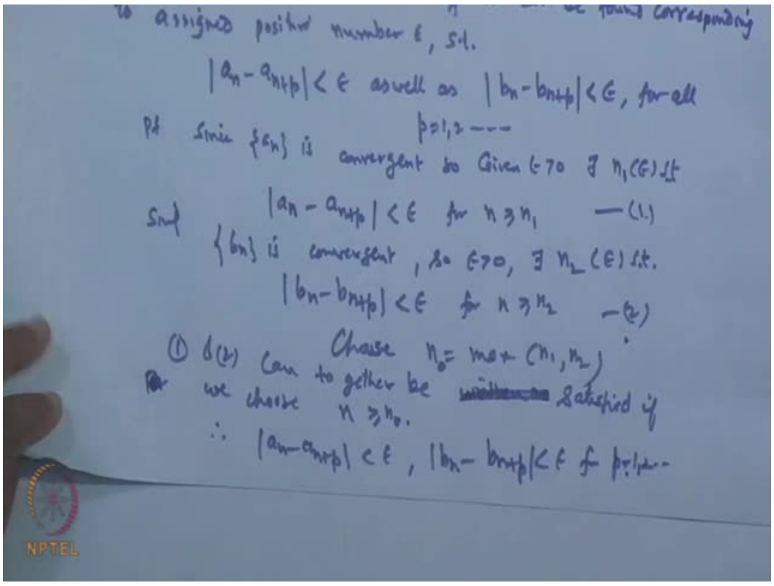

Then choose n to be the maximum of n 1 and n 2 or say n naught, maximum of this. Then what happened 1 and 2 can it not be combined, so if 1 and 2 can together be written like this as that for a given epsilon 1 and 2 together can be written as or together be satisfied as a, be satisfied. If we choose all n greater than equal to n naught because this is 2 for n is greater than n 1, n naught is greater than n 1. So, for all n greater then n naught is also true, this is also n is greater than n naught. So, if we take n greater than n naught then both the conditions, what the 1 and 2 in equations are satisfied.

Therefore, we get this result. Therefore, mod n minus n plus p remains less than epsilon as well as mod of b n minus b n plus p is less than epsilon, for all p 1, 2, 3. That is what he said, so this. So these are few properties of the convergent sequence of rational numbers. Now, before going for the convergent sequence, what are the, we also develop some criteria to just whether the given sequence is a convergent or divergent.

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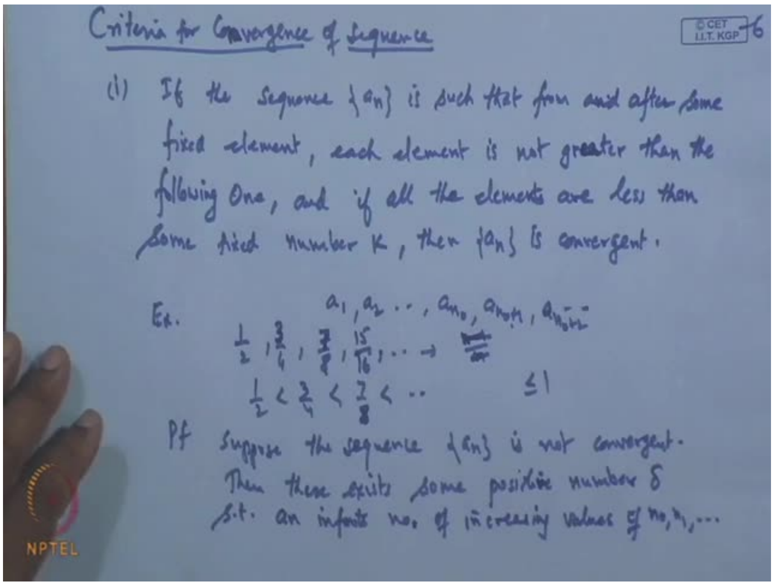

So the criteria for the convergence, the criteria for testing the convergence of sequence of rational numbers, let it be we are all dealing with the rational first. So, we do not, if the sequence a naught a first criteria a, if the sequence a n is such that from and after such that from and after some fixed.

Some fixed element, each element is not greater than not greater than, the following one, is not greater than the following one and if and if all the elements are are less than some fixed number K, some fixed than sequence a n is convergent sequence, convergent. So, what this says is if a sequence a n is such that from and after some fixed element, each element is not greater than the following one, each element is not greater than the following one. It means that after say, a n naught. If I take sequence a 1, a 2 a n naught a n naught plus 1 and so on.

This is our sequence and suppose after this n naught, what he say is each element is not greater than the following one, means a n naught is not greater than a n naught plus 2, it means a n naught is less than a n naught plus 2, a n naught is less than a n naught plus 2 which is less than a n n naught plus 3 and like this and if all the elements are less than some fixed number K if all the elements are less than K, then the sequence will be convergent. For example, suppose I take this, say half, 1 by 4, 1 by 8 like this. Now, this sequence is such that each element is not greater than the following one is half. So this is not greater than so four is four is not greater than, no it is wrong.

Now, so we will take no. We should take half, 3 by 4. Let us take 3 by 4, then 7 by 8 and then 15 by 16. Let us take this example, if we take this example then what happen is the first term is half, second term is 3 by 4. So, 4 and this 6, it means this is like this. It is half, is less than 3 by 4, 7 by 8, is it not less than like this like this. So, each term element is not greater than the following one. This is not greater than this number and like this and all the terms of the sequence are bounded by some number k. All the terms are less than 1, you see because this sequence is basically what n minus 1 by n type.

In fact it will not be like this, n minus 1 because the difference is something, coming to be 1, 2, 4, 6, is 2 n 2 to the power n and here it will come something else, but the denominator is low than the 1, term lower than the numerator, the numerator is 1 term lower than the denominator, 1 unit. So, we are getting this and each will dominate less than equal to 1, so if this sequence will converge and converge to 1. Now, proof is like this, why it is. So, since a n sequence is such that from and after some term each element is not greater than the following 1 and if all the elements are less than fixed number then an convergent.

Suppose this is not convergent, suppose the sequence is a n is not convergent, it is a divergent sequence. Then we will lead, it will lead a contradiction. So, if it is not convergent then there exist are, then there are exist some positive number there exist some positive number delta. Such that such that an indefinite number of the increasing values such that an infinite number infinite number, of infinite number of increasing values of n of infinite number of increasing values of n naught, n 1, n 2 and so on can be found can be found.

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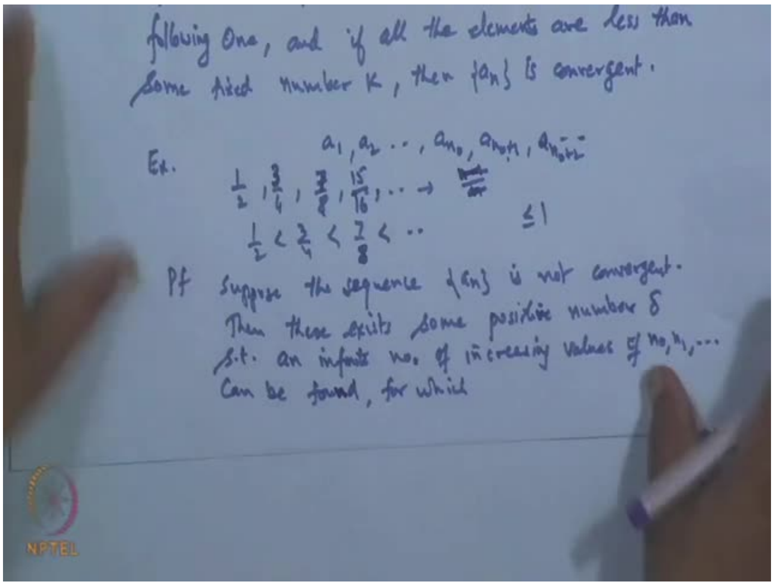

For which the difference mod a n 1 minus a n naught is greater than or equal to delta mod a n 2 minus a n 1, is greater than equal to delta and so on. So, let us see why it is. So, when the sequence is convergent, so according to the definition a sequence is convergent means the difference between a n and n plus p is less than epsilon means whatever the house able small epsilon is choose, one can always identify n 1.

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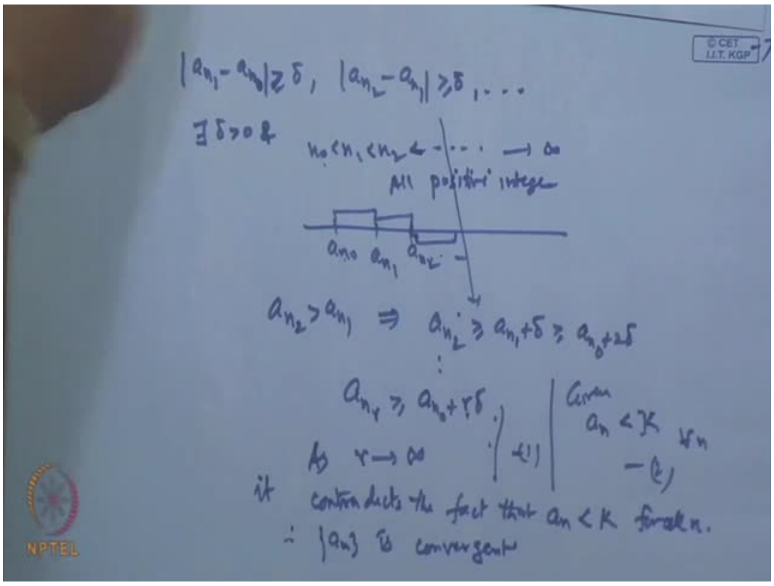

Such that from this term onwards the difference can be made as a small, as we please. If it divergence divergence means that the difference should not be, should not go to 0. It keeps on increasing, if the difference keeps on increasing then the sequence will be a diverging sequence. So, here we what we are saying if suppose this is not a convergent. So, let us see the contradiction part of it, so what we are saying is that we can find out a number delta for a given, some number. Delta can be identified and a sequence n 1, n 2, n a number delta and a sequence n 1, n naught, n 1, n 2 and so on. Such that which is increasing nature n 1 is less than n 2 less than n so on and it goes to infinity of all the integers, all positive integers.

So, there exist a delta greater than 0 and a sequence of and positive integer, which you have increasing nature. Such that corresponding to this sequence, the terms corresponding to these term, we are having the term a n 1 minus a n, a n 1, a n naught, a n 2 etcetera. If I take the difference between these 2, a a n naught, a n 1, a n 2 and so on. If I take the difference of these two difference of these two and like this difference of these two. These should be greater than equal to delta, it means the difference between two terms can never go to 0, because it is always will be greater than equal to delta. So, that is why that sequence will not be convergent, it will be a diverging one.

So, criteria for diverging is that if a sequence diverges we can also write in a form that there exist a positive number delta such that the, there will be sequence and not increasing sequence of integer, such that the difference, consecutive difference of the term will be greater than equal to that positive number delta. So, if it, so then what happened is. If suppose n naught, n 1 and n 2 are the corresponding n 1, a n naught etcetera. If a n 1 is greater than a n naught, a n 2 is greater than a n 1. Say, then from here we can say a n 2, let it be greater than a n 1. Then what happens is, we can write this thing as from here we can write a n 2 is greater than equal to a n 1 plus delta, but a n 1is also greater.

So, this is further greater than equal to a n naught plus 2 delta, like this. So if I continue this then a n r is greater than equal to say, first term, a n naught plus r delta this we get it, but what is given is the given is that a further it is given if all the elements are less than the fixed number, K means all the elements of this this is given all the elements of a n are less than the fixed number K are less than some fixed number K for each n this is given, but here we are getting a n r is greater than a n naught plus r plus delta a naught plus r into delta.

Now, if I choose r to as infinity or tend to infinity then this a n naught can exceed any value, whatever. Suppose I take K then it will definitely exceed this a n r will definitely exceed K because of the term of the r is going to infinity. So, this value can be as large as we can, as your imagine so this will be all violated.

So, this condition condition one and condition two are contradictory. So, as r tends to infinity it contradicts it contradicts the fact that a n's are less than K for all n and this contradiction is reached because our wrong assumption that sequence should not convergent. Therefore, our sequence a n is convergent so that is why. So, this is the first criteria that if the terms of the sequence are basically all of increasing in nature, this is the increasing in nature these terms are increasing in nature, but it is dominated bounded upper bounded by some number K. Then such a sequence will be a convergence sequence so in other words also we say.

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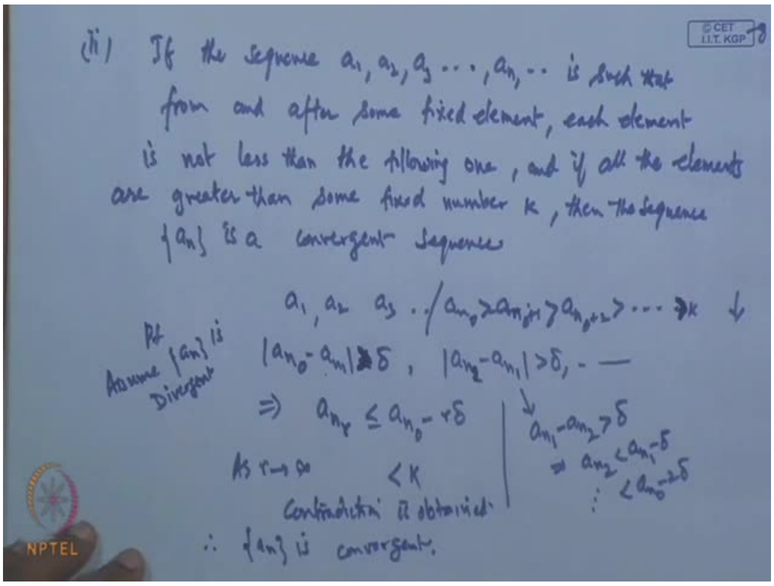

A sequence which is a monotonically increasing sequence and bounded sequence than it must be a convergent sequence. So, that is what the second criteria says. If the sequence if the sequence a 1, a 2, a 3 and so on, a n is such that is such that from and after some and after some fixed element fixed element each element is each element is not less than each element is not less than the following one following one and if all the elements are all the elements are greater than some fixed number small k, is small k then the sequence is, then the sequence a n is a convergent sequence.

So, what is this sequence, but we are getting is if we have a 1, a 2, a 3, a n etcetera. These are the terms of the sequence and this satisfy the condition from and after some fixed element, say a n naught. Each element is not less than following one, it means a n naught is greater than a n naught plus 1 greater than a n naught plus 2, greater than and so on which after this the terms are of decreasing nature, this is of decreasing nature and not only this. If they are decreasing nature, but all the element of this all are greater than some numbers, mean they are not decreasing. They are decreasing to some number greater than or equal to to K, greater than some fixed a small k, greater than the small k the terms all of decreasing, but it will decrease and it will not cross the k.

So if all the terms, all are after certain stage are of decreasing nature and after certain stage if all the elements are greater than some fix number then such a sequence will be a convergent one. So, it is basically lower, the sequence is bounded below when n is sufficiently large is bounded below by K because the terms are not going beyond this, lower than this. So it is bounded below by K, by and why it is after some term onward because these first few terms, they does not affect the convergence or divergence of the sequence.

If the limit of the sequence is convergent that is the limiting value is some finite quantity then whether these first few terms are very large or maybe the infinity, we would not care for it. Because as n is sufficiently large the behavior of the convergent sequence is tested so that is why the first few terms, even they does not behave which does not satisfy this criteria maybe the few term a 1 is less than a 2, a 2 is greater than a 3. Hardly matters, but after certain stage if all the terms of the sequence follow this criteria either they are decreasing or they are increasing. If it is decreasing than all these terms are also be should be greater than K and if it is increasing than all sub will be bonded by k.

Then in the both the case the series will be convergent and the proof is just by follows on the previous sign we have seen that a n naught a n naught minus a n 1 is less, is greater than assuming that this is diverging, assume sequence a n is diverging, divergent. Then there exist a delta and a sequence of the integers n naught n 1 into n, increasing sequence such that a n naught minus a n 1 is greater than delta. a n one minus a n or a n 2 minus a n 1 is greater than delta and so on. So, later down equal to high, so from here if a n 1 is greater than a n naught or a n 2 is greater than a n 1 etcetera.

We write from this say negative sides, so we… What we can do is we can put it a n r, can be said that a n r. This can be written as less than or equal to a n naught minus r r delta. Why because a n naught is greater than a n 1 a n 1 is greater than a n 2. So, from here if I take, the reason is because from here if I take we get a n one minus a n 2 becomes positive, because a n 1 a n a naught is greater than a n 2. Is it not, so a n 1 is greater than a n 2. So, a n 1 minus a n 2 will be positive so this is greater than delta. So this shows a n 2 is less than a n 1 minus delta. Similarly, here if I take substitute a n naught then this will come a n naught minus 2 delta and continue this we get this term.

Now, as r tends to infinity then this number r delta will be very, very large. When r is sufficiently large this number is very large a n naught is a fixed rational number. So, when you sub take from a naught a very large number than obviously the quantity will definitely exceed by K, cannot be greater than K cannot be greater than K. It will always be less than K for large enough. So, it will contradict the whole thing that all the terms of the sequence are greater than K, which is not satisfied. Therefore, a contradiction contradiction is obtained and this contradiction is because our wrong assumption, that is. Therefore, the sequence a n is convergent, so this criteria and the sequence which I already given one example the, in fact that was the example for this. Half 1 by 4, is it not. 1 by 8 and so on.

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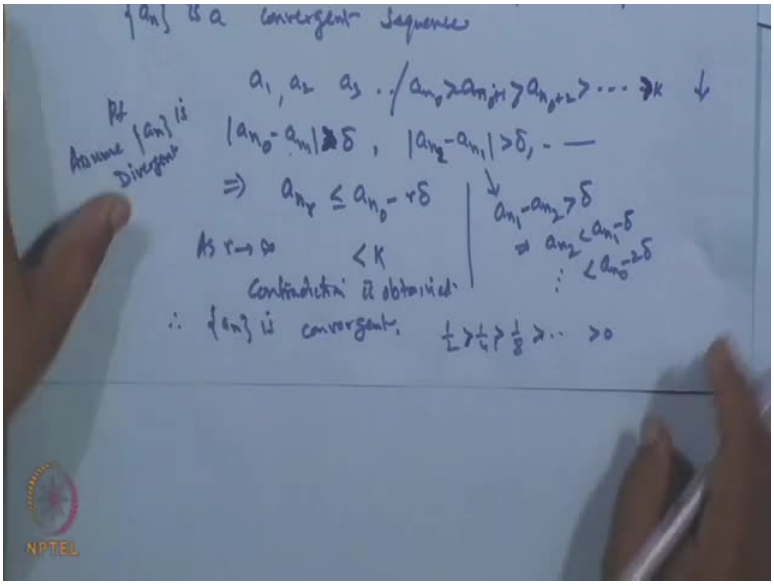

So what happen is this half is greater than 1 by 4, greater than 1 by 4 is greater than this and like this, clear. So, this will be like that and this will be, always be greater than 0 this is always greater than 0, some positive quantity, so which convergent, this one. Now, once we have the concept of the convergent sequence then similarly, just like a cards. We have already added the 2 cards and how to combine these, multiply the two cards. Here also we can have this similar type results the addition, subtraction, multiplication of this.

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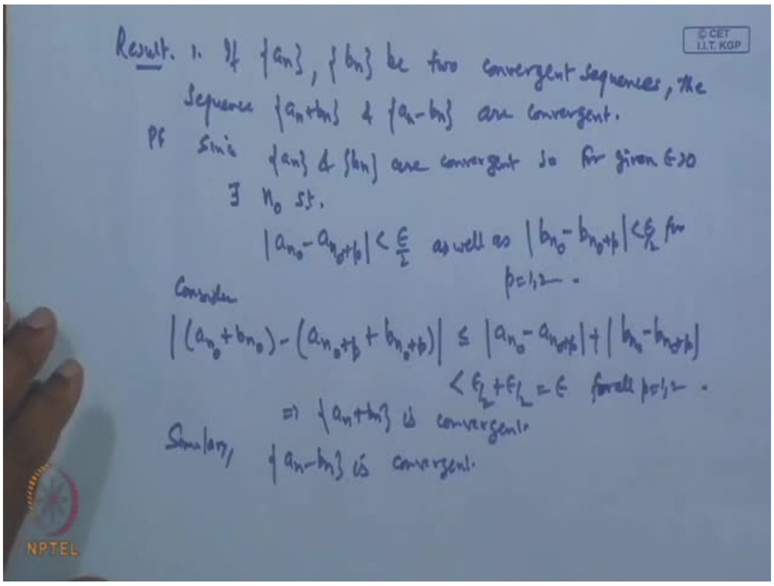

So few results which will be used, the first result is if a n and b n a n b n with two convergent sequences. Convergent sequence the sequence a n plus b n and a n minus b n in which the elements of the summary are also convergent, are convergent and this proof is very simple, just we wanted, the a n plus b n minus b n to be convergent. So, what we do is since a n b n are convergence.

Since sequence a n and b n are convergent so for a given epsilon greater than 0 there exist an n naught such that mod of a n naught minus a n naught plus p is less than epsilon by 2, as well as mod of b n naught minus b n naught plus, p less than epsilon by 2, for p equal to 1, 2, 3; because we can find a common term, that is we have seen. Now, consider consider mod of a n plus p, a n plus b n naught minus a n naught plus p plus b n naught plus p.

Now, this will be less than equal to mod a n naught minus a n naught plus p plus mod b n naught minus b n naught plus p and this is less than epsilon by 2, this is so this is less than epsilon by 2. So, this is less than, so this is less then epsilon, for all p 1, 2, 3. Therefore, the sequence a n plus b n is convergent.

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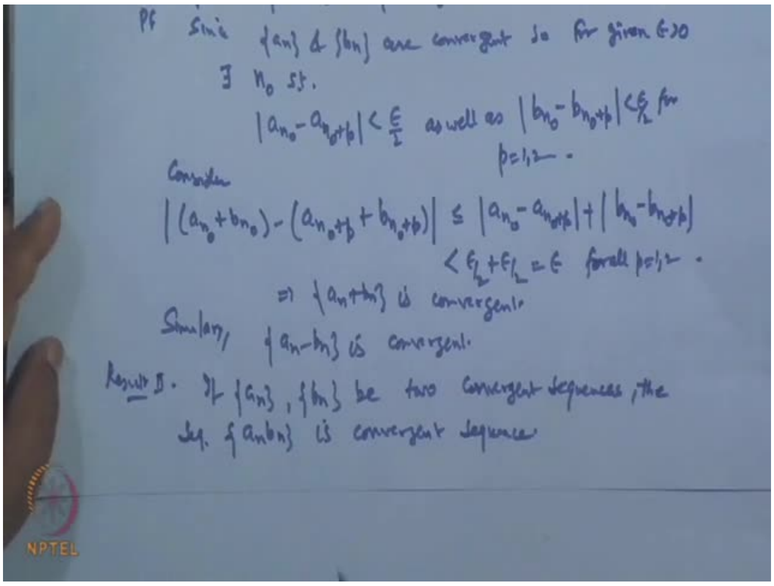

Similarly, we can show that a n minus b n is also convergent. In a similar way, the second result is about the product if a n, b n are, if a n, b n are be the 2 convergent sequence then if a n b n be 2 convergent sequences then the sequence a n into b n a n into b n are also, is also convergent sequence, is convergent sequence. Similarly, when you go for the division, if a n by b n, will also be provided the b n is greater than some number K. So, proof we will see next time, thank you, thanks, clear. So, let us stop it here and the prove, we will go next time.

Thank you very much.