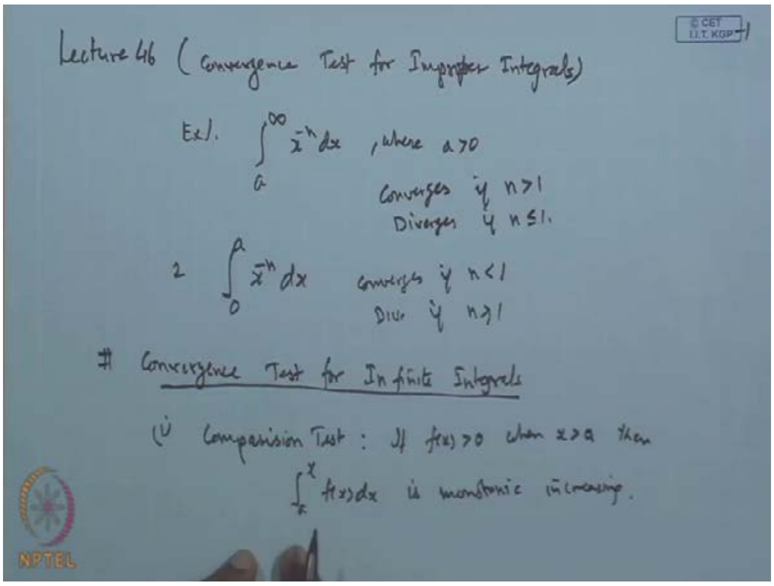
A Basic Course in Real Analysis
Prof. P. D. Srivastava
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 46
Convergence Test for Improper Integrals

Today, we will discuss the few tests, which will decide about the convergence or divergence of the improper integral.

(Refer Slide Time: 00:30)

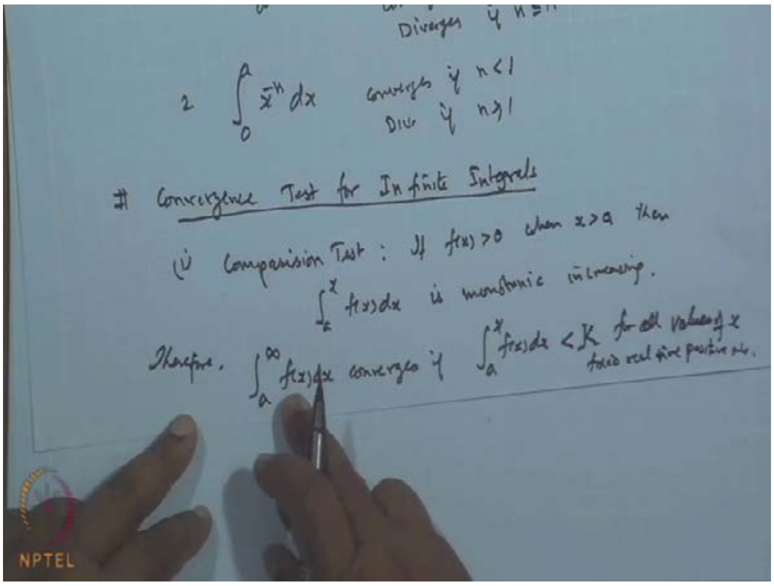


We have already discussed these two examples. The first example which we have discussed a to the power a integral a to infinity x to the power minus n dx. This is where a is greater than 0. This is improper integral of the kind one, and we say this integral converges converges, if n is strictly greater than 1 and diverges if n is less than equal to 1. So, this we have seen as an example and it will be used for future work.

Another example which we have seen, if the function is not defined at the end points, that is function becomes unbounded at the either a or b, then in that case, we can also have a convergence test and for which this example will be helpful. So, we have seen this example 0 to a x to the power minus n dx where the function becomes undefined; is not defined at 0. Hence this is an improper integral of the kind two, and we have seen this integral converges if n is less than 1 and diverges if n is greater than equal to 1. So, these two examples we have seen, and then we wanted to generalize this (( )) particular this result for a for a improper integral to test whether it is convergent or not. So, this example will be used.

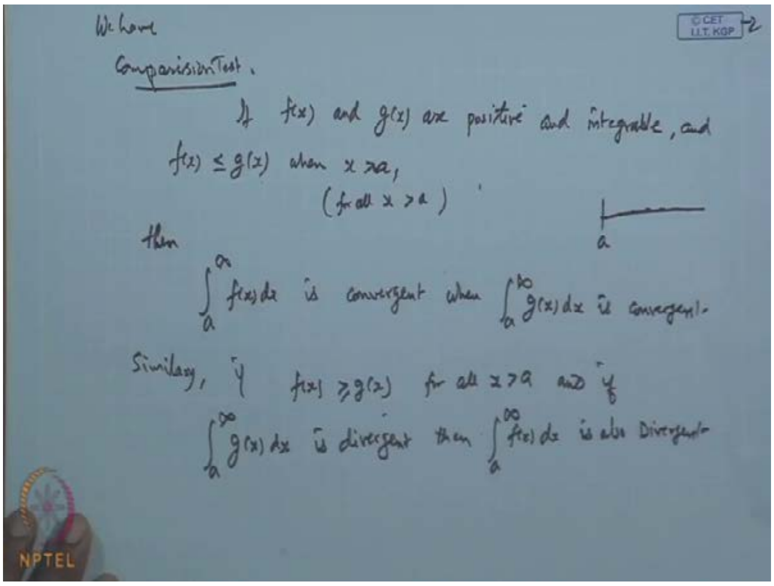
So, first we will go with the convergence test for indefinite integral sorry for infinite integrals. So, the first test we say is a comparison test. The comparison test, say we know, if fx is greater than 0, when x is greater than a, then this integral a to x fx dx is monotonic monotonic increasing is monotonic increasing. That means when x increases, the corresponding value of this integral will increase. Therefore, if we say that this integral converges; this gives an idea.

(Refer Slide Time: 03:17)



Therefore, we say the improper integral a to infinity fx dx converges if the integral a to x fx dx remains bounded by k; that is the value is always be less than k for all values of x, values of x, where k is a fixed number; this is a fixed real number; some positive number positive number. So, if this is true, then we say this integral converges because this is a monotonic increasing function and for each x this is bounded by k is less than equal to dominated by k; k is a fixed. So, the value of this integral will be a finite quantity. So, it is convergent and this suggests the test which is known as the comparison test.

(Refer Slide Time: 04:20)

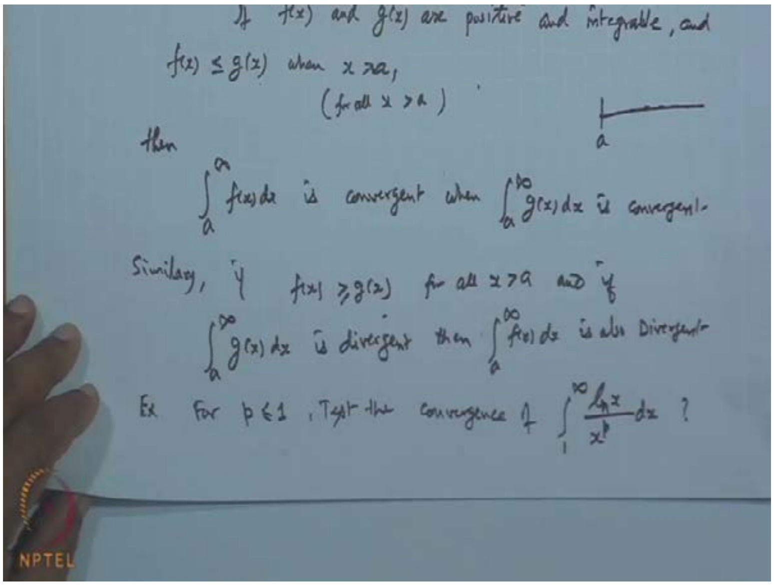


So, we have this test which is known as the comparison test. What this says is if fx and gx are positive are positive and integrable, and fx is less than equal to gx throughout the interval, when x is greater than or equal greater than a. It means from a onward whatever the point x is, this condition is satisfied for all x; this is true for all x; even this is true for infact all x is strictly greater than a. Then the behavior of this integral, then we say integral a to infinity fx dx, this integral is convergent when integral a to infinity gx dx is convergent. It means when the function f and g are two functions which are positive and integrable, and if the right hand side integral of gx a to infinity converges, then left hand side integral a to infinity fx dx converges.

It means if we want to test the nature of the convergence of this integral and we are aware to construct a function g for which this condition is satisfied from the interval x greater than a, then convergence of this integral will imply the convergence of this integral. Similarly, if suppose if fx is greater than equal to gx for all x greater than a and if integral a to infinity gx dx diverges, is divergent, then the integral a to infinity fx dx is divergent, is also divergent. So, this is known as the comparison test. It means with the help of the function f, if we are able to get g such that either this inequality holds for all x or this inequality holds for all x, then the behavior of the integral a to infinity fx dx can be judged and it will depend on the inequality.

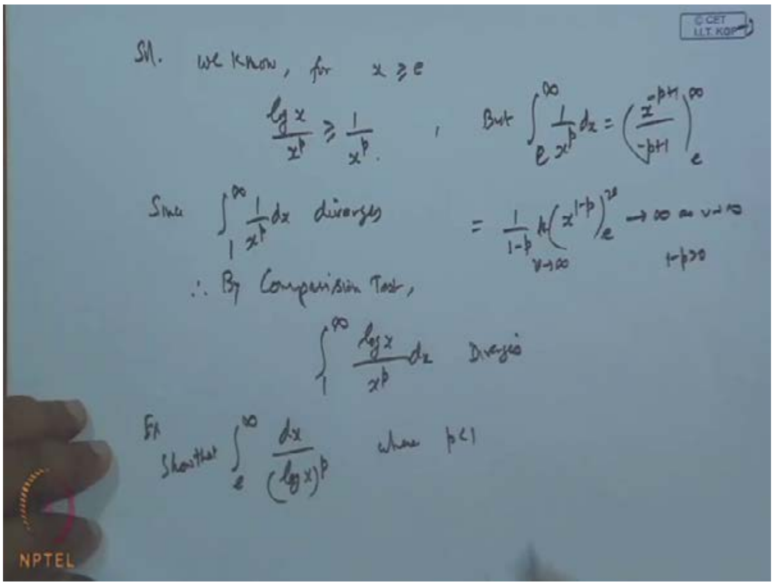
If this inequality holds then if this integral is convergent, then this integral will converge. Similarly this inequality hold and this integral will diverge then this integral will also diverge for it. So, that is what we get it from here.

(Refer Slide Time: 08:00)



Another test which we have or let us take an example here, that is better that if we take say for example, yes suppose, I take example p is greater than p is less than equal to 1, suppose I take for p is less than or equal to 1, test the test the convergence of convergence of the integral 1 to infinity log x l n x divide by x to the power p dx convergence test the convergence for this integral; p is less than equal to 1.

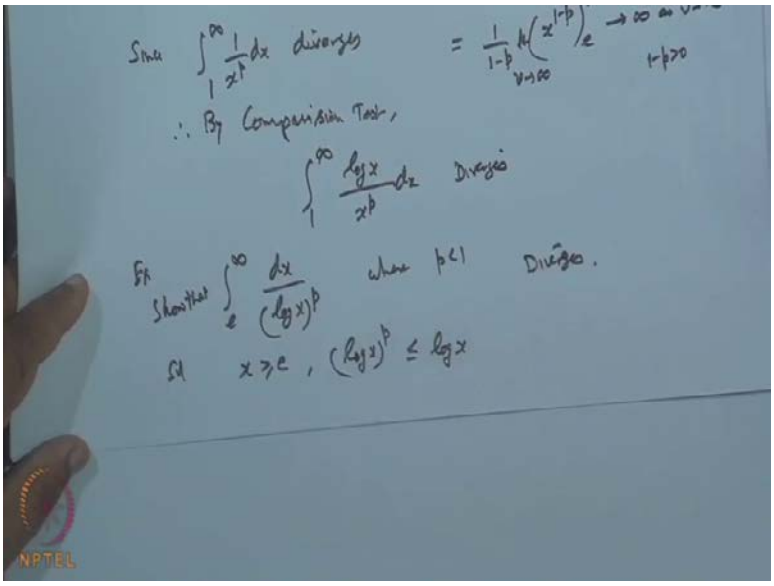
(Refer Slide Time: 08:52)



Now, we know, for x we know for x greater than equal to e log of x log of x divided by x to the power p will be greater than equal to 1 by x to the power p; 1 by x to the power p, but integral 1 to infinity integral 1 to or e to infinity e; I will say e to infinity one upon x to the power p d x; p is greater than less than equal to 1. This is what? this is nothing but x to the power p minus minus p plus 1 over minus p plus 1 e to the power e and infinity. So, it will go to, when you take p, 1 minus p, it is negative. So, we get from here is this will be equal to 1 by 1 minus p and then this is x to the power 1 minus p; p is negative. So, it is basically e to v and then take the limit v tends to infinity; is it not. So, it will go to infinity as v tends to infinity because this is positive; 1 minus p is positive; p is less than or equal to 1. So, this is infinity. So, it means this integral will diverge. And 1 to e is finite sum; so obviously, 1 to e does not contribute; this diverges then the total thing will diverge.

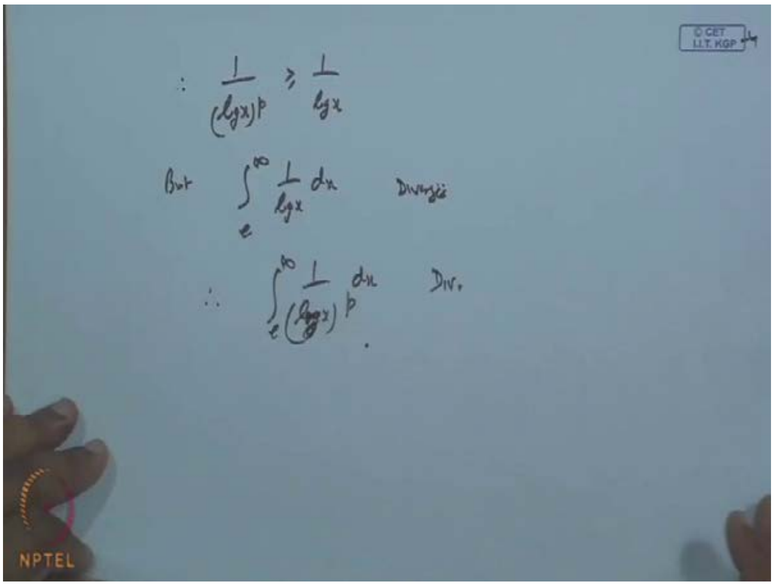
So, the right hand sides, so since the integral 1 to infinity 1 by x to the power p dx diverges. Therefore, by comparison test comparison test integral 1 to infinity l dx or logx divide by x to the power p dx diverges. So, this way, we can find out. Then another one example is also, suppose I take this one, say another example if we take, say let us take e to infinity e to infinity log x to the power p; integral e to infinity e to infinity d x by log x l n x to the power p, where p is less than 1 so that this integral diverges diverges.

(Refer Slide Time: 12:13)



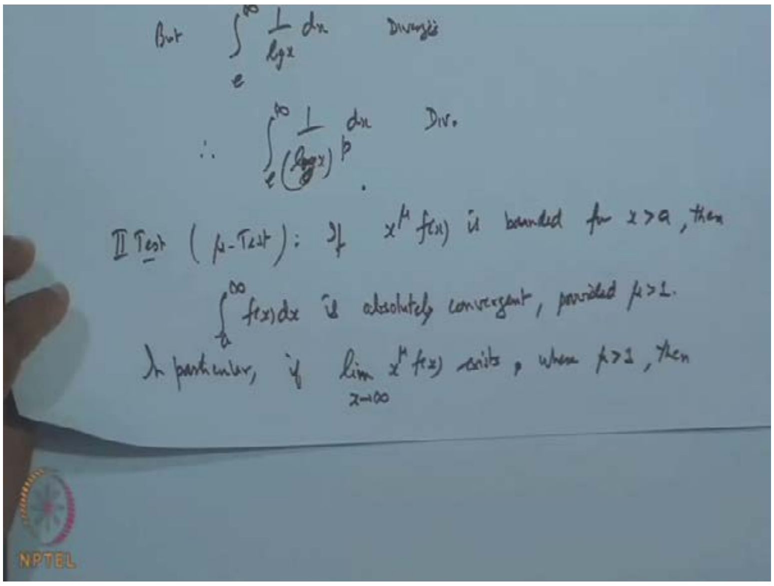
So, again we see here solution. When x is greater than equal to e, the log of x to the power p is less than equal to log x is less than equal to log x. Therefore, why? Because the x is greater than equal to e log of x to the power p is lying between less than 1. So, it will be positive quantity; log 1 is greater than 0 of course. So, it is positive and then for p, p is less than 1. So, we get the log x to the power p is less than log x.

(Refer Slide Time: 13:03)



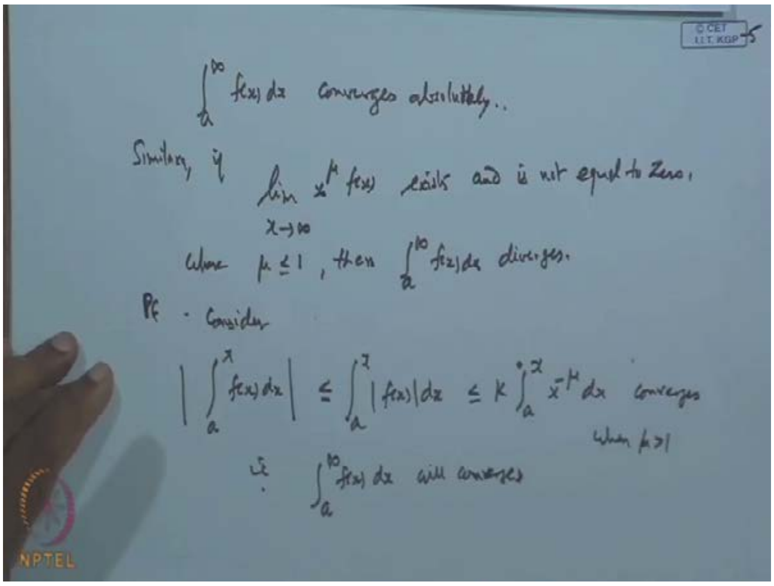
Now, 1 upon this thing log x to the power p, therefore 1 by x log x to the power p is greater than equal to 1 by log x. But what is the 1 by log x, but integral e to infinity 1 by log x 1 by log x and that is integral diverges; is it not. So, this diverges, d x; it diverges; so this integral will diverge. Therefore, it diverges; therefore, this diverges; e to infinity to the power d x diverges. So, that is what the comparison test will be very useful, but only thing is we have to identify the relation. If we are unable to identify such a relation, either fx is less than equal to 0 or fx is greater than equal to gx throughout the interval, then only we can decide about the nature, otherwise not.

(Refer Slide Time: 14:11)



The second test is mu test. What basis is, if x to the power mu x to the power mu fx is bonded is bounded for x greater than a, then then integral a to infinity fx dx is absolutely convergent absolutely convergent, provided mu is greater than 1 and diverges, and if if let us, or we can say in particular, in particular if the limit of this, limit of x to the power mu fx when x to tends to infinity exist exist when mu is exist, where mu e is greater than one.

(Refer Slide Time: 15:51)

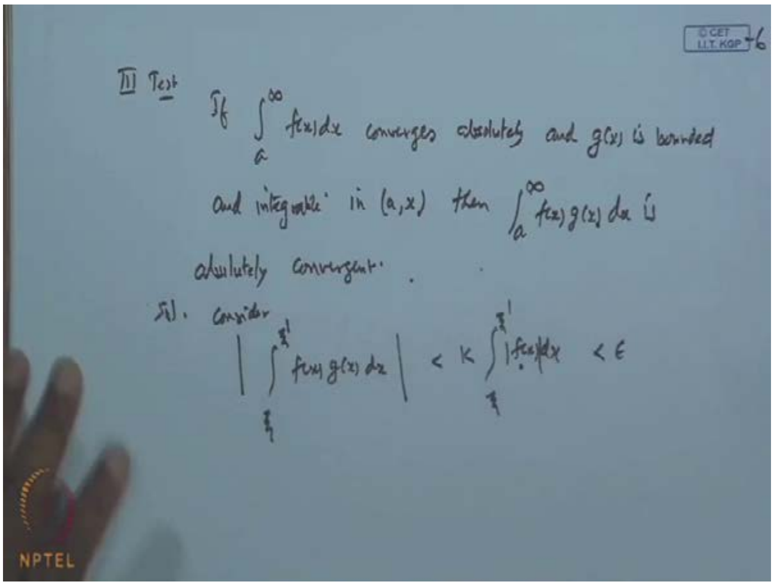


Then the integral a to infinity a to infinity fx dx converges absolutely absolutely absolutely converges absolutely. Similarly, similarly if limit of this x to the power mu fx when x tends to infinity exist and is not equal to and is not equal to 0, different from 0, where mu is less than or equal to 1, then integral one a to infinity fx dx diverges. of course, proof is simple.

What we knew is, consider this modulus integral a to x fx dx. Now this is less than equal to a to x mod of fx dx. Now, what is given is that if x to the power mu fx is bounded, so modulus of this is less than equal to say k bounded; that is, if this is less than equal to k, that is x to the power mu or mod x to the power mu fx is less than equal to k; k is some constant, for some constant k.

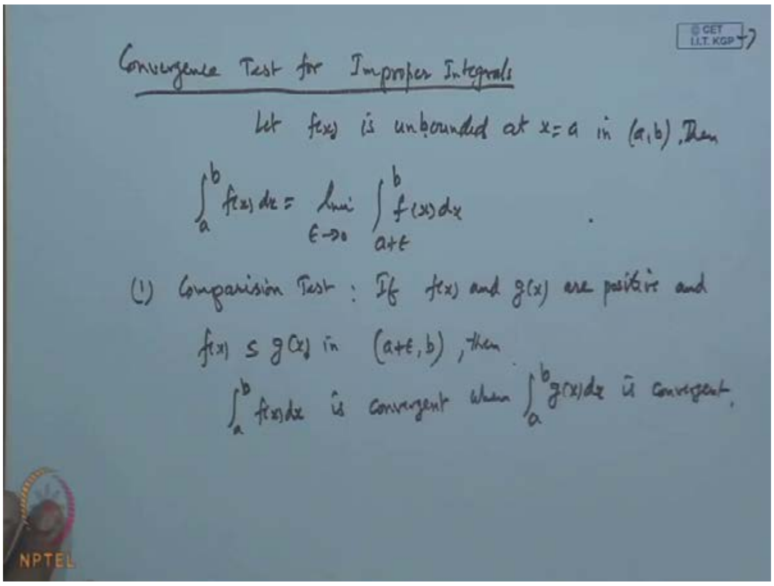
So, we can write mod of fx is less than equal to k times integral a to a to x, x to the power minus mu dx; a to x x to the power minus mu. Now this integral is of the first type; integral a to infinity x to the power n type, minus n type. So, if n is we have seen that this integral will converge when mu is greater than 1. So, it converges; therefore, the integral converges; converges when mu is greater than 1; is it not? When mu is greater than 1, when x tends to infinity this that is this integral a to infinity fx dx, this integral will converge will converge will converge. Here, this integral converges; therefore, this integral will converge. So, we get that thing. Similarly, if we take the limit this in particularly limit of this exist and if greater than 1, then this integral converge absolutely. So, opposite of this will be the diverging one.

(Refer Slide Time: 19:14)



The third test, of course it is not very useful, but I will just simply say, if fx if integral a to infinity fx dx, this converge converges absolutely absolutely and gx is bounded gx is bounded and integerable and integerable, and integerable in the interval a to x, then for each x then integral a to infinity fx gx dx is absolutely absolutely convergent. Why? Because the reason is consider modulus integral xi xi dash fx gx dx necessary sufficient consider for the convergence of the integral. So, this is less than equal to k times integral xi to xi dash mod of fx dx because this is given to be bounded; so let it be bounded by k and this. Now, if this integral fx converges absolutely is given; so this is fine; less than equal to some quantity. So, this will be less than epsilon. So, this shows necessary less than epsilon and this shows that this is convergent epsilon. Now, these are the general case. In particular, we say the convergence of the all show case.

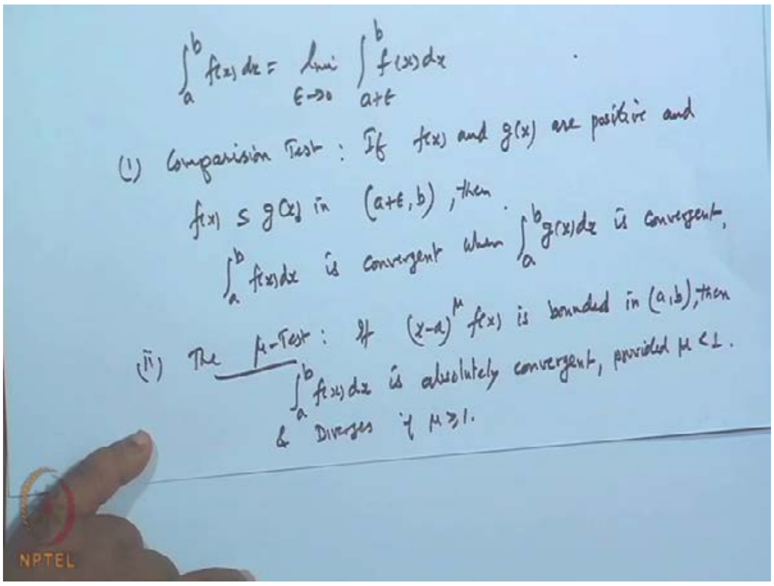
(Refer Slide Time: 21:19)



So, in particular, we can see the cases when convergent test for improper integral. This is a particular case. So, I suppose it is sufficient to consider. Let fx suppose let fx is unbounded, unbounded at x equal to a; improper kind two. Of course, x in the interval a b; then this integral a to b fx dx we can write as limit epsilon tends to 0 a plus epsilon to b fx dx and this if this limit exist, then we say improper integral of kind two exist or convergent. So, in instead of going for the limit, we go by the test and the first test is comparison test.

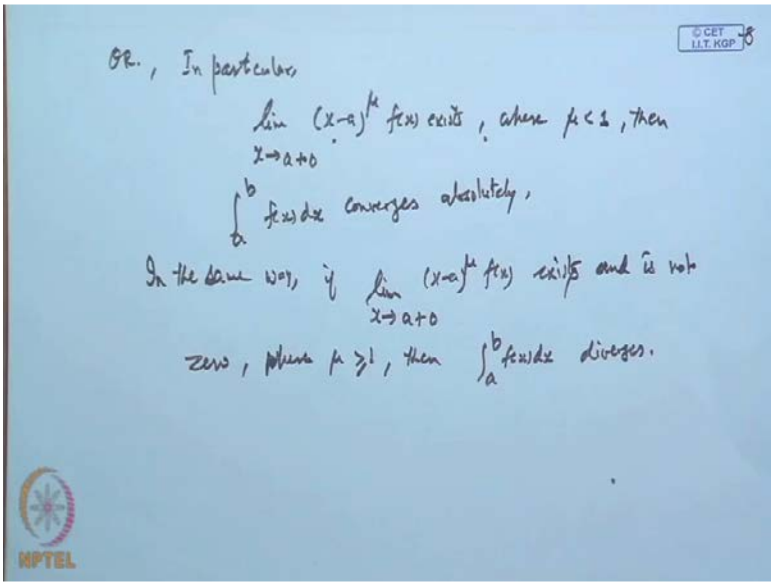
What this test says? If fx and gx are positive are positive and fx is always less than equal to gx in the interval a plus epsilon to b in this interval, then the integral a to b fx dx is convergent is convergent when the integral a to b gx dx is convergent. And the proof is just by the previous thing because if you take this thing a to b fx dx, then this is less than equal to gx and gx less than equal to a plus epsilon when epsilon tends to 0, this will converge; therefore, this integral will converge. So, it is not that.

(Refer Slide Time: 23:42)



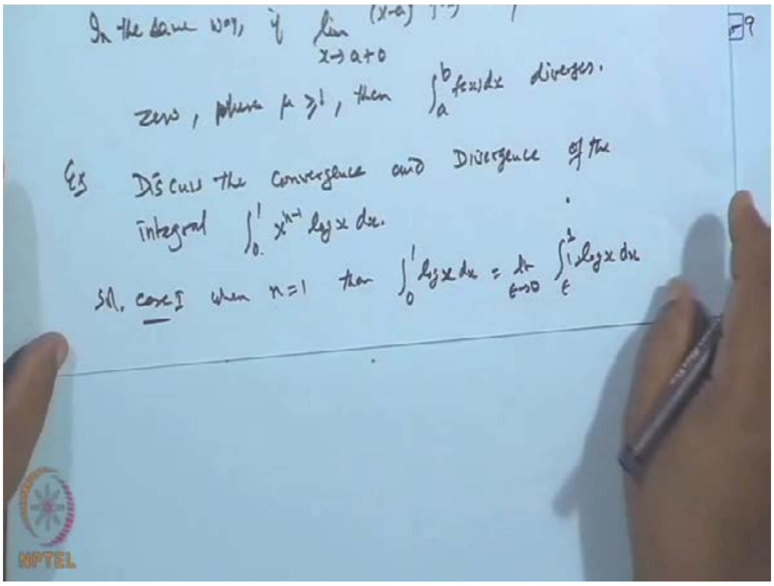
The second test is the mu test. The mu test, if x minus a to the power mu fx is bounded is bounded in the interval ab, then integral a to b fx dx is absolutely convergent, if provided mu is is strictly less than 1 and diverges and diverges if mu is greater than or equal to 1. So, this will be our …

(Refer Slide Time: 24:49)



Or equivalently we say in terms of the limit or equivalently or equivalently in particular if the limit exist, then if limit of this, as x tends to limit of this as x tends to a plus 0 x minus a to the power mu fx exist, where mu is less than 1, then integral a to b fx dx converges absolutely. And in the same way, in the same way, if limit of this x minus a to the power mu fx when x tends to a plus 0, fx exist and is and is not zero and is not zero, where mu is greater than or equal to 1, then the integral a to b fx dx diverges. So, these tests are very important and we will make use of these tests very frequently.

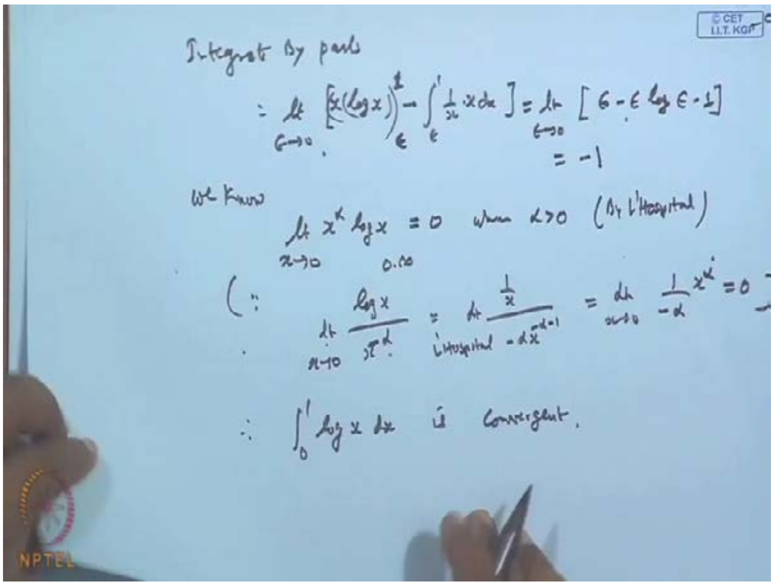
(Refer Slide Time: 26:33)



Let us see some problems, discuss the convergence, the convergence and divergence of the integral of the integral 0 to 1 x to the power n minus 1 log of x dx. Now, this integral if you see, at x equal to 0, the integral because n, if it is less than 1, then this will come in the denominator; so at x equal to 0, it is not defined; unbounded. Therefore, we can discuss is into two phases: when n is lying between 0, less than 0, when x is, n is equal to 1, and when n is greater than 1. These sequences will be there.

So, let us see the solution. Case one: When n is equal to 1: Just first, simple case first. What happened when n is equal to 1? This integral 0 to 1 reduces to 0 to 1 log x dx. Now, this we can write it as epsilon limit epsilon tends to 0 because when we take x tends to 0, it is going to be unbounded. So, let epsilon tends to 0, epsilon to 1 log of x dx integrate by part. So, when you integrate by part, we are getting the value of the integrate by parts.

(Refer Slide Time: 28:29)

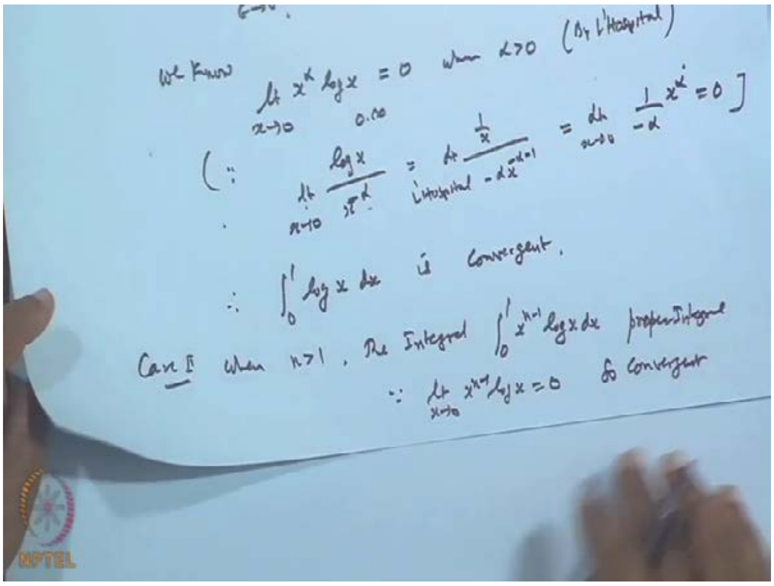


So, we get from here is limit epsilon tends to 0; first function integral of the second; first function integral of the second say first function is log x; integral of the second is x minus integral dc of the first; first is 1 by x integral of the second this, and then epsilon to 1 and like this. So, when you are again integrating, finally you are getting limit epsilon tends to 0; you will get the value to be epsilon minus epsilon log epsilon minus 1; is it not? That means epsilon 1, 1 minus and then we are getting to be this minus integral; this will be the first integral. So, here is x minus 1 and then plus epsilon and here epsilon log epsilon. Then first integral of the second; so you are getting this way; is it not? Because log 1 is 0 this is 0 to 1, I am sorry this is 0 to sorry epsilon to be 1. So, we get this now when epsilon is 0, this is 0.

What about this? This we know that x to the power alpha log of x when x tends to 0 is always be 0. Why? Where alpha is greater than 0 when alpha is greater than 0 because by using the roots; if we apply the root, you will get the immediately the things because this can be written as because this is 0 into infinity form; 0 into infinity form; so it can be written as logx divided by x to the power minus alpha and limit x tends to 0. Now, when you differentiate apply the root. So, logx is 1 by x; this will give minus alpha x to the power alpha minus minus 1. So, when you get from here is and limit x tends to 0. So, limit x tends to 0. This is minus alpha and then x to the power alpha will go up; otherwise a; so it is basically 0. So, this is always be 0; it means this is 0. So, the limit of this comes out to be minus 1.

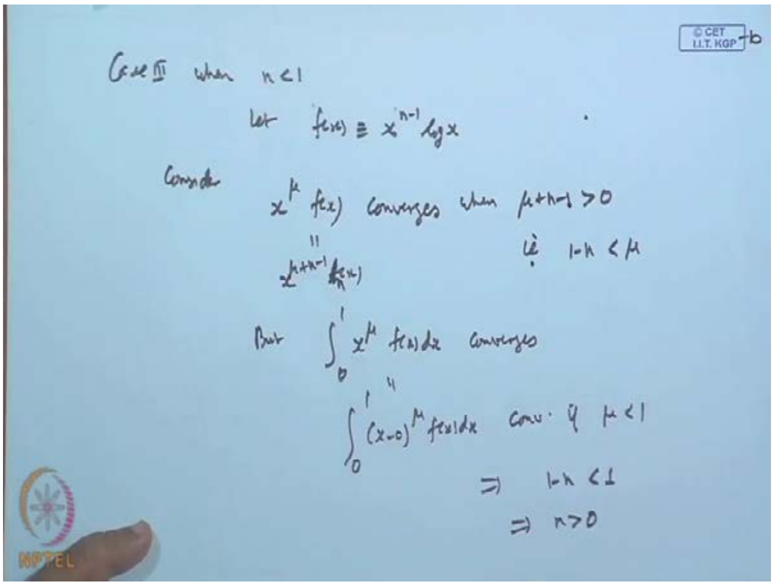
It means the value of this integral is comes out to be finite. Therefore, this converges. So, integral 0 to 1 log x dx convergent is convergent.

(Refer Slide Time: 31:47)



Now, case two: When n is greater than 1. Case two: When n is greater than 1, n is say lying between 0 and 1. If integral proper… So, when n is greater than 1, the integral integral 0 to 1, x n minus 1 log x dx. Now, here, the integrant when n is greater than 1, this basically comes in the numerator only. This comes to in numerator; therefore, the integral is proper. This integral is proper integral because the limit of this x n minus 1 log x as x tends to 0 is 0. So, this limit is 0; means, it is finite; does not go to unbounded. So, it is a case of proper integral. So, it has be convergent; so convergent.

(Refer Slide Time: 32:59)

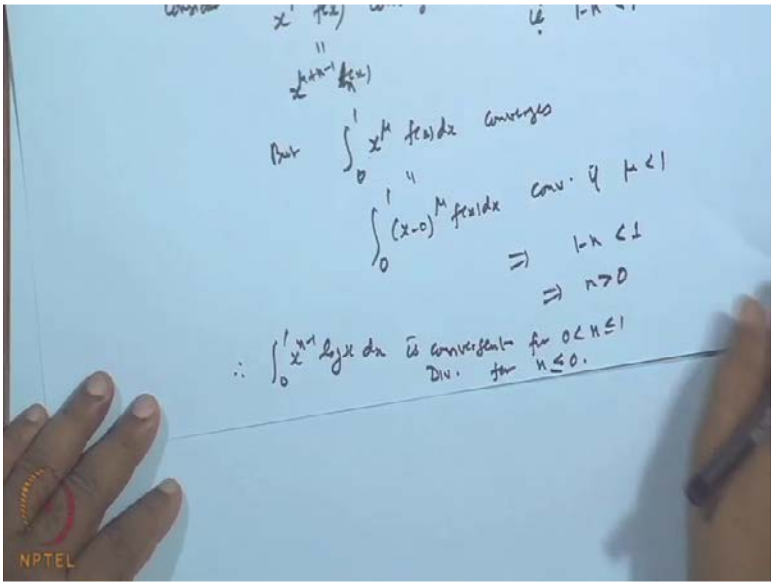


So, nothing to prove when n is less than 1, n is less than 1. Then let us denote let fx stands for x to the power n minus 1 into log x. Now, consider x to the power mu fx. Now, one thing is when you take x to the function fx, what is the result? If you remember the result was mu test. The mu test was, yes if this is bounded in the interval, bounded in a b, then this is absolute convergent provided mu is less than 1. So, we can use this criteria here.

So, when n is less than this, in this x to the power mu dx fx converges; converges when mu plus n minus 1 is greater than 0 because this is equivalent to what? This is equivalent to x to the power mu plus n minus 1 into fx; fx means log x; log x that is and we know the limit of this is 0 if this is greater than the positive quantity. So, the limit of this, it will converge. Therefore, this converges when mu plus n minus 1 is greater than 0. So, this implies that is the 1 minus n is less than mu, but integral converges, but this integral 0 to 1 x to the power mu fx dx.

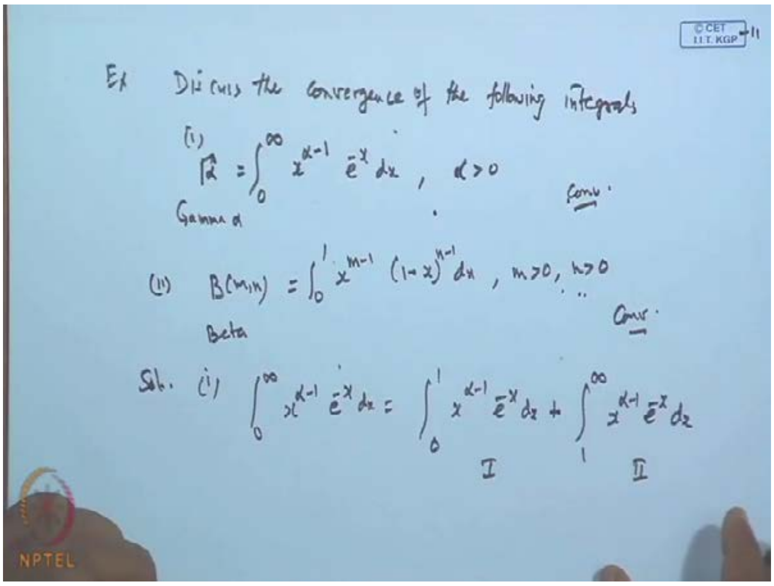
So, this is a bounded function. So, this integral converges by mu test because it is equal to the same as integral 0 to 1 x minus 0 mu fx dx. So, this integral converges if mu is strictly less than 1. So, from here, this show 1 minus n is strictly less than 1; this implies n is greater than zero. So, this integral converges when n is greater than equal to 0, greater than 0. Already we have seen when n is greater, lying between less than equal to 1 is convergent and diverges when n is less than equal to 0.

(Refer Slide Time: 35:37)



So, what we see here, therefore, integral 0 to 1 x to the power n minus 1 log x dx convergent for 0 less than n less than equal to 1 and divergent for n less than or equal to 0 because when mu is greater than equal to 1 it is diverging. When mu is greater than equal to 1 means n is less than equal to 1; so divergent. So, that is the result.

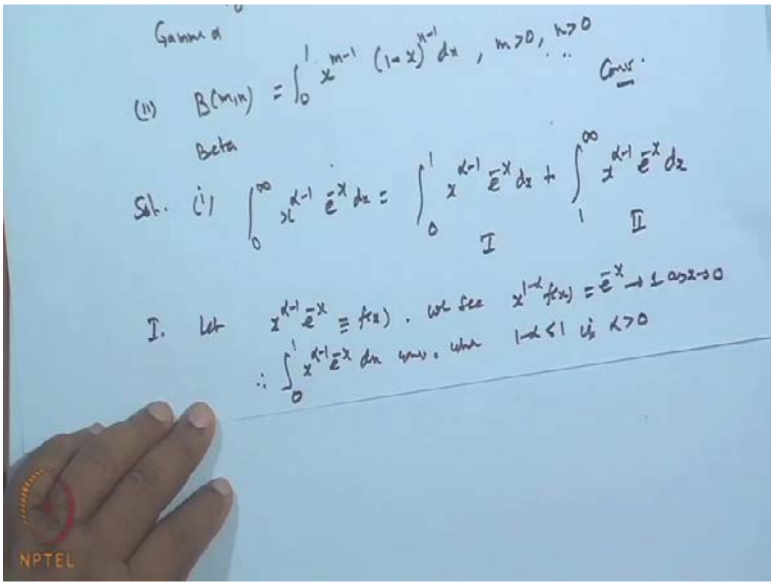
(Refer Slide Time: 36:12)



Another example, we will discuss discuss the convergence discuss the convergence of the following integrals following integrals. The first integral is 0 to infinity x to the power alpha minus 1, e to the power minus x dx and alpha is greater than 0. In fact, this is denoted by gamma alpha. So, we will use the later on, but gamma alpha. Gamma alpha second integral say is 0 to 1 x to the power m minus 1 1 minus x to the power n minus 1 d x when m is greater than 0 n is greater than 0. So, this integral we denoted by the gamma beta m n; m n is the beta function; beta m n - this converges here. So, this basically when alpha greater than 0, this will converge. When m is greater than 0, n is greater than zero; so basically this is convergent; this is convergent; this we will test it.

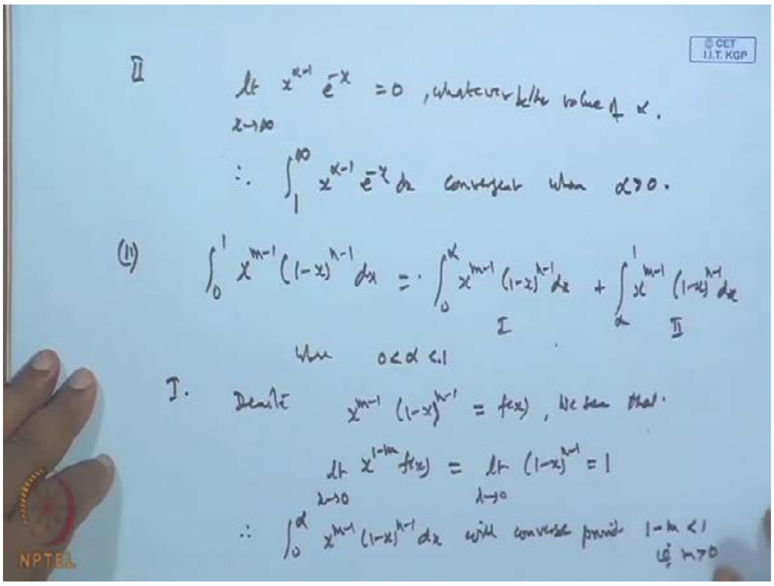
Solution is, first one: Now, we look this integral; alpha is greater than 0; so when alpha is lying between 0 and 1, this x will come in the denominator. So, it becomes unbounded when x is 0 at the lower limit. And then because of the upper limit, it is mix; again a improper integral. So, we will divide this integral 1. 0 to infinity x to the power alpha minus 1 e to the power minus x dx as 0 to 1 x to the power alpha minus 1 e to the power minus x dx plus 1 to infinity x to the power alpha minus 1 e to the power minus x dx. So, this is first integral; this is second integral; two parts. Now, check the first part first integral.

(Refer Slide Time: 38:40)



Let us denote let x to the power alpha minus 1 e to the power minus x is fx. So, what we see here, we see that x to the power one minus alpha fx; that is equal to what? e to the power minus x and that goes to one as x tends to 0 therefore, this integral is convergent; therefore, integral this fx dx will converge. therefore, this integral fx dx x to the power alpha minus 1; this integral converge; alpha minus 1 e to the power minus x 0 to 1 the d x converges when 1 minus alpha is less than 1; that is alpha is greater than 0.

(Refer Slide Time: 39:51)

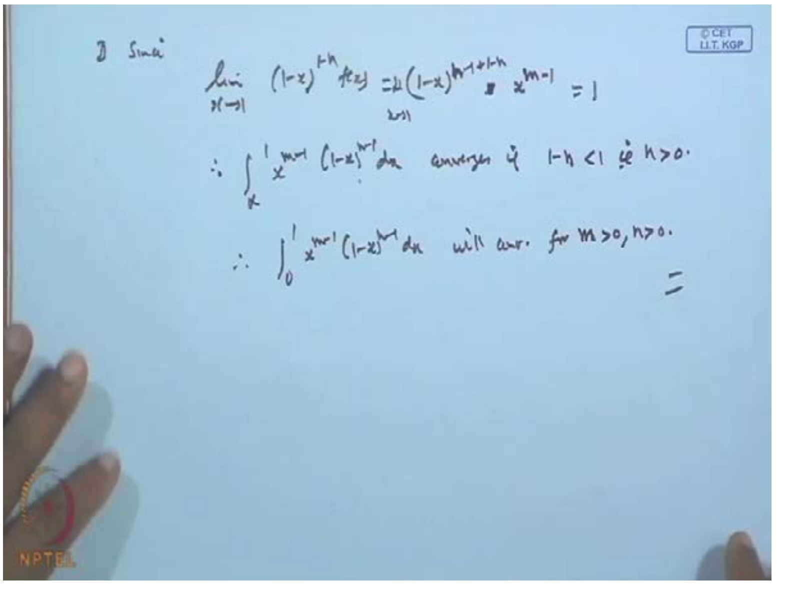


The second part of this, second part of this, the function limit of this x to the power alpha minus 1 e to the power minus x when x tends to infinity, this is always be 0. Whatever the alpha may be, this power x to the power e to the power x will, minus x, will go in the denominator; so it will be always 0; therefore, whatever the value of whatever be the value of alpha; therefore, the second integral 1 to infinity x to the power alpha minus 1 e to the power minus x dx will be convergent when alpha is positive. So, this will be greater than 0; therefore, this integral converges for all.

Now, second part is full; second integral is again 0 to 1 x to the power m minus 1 1 minus x raised to the power n minus 1 dx. Here, again it depends on suppose m is less than 1, then at this point, it is convergent and when n is less than 1 at the upper point it is again divergent. So, what we do? We break up into two parts 0 to alpha x to the power m minus 1 1 minus x n minus 1d x and plus alpha to 1 x to the power m minus 1 1 minus x n minus 1 dx. So, we will look the convergence of both; both integral converges then only this integral will converge.

So, let us see the first where alpha is where alpha is, obviously, is lying between 0 and 1. clear now let us see the first word denote x to the power m minus 1 1 minus x to the power n minus 1 is suppose fx. Then what we see here is, we see that the limit of this function fx when multiplied by 1 minus m fx as x tends to 0, is basically what? This is the same as limit x tends to 0 1 minus x to the power n minus 1; so that comes out to be 1; it means it is finite. So, the first integral therefore, the integral 0 to alpha x to the power m minus 1 1 minus x n minus 1 dx will converge provided what? Provided this 1 minus m, by which you are multiplying, is strictly less than 1; that is m is greater than by mu test. So, this converges, provided this.

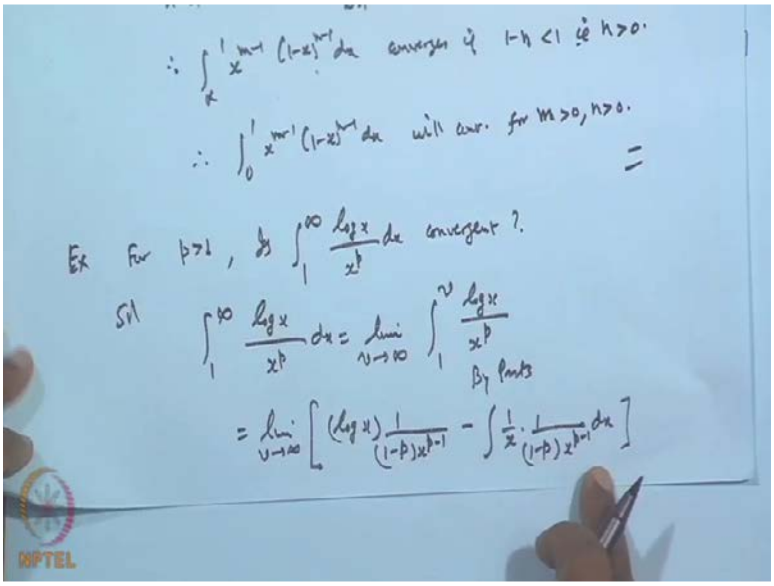
(Refer Slide Time: 42:58)



Now, since again second part is since limit of this 1 minus x to the power 1 minus n fx as n x tends to 1 is nothing but what? This will be the same as 1 minus x to the power n minus m minus 1 as x tends to 1. So, it comes out to be 0. Why 1 minus x tends to 1 fx what is our fx? fx is denoted by this one; is it not? So, when you multiply this by 1 minus x 1 minus n. So, that becomes 0 and when x to the sorry this will go n minus 1 plus 1 minus n and then multiplied by x to the power 1 minus x to the power m minus 1. So, this limit comes out to be 1.

So, once this limit it means it is a finite quantity. Therefore, the second part will be convergence. So, second part alpha to 1 x m minus 1 1 minus x n minus 1 dx converges if this 1 minus n is strictly less than 1; that is n is greater than 0. Therefore, whole integral 0 to 1 x m minus 1 will converge for m is greater than 0, n is greater than 0 because this is true for all n and for all m; n is greater than 0 for all m and earlier m is greater than 0 for for all n. So, common factor is m is greater than 0 and this is known as the beta function. it will also be very useful functions. Apart from this, we will now look few more examples that is also...

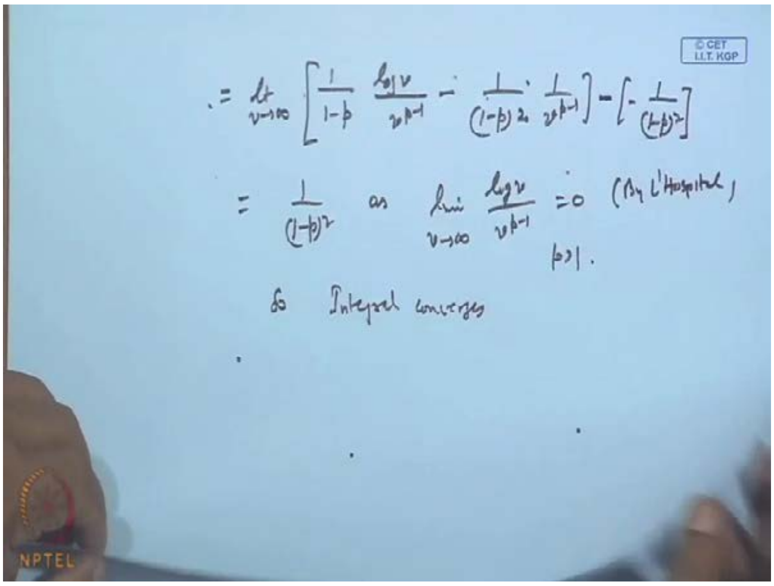
(Refer Slide Time: 45:02)



Let us see for p greater than 1, for p greater than 1, is integral 1 to infinity log x over x to the power p dx convergent? This is our… Now, this we will show it by first definition itself. So, let us suppose we can write it this for convergence part is 1 to infinity log x over x to the power p dx can be written as limit v tends to infinity 1 to v log x over x to the power p because x vary from 1 to infinity; so it is well defined at this point; 0 is not available. Only point when x tends to infinity, this because of the infinity it is improper. So, if this limit exists then we say the integral will be convergent. Now, this you integrate by part by part by parts. So, if you apply the integration by parts taking the first function, this second function this.

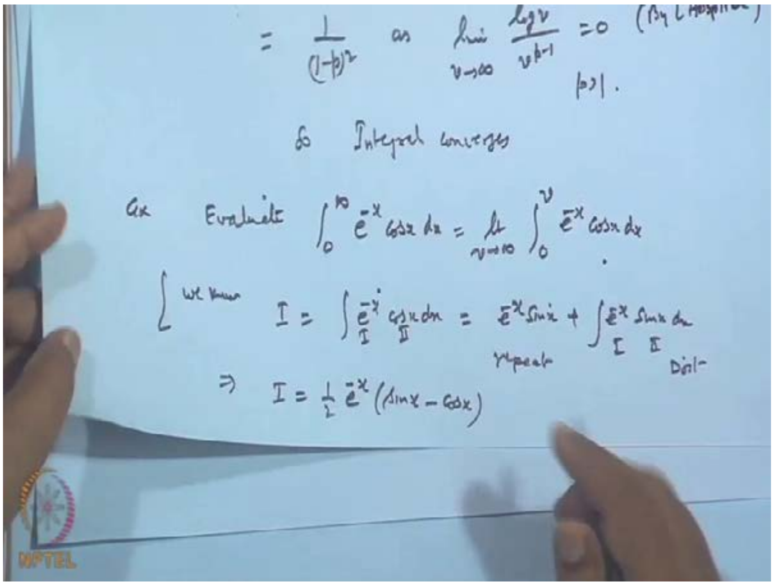
So, we get first function first function is 1 upon x to the power p; first function is sorry log x. First function integral of the second. So, integral x to the power minus p plus 1 over 1 minus… So, we get 1 minus p x to the power minus p plus 1. So, p minus 1 first function integral of the second; integral of the second is log of minus p; that is 1 by p first function integral of this; integral dc of the function dc of the first 1 by x and integral of the second will be 1 minus p x to the power p minus 1 x to the power p minus 1 and that will be equal to what? dx. So, when we again apply this integration, now it is simple integration.

(Refer Slide Time: 47:22)



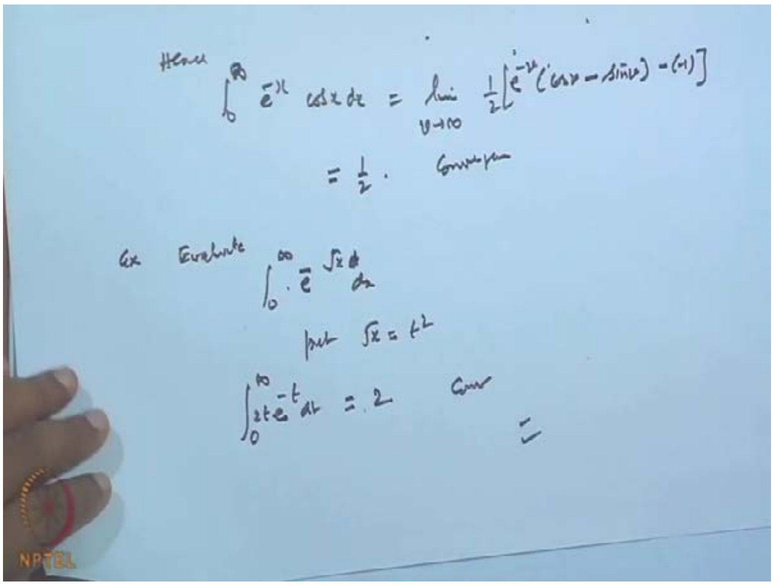
So, when we get this integration, what we get from here is that this comes out to be limit v tends to infinity 1 over 1 minus p log of v over log of v over v to the power p minus 1 minus minus 1 over 1 minus p whole square into 1 by v p minus 1; then minus minus 1 over 1 minus p square. just simple integration we will see. So, what we get when v tends to infinity? Now, this part, only thing, limit of this is because this will go to 0. So, what we claim is the limit of this will be 0; limit of this will be 0. So, basically, the value will come out to be 1 over 1 minus p whole square as limit of this limit of log v over v to the power p minus 1 v tends to infinity is 0 by L'hospitals rule. That you can just verify as value p greater than 1; so this proves that. So, the integral converges, integral converges.

(Refer Slide Time: 48:54)



Now, next, if we look this again, integral second, evaluate this integral integral 0 to infinity e to the power minus x cosine x dx. So, what we do is we are first taking this as the limit v tends to infinity integral 0 to v e to the power minus x cos x dx. Now, what I say, we know that if I take I to be the integral e to the power minus x cos x dx and then taking first and second function, first function integral of the second minus integral dc to first integral integral of the second minus Integral of the second sin x minus integral dc of the first and integral of the second is sin x dx and again first function and second function, and again repeat this thing; what we get? We will get the I again. So, finally, we are getting I. So we do it again. So, what we get? I comes out to be half half e to the power minus x sin x minus cosine x. So, now 0 to infinity of this means I 0 to v is; is it not? So, this will be…. Therefore, what we want is the limits.

(Refer Slide Time: 50:37)



So, we get from here is hence 0 to infinity 0 to infinity e to the power minus x cos x d x that will be equal to what limit? limit v tends to infinity is half e to the power minus v minus v so sine of v minus sin of v minus minus 1 because this will be when you go for this 0 to v 0 to v. And then finally, you will get something here. Integration sorry integration when you do it, this part when you do for integration you are getting this thing minus v and 0.

So, finally, integration will be from here; this will be 0 to v. When you are writing integral 0 to v e to the power minus x cos x d x, then you will get this result, which while I am doing this one; half of this this is half; e to the power minus v e to the power minus v cosine v minus sin v and then then minus minus plus half. This you will get it. So, when you take the limit of this v tends to 0 to infinity get this thing. So, we get from here, what we get is now, the answer will be when v tends to infinity this is bounded function; bounded by 2. mod cos x is less than cos v is less than equal to 1; mod of this less than v equal to 1. So, this part will tend to 0 and here you are getting is half; so that is the answer; so convergent; convergent one; so this we get it. Now, one more example is, let us evaluate integral 0 to infinity e to the power minus root x dx. So, here, if we substitute root x equal to t square, then everything will be fine and we get 0 to infinity e to the power minus t and twice t dt into 2 t dt 2 t dt and that is all. So, when you integrate you are getting the value 2. So, it is convergent and that is it.

Thank you very much.

Thanks.

So, this completes your course. Fine.