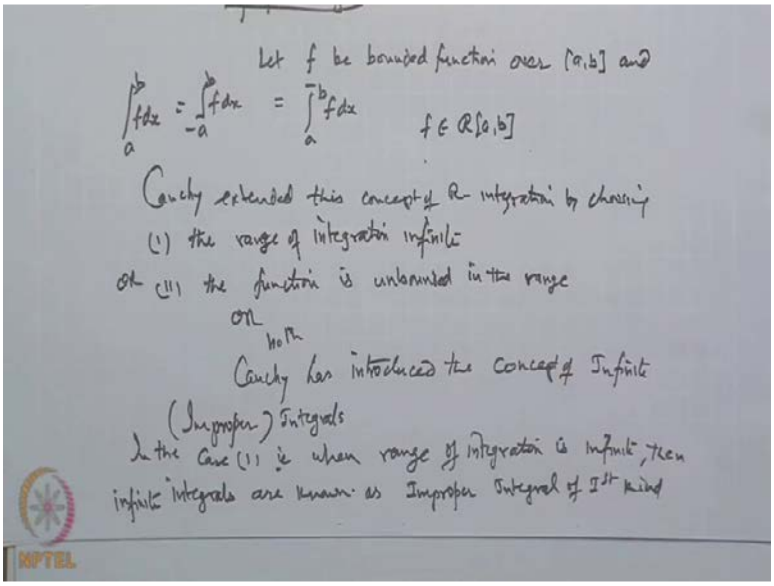
A Basic Course in Real Analysis

Prof. P. D. Srivastava
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 45
Improper Integrals

(Refer Slide Time: 00:27)

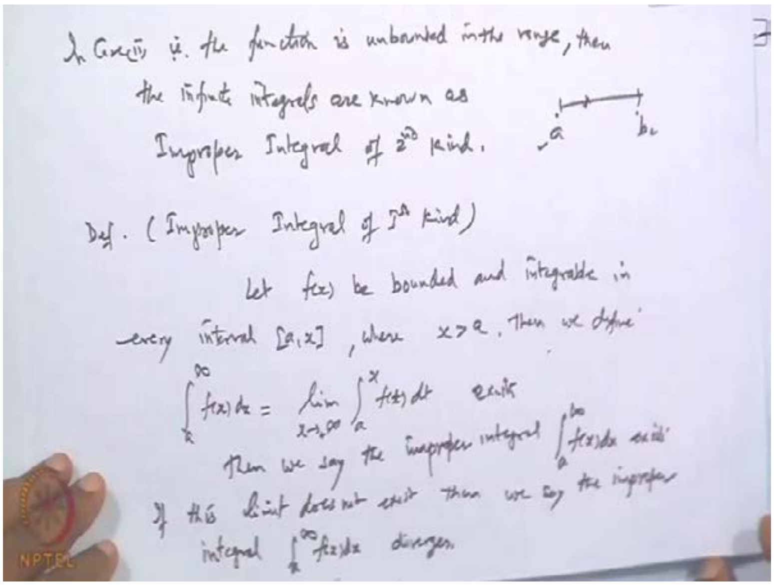


So, improper integrals; so, we will discuss the improper integral today. We have seen already the Riemann integral; when the function f be a bounded function over the closed interval a, b and if the lower Riemann integral a bar f d x and b bar upper Riemann integral when both exist and coincide, then this we denoted by a to b f d x and is called we say f is Riemann integral function over the interval a, b.

So, for a bounded function, the Riemann integral of a function f x, which is bounded over a close interval can be defined, provided this limit exists. The idea of extending this limit, why we consider only a to b? It may be a to infinity or minus infinity to b or may be minus infinity to infinity; so this idea of extending the integration, the range of the integration is done by Cauchy, and he has introduced the infinite integrals by taking the range of the integration infinite; and second be the function is unbounded in the range.

Because here is Riemann integration, we consider f to be a bounded and the range of integration is take to be finite but what the Cauchy extended this concept of Riemann integration; by choosing the range of the integration infinite or the range of the or may be both function is unbounded range. So, by this way he has introduced the concept of infinite integrals. So, Cauchy has introduced the concept of infinite or we are also calls it as improper integrals. The first one when the range of integration is finite in the case one; that is when range of integration is infinite then the infinite integrals is known as improper integral of first kind.

(Refer Slide Time: 04:18)

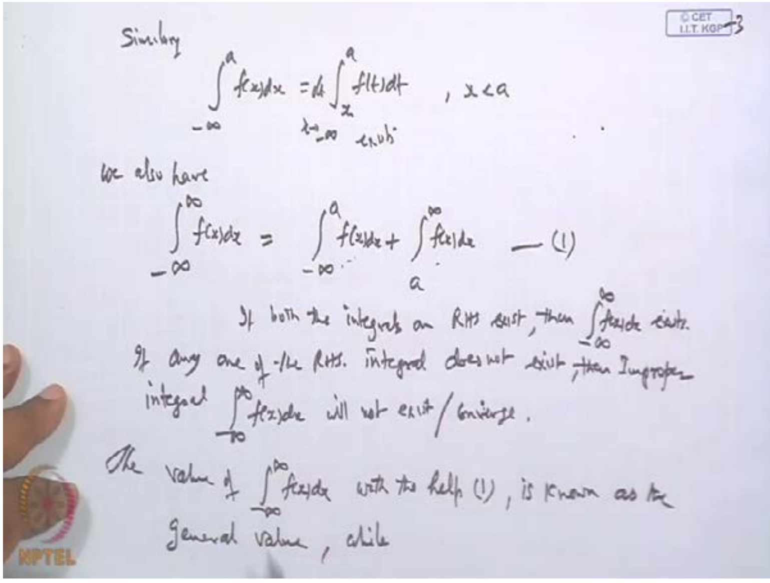


And in the second case; in case two, that is the range of the function is unbounded in the range; that is between a to b the function have a point of discontinuity at the point either a or at the point b or may be in somewhere inside c. So, function becomes unbounded discontinuity of the first kind, it becomes unbounded at either in the lower limit or may be a upper limit or may be any point in between this.

So, if the function is unbounded in the range, then the infinite integrals are known as improper integral of second kind and third case when both may be mixed out. So, it is mixed type of improper integral will also be there. So, we will take up this. Define the improper integral of the first kind and second kind as follows. First, improper integral of first kind; let us suppose f x be bounded and integrable in every interval a, x where x is greater than a. Then we define integral a to infinity f x dx as limit of this a to x f t dt you can say when x tends to infinity; that is in x tends to plus infinity, then this plus infinity then we get.

Now because this integral a to x f t dt exist because the function f is bounded and integrable in every interval a to x. It means integral a to x f t dt this Riemann interval will exist, for each x which is greater than a. So, the limiting behavior of this integral when x tends to infinity; if this limit exist and comes out to be a finite quantity, then we say a to infinity f x dx this improper integral of first kind exist. So, then this improper integral exist, then we say the improper integral a to infinity f x dx exist or converges. If this limit does not exist, we say diverges. If this limit does not exist, then we say the improper integral a to infinity f x dx diverges. Why is they either come to be infinity or may be the limit does not exist, we say it is divergent.

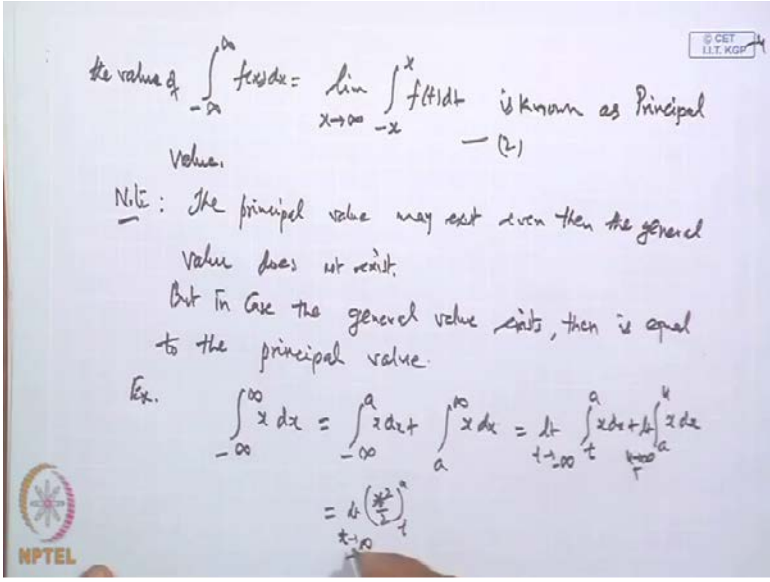
(Refer Slide Time: 08:54)



In a similar way we define; similarly, we define integral minus infinity to a f x dx as limit x tends to minus infinity say x to a f t dt where the x is always less than a and assume that this integral will always be well defined and exist. So if this limit exists, then we say this integral converges. If they do not exist, we say the integral diverges. The third case may be that we also have the integral minus infinity to infinity f x dx and this is defined as the integral minus infinity to a f x dx plus integral a to infinity f x dx.

And if both the integrals on the right hand side exist, then the integral minus infinity to infinity f x dx will exist. If any one of the right hand side integral does not exist, then the improper integral minus infinity to infinity f x dx will not exist or we can say will not converge; that is divergent. The value of this is 1, the value of the integral 1; the value of the integral minus infinity to infinity f x dx with the help of equation one is known as the general value.

(Refer Slide Time: 11:44)



While the value integral minus infinity to infinity f x dx which can also be written as limit minus x to x when x tends to infinity f t d, this will also give minus infinity to infinity. The value of this while value of this integral calculated with the help of this is known as the principle value of the integral. Now as a note; the principle value of an integral may exist, even then the general value does not exist. But in case the general value exists, then it is equal to the principle value. So, what we have seen is that improper integral of the first kind can be written in three types of ways; either a to infinity or minus infinity to a or may be minus infinity to infinity.

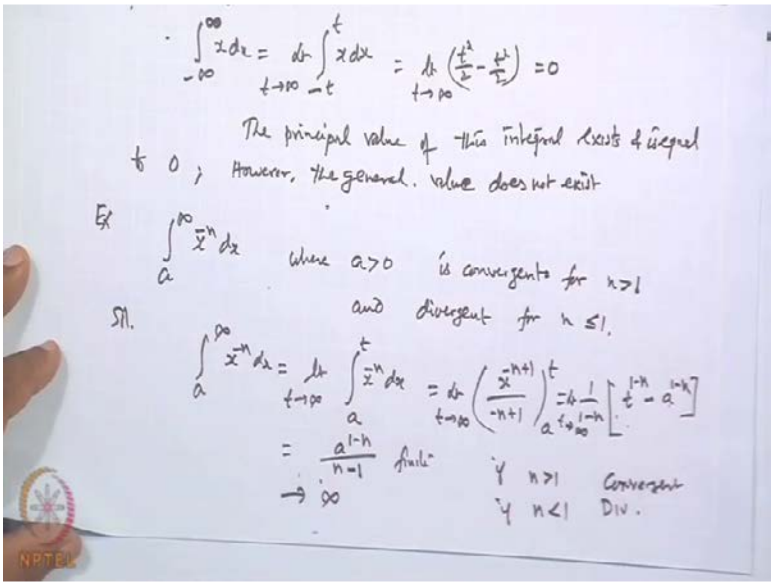
In first two cases, just to test whether it is limited integral exist or not, consider a to x f t dt as if it is a definite integral, find out the value, get the terms of x in terms of x and then let x tends to plus infinity. If this limit exists then we say this integral has a value equal to the value of this limit and we say the integral converges. If limit does not exist or it goes to infinity, then we say this integral diverges. The same case happens with the second case when minus infinity to infinity and so on. So, these two are different case and is clear cut the convergence or divergence will come.

In the third case minus infinity to infinity f x dx as; this can be written in two ways, one is minus infinity to a f x and then a to infinity, when one of the limit is infinity or minus infinity and other limit is finite. So, these two integrals in the right hand side, they are basically the improper integrals of the kind one. Then the existence of this integral will depend on the limiting value of this first as well as the earlier one. So if both the limit exists, both the integral exist, then only we say this improper integral minus infinity to infinity f x dx will exist. If any one of the limit does not exist, then this improper integral will not exist.

Now since the same integral we can also put it in the form of minus x to x f t dt that when x tend to infinity it also give the minus infinity. So, what is the difference between these two; this is the first, this is second. First is this, the way in which written and second one is this. So, if the limit of the first exist if the integral where the both the limit exist, then the limit of the second will definitely exist and equal to the value of the integral. But if the second exist, then first may or may not exist. So that is what it says, the principle value may exist even the general value does not exist.

Now let us see for example, suppose I wanted to take this integral minus infinity to infinity x dx. Now let us look, this can be written as minus infinity to a x dx plus a to infinity x dx. Now both the integrals when you find x square by two n, then this can be written as limit t tends to minus infinity t to a x dx plus limit u tends to plus infinity a to u x dx. Now if I calculate it what you get, x square by 2; so t square by 2 and then x square by 2 t to a and limit t tends to minus infinity. So, this will be this and then second one is x square by 2 under the limit a to u and limit u tends to plus infinity. So obviously, it will go to infinity. So, limit this will value does not exist. So, the integral diverges.

(Refer Slide Time: 17:42)

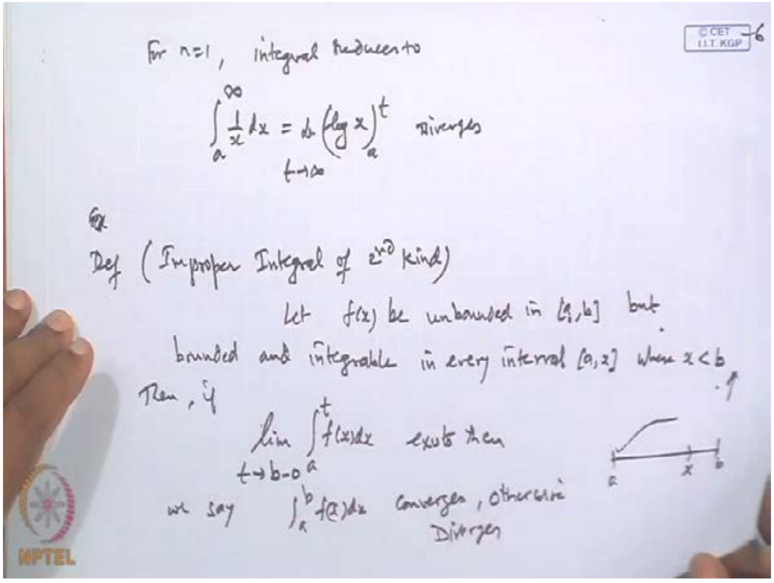


However, if I take the principle value; however, if we take minus infinity to infinity x dx as integral minus t to t x dx limit t tends to infinity, then what you get. This you are getting t square by 2 minus t square by 2. Is it not, x square by 2 and then limit lower and upper limit, limit t tends to infinity and that comes out to be 0. So, what it says? The principle value of this integral exists and is equal to 0; however, the general value does not exist. So, this verifies our claim that though the principle value sometime may exist but general value may not.

Now, let us take some few examples where, say, we claim; this example will also be used as a standard result to compare the function to be tested. So, first is that integral a to infinity x to the power minus n dx where a is greater than 0. We claim that this integral is convergent for n greater than 1 and divergent for n less than equal to 1. So, let us see the proof. Solution, a to infinity x to the power minus n dx; this can be written as limit x tend to minus n dx a to t t tends to infinity. So, this will be equal to x to the power minus n plus 1 over minus n plus 1 and then integral limit a to t and t tends to infinity. So, that will come out to be 1 over 1 minus n t to the power 1 minus n minus a to the power 1 minus n and limit as t tends to infinity.

So, case 1; if our n is greater than1, then in that case n is greater than 1. Then this 1 minus n becomes negative. So, it will come in the denominator. So, when t tends to infinity it will go to 0. So, only left out is minus of this; it means the value will come out to be a to the power 1 minus n over n minus 1, that will be the value. So, it is nothing but a finite value. So, it is convergent. Now on the other hand if I take n to be strictly less than 1, then this is positive. So, it will go to infinity. So, value will go to infinity; therefore, it diverges.

(Refer Slide Time: 21:36)

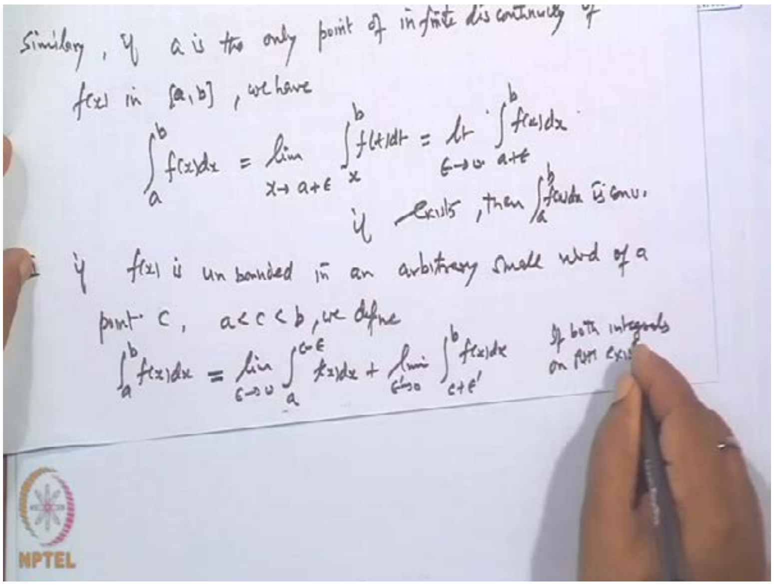


When we take n is equal to 1; for n is equal to 1, the integral reduces to a to infinity 1 by x dx and that is nothing but log of x and then limit a to t and t tends to infinity. So, this will diverge. So, the result follows that one upon x to the power n is convergent, when n is greater than 1; diverges when n is less than equal to 1. So this can be treated and extend a result. Then another example is we can also use as the extended result, but prior to this let us come to a definition for improper integral of second kind. Now what we have stored; that in this second kind their limit of integration or the range of integration is finite a to b.

But the function need not be a bound; it will not be a bounded function in fact. Because if it is bounded it is finite, then one can say the Riemann integral of this function will exist, provided the lower and upper limit is exist and identical. But the function is not bounded and it has a point of discontinuity as said it as a sudden jump; it goes to plus infinity or minus infinity, then function cannot be we cannot say a to b f x dx a Riemann integral function, because the condition is function must be a bounded function. So, in such a case we get the concept of the improper integral of second kind. So, let us assume let f x be unbounded in the closed interval say a, b but bounded and integrable in every interval a to x where x is strictly less than b. So, the function is like this is our a, b.

So, so far we take any point x the function is bounded. But as soon as it reaches to b, it is unbounded; it goes to infinity all of sudden. So, what we say is then if limit of this a to t f x dx when t tends to b say here b minus 0 from the negative side; means from the left hand side I am approaching b minus 0. If this limit exist if, then we say then we say the improper integral a to b f x dx this limit converges; otherwise diverges. So, that is what. So, here b is the only point of infinite discontinuity of f x.

(Refer Slide Time: 25:43)

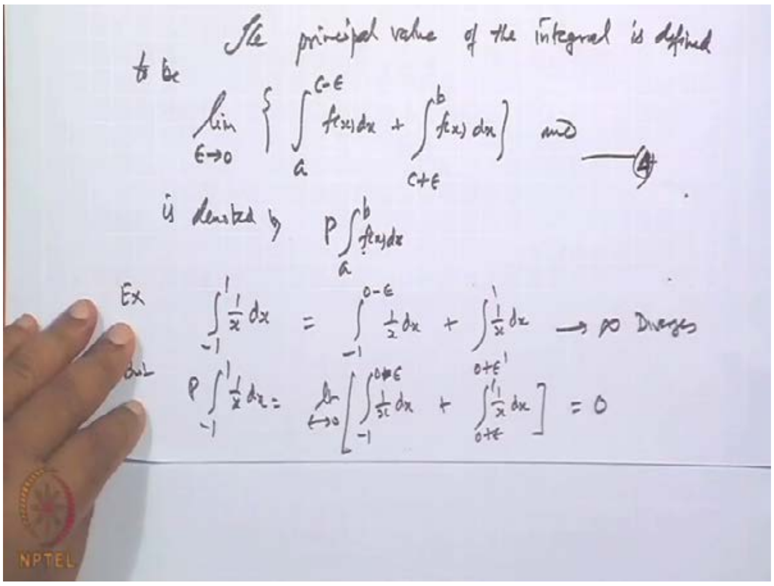


So what we do is in that case, we basically write; thus in this case, we have here a to b f x dx. This we can write as a limit epsilon tends to 0 and then a to b minus epsilon f x dx. Now up to b minus epsilon, this function f x is bounded and integrable. So, the value of this can be obtained, but it depends on epsilon. So as soon as you take epsilon tends to 0, it means we are approaching the b from the right hand side; this is b minus epsilon. So when epsilon tends to 0, we are approaching to b from the left hand side towards this. So, when we are approaching if this limit exists, then we say this integral exists and convergent; otherwise it will diverge.

Now similarly, if a is the only point of infinite discontinuity; the function is unbounded at this point of infinite discontinuity of the function f x in the interval a, b, then we have integral a to b f x dx is the limit x tends to a plus epsilon integral x to b f t dt and this is the same as limit epsilon tends to 0 integral a plus epsilon to b f x dx. So, if this limit exist, then integral a to b f x dx is convergent; otherwise it will diverge.

The third case, if the function f x is a unbounded in an arbitrary small neighborhood of a point c which lies in between a and b. Then we define the improper integral of the second kind as a to b f x dx as the limit of this sum of the two integral a to c minus epsilon, epsilon tends to 0 f x x plus limit epsilon tends to 0 integral c plus epsilon to b and then f x dx. In fact, this will be epsilon dash, epsilon dash so that you are not getting same epsilon. Now if both integrals on the right hand side exist.

(Refer Slide Time: 29:44)

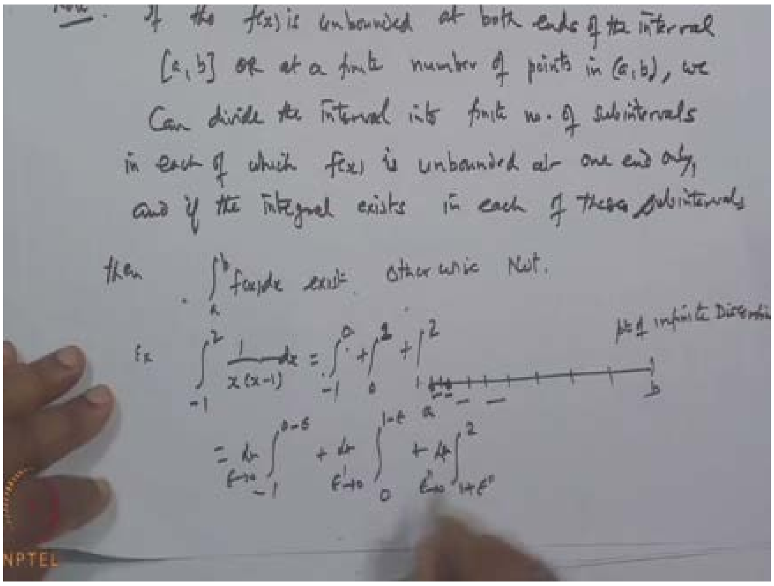


Then integral a to b f x dx will exist and is said to be convergent and its value is known as the general value. While the same thing, the principle value; in this case, the principle value of the integral is defined to be limit epsilon tends to 0 integral a to c minus epsilon f x dx plus integral c plus epsilon b f x dx and is denoted by the principle value of this integral a to b f x dx, which has a discontinuity at a point c; infinite discontinuity at a point c. Now again this limit, this is say 4 and this one is say suppose 3.

Again the same case as in case earlier; the general value if at exist, the principle value has to exist. But if the principle value exists, then the general value may or may not exist. For example, if suppose I take this integral minus 1 to 1 1 by x d x. Now this integral, the limit of integration is finite, but the function is not defined at x equal to 0. So, it is an improper integral of the kind two; therefore, we can calculate this value of the integral as minus 1 to say 0 minus epsilon 1 by x dx plus 0 plus epsilon dash and 1 by x dx to 1.

And when you compute this value, what happens; this is log x and log of x at the point one is 0, but when epsilon is taken epsilon tends to 0, it goes to infinity minus infinity. So, this tends to infinity that diverges, limit does not exist. But if we take the principle value of this integral minus 1 to 1 1 by x dx, this is the same as the limit epsilon tends to 0 minus 1 to minus epsilon 1 by x dx plus 0 plus epsilon 1 by x dx to 1, what we get; the value will come out to be 0. Just integrate and substitute and get that answer. So, this value will come; the principle value will be there.

(Refer Slide Time: 33:20)



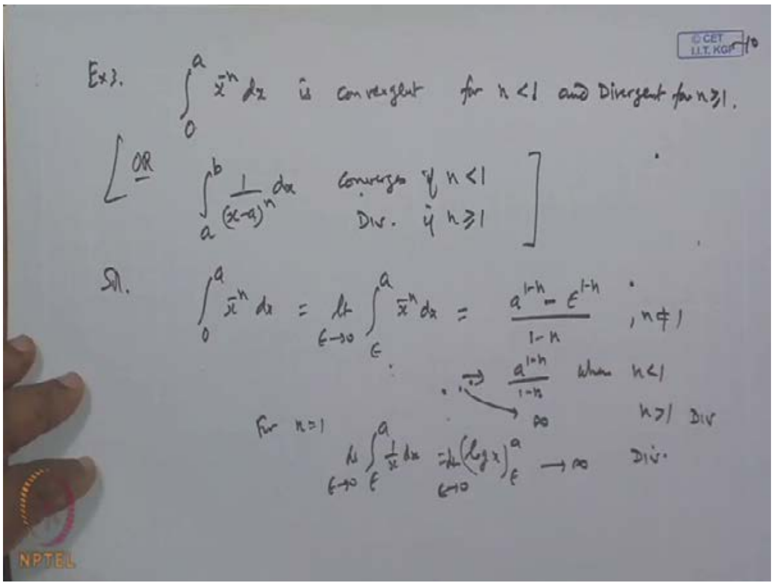
Note: If the function f x is unbounded at both ends of the interval a, b or at a finite number of points in the interval a, b, then we can divide the interval into finite number of subinterval in each of which f x is unbounded at the one end only, and if the integral exists in each of these subintervals, then the integral a to b f x dx exist; otherwise not. What is the meaning of this is, suppose we are having this interval a, b and the function is such which has a point of discontinuity at finite numbers of points; these are the points of infinite discontinuity. It means the function is unbounded at this points discontinuity including the point a and b also. So what we do is we divide this into the intervals.

So, we take the sub interval like this; suppose at this point, then we can take this interval where the function is only discontinuity at this point or may be at this point also, then we can say m minus epsilon and then some point between we can find out so that at only one point, it has a point of discontinuity. Similarly in this interval, only at this point is infinite discontinuity and like. So, over these subintervals we can find out the value of these integrals and take the limit. If all these limit exist; that is a right hand side integral exist ,then left hand side integral will exist and we say this is integral convergent; otherwise not.

For example, if suppose I take the interval, say, 1 over x into x minus 1 dx; suppose I take this integral. So, 0 and 1 are the two points. So, I am taking 1 and 2; this integral is I have taken, so obviously at the point 0, the function is not defined. It goes to infinity, x equal to 1 also the function is not defined. So, what we will do is we will break away into 0 to 1 plus 0 to 2. So, what 0 to 2 at the point 1 is there, so we can change it 1 and then 1 to 2. So what happen is here we take the minus epsilon, here we take the plus epsilon and in that case 0 minus epsilon and then epsilon tends to 0.

So, this is the same as minus 1 to 0 minus epsilon limit epsilon tends to 0 plus limit epsilon dash tends to 0 0 to 1 minus epsilon and then limit epsilon double dash tends to 0 1 plus epsilon double dash to 2. And thus see the value if the integral exists all the three, then this integral will exist; otherwise not. So, this is what. Again as I showed there are few examples further which are treated as standard results and can be used to compare the unknown integral with these known functions so that one can identify whether the given improper integral converges or diverges.

(Refer Slide Time: 38:29)



So, let us take one example here. We claim that this integral 0 to a x to the power minus n dx is convergent for n is less than 1 and divergent for n is greater than or equal to 1. Now if you see this example and the previous examples which we have seen. Earlier the example was a to infinity x to the power minus n. So, here because of the upper limit was infinity, the integral becomes improper and that is called the improper integral of the first kind. And in that case the result was when n is greater than 1, this integral converges. When n is less than equal to 1, diverges.

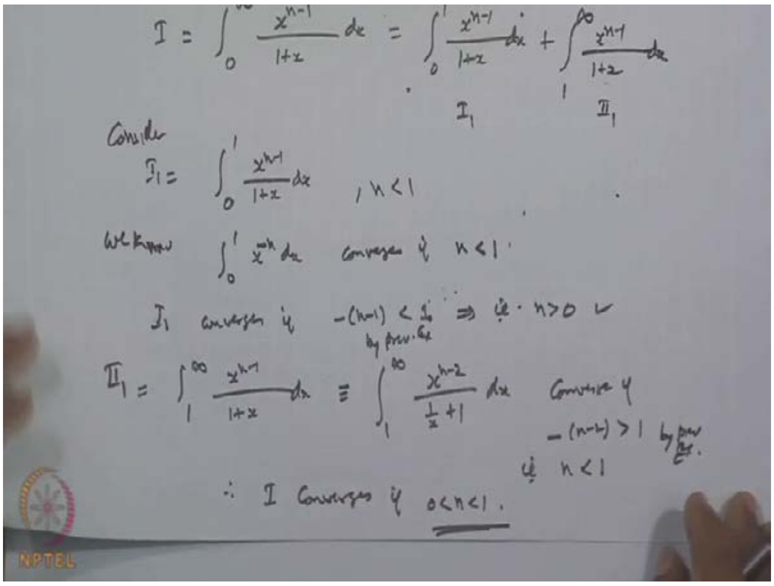
But here the point of discontinuity is 0. So, the result is something different, means do not confuse with that. This is at the point 0 the function b becomes unbounded. So the result says that when n is less than 1, it is convergent; greater than 1, divergent or we can also write like this. If suppose we have integral a to b 1 over x minus a to the power n dx, then this integral converges if n is strictly lx than 1, diverges if n is greater than equal to 1. So this clearly shows that x equal to a, the function becomes unbounded. So, let us see the one we will prove this and this is just a generalization; when you substitute a equal to 0 you get that result like this. So, let us see the solution for it.

So, what we do is we start with the 0 to a x to the power minus n dx and this can be written as what; limit epsilon tends to 0 epsilon to a x minus n dx. If this integral exists, then this integral will exist convergent. So, that value will come out to be what; a 1 minus n minus epsilon 1 minus n divided by 1 minus n where n is not equal to 1. So, this limit will tends to a 1 minus n over 1 minus n, when n is less than 1. Because n is less than 1, this becomes positive.

So, epsilon tends to 0 means it will go to 0 and we get this value. But when n is strictly greater than 1, this is negative; so, 1 by epsilon will go infinity. So, it will go to infinity and diverges and for n is equal to 1, again this integral reduce to 1 epsilon by a 1 by x dx epsilon tends to 0; that is equal to log x and then epsilon to a epsilon tends to 0 it will go to infinity, so diverges. So, this proves the result for it. The second example again, say, suppose I take this integral 0 to infinity x to the power n minus 1 over 1 plus x dx when this integral when n is less than 1 is convergent, if n is greater than 0. It means this integral converges for all n; that is the integral I converges if n lies between 0 and 1.

Now if you look at this integral, this is a mixed type of integral; mixed type means limit of the integration is infinity. So, it is an improper integral of the first kind, but the function itself x to the power n minus 1, when n is less than 1 this x will come in the denominator. So, when around the point 0 or x tends to 0 the function is undefined; therefore, it is also an improper integral of the second kind. So this integral is a mixed type integral, which is improper integral first and second kind. So, in order to judge the convergence part of this integral, we will break up our two integral first. In the first case when the continuous function becomes unbounded at 0, another case when function is bounded, but at the point infinity only we have to test the convergence part.

(Refer Slide Time: 43:28)



So what we do is, we consider the integral I which is integral 0 to infinity x to the power n minus 1 over 1 plus x d x. This we can break up in to 2 parts; 0 to 1 x n minus 1 plus x dx plus integral 1 to infinity x to the power n minus 1 1 plus x dx. Let it be first and second; so, first and second. So, first integral has a point of discontinuity unbounded at the 0, while the second integral is for 1 to x it is finite bounded. But at infinity problem is there. So, let us test this. So, consider say I 1 and then I 2. So, consider I 1; I 1 is 0 to 1 x to the power n minus 1 over 1 plus x dx. Now what happen; this 1 plus x is not going to affect the whole integral.

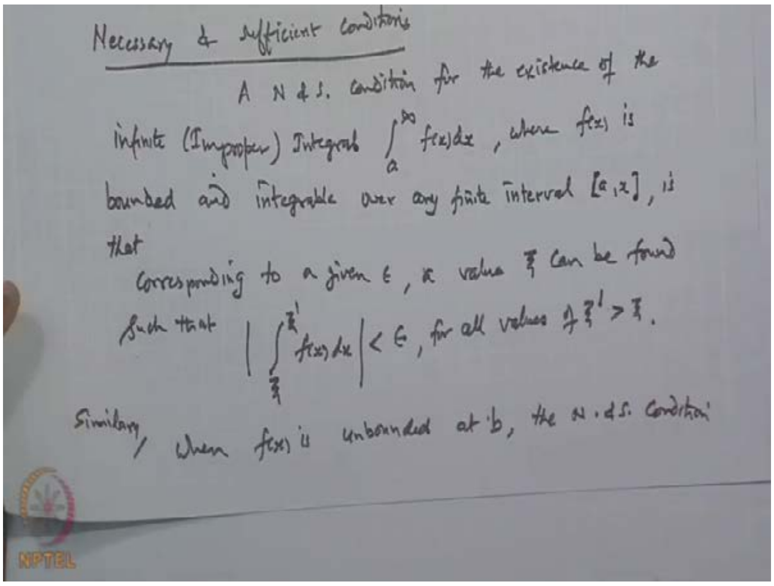
This becomes improper because of this term x to the power n minus 1 and n is strictly less than 1. Therefore for this integral, we have to test it with our previous knowledge. What the knowledge is we have seen; we know that integral 0 to 1 x to the power minus n d x converges if n extends to a, converges if the n is less than 1; is it not, if n is strictly less than one. So, what we say is that this integral is convergent when n is strictly less than 1. So, here this is nothing but what n minus 1. So, I 1 converges if minus times of this is strictly less than 1. But this minus time implies n is greater than 0. So, this is part for convergence. Second one is second I 1; this integral 1 to infinity x to the power n minus 1 1 plus x dx.

So, again this integral converges when this is greater than 1, is it not. So, this portion that is x to the power n minus 1 is equivalent to it should be bounded also, is it not. So, what we do is 1 to infinity x to the power n minus 2 1 by x plus 1. So when x tends to infinity, this denominator will not create the problem; this is 0 and this will be 1 only. So, now, you are getting this part. So when 1 to t, it is bounded functions like this. So now, this integral converges if minus times of n minus 2; that is minus x to the power n is greater than 1, that is n is less than 1. So, you can say by previous example and this is also by previous example. So this implies that I converges, when if n is lying between 0 and that is also considered to be a standard wizard for it.

Now so far we have discussed many theorems problems, improper integral first kind and their convergence part. But always what we did is we have computed the value of it inside by choosing epsilon and then epsilon tends to 0 or epsilon tends to plus 0 or something. Then if the right hand side has a finite value, then we will say that integral converges. But if the limit does not exist, then in diverges. Now it may not be always be possible or may be very difficult to compute always the integral and then taking the limit when epsilon tends to 0 or may be t tends to infinity. So, to avoid this thing we have developed certain results or formula, which are helpful in identifying whether the given improper integral converges or diverges.

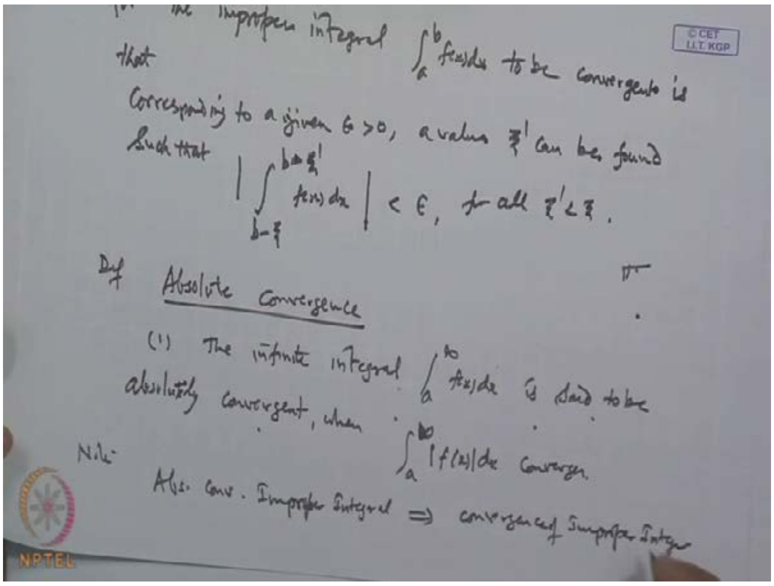
Just by looking the integral and the standard results which we have already shown; this four example which we have discussed; basically I told these will be the weight bound, these will be used as a standard results. So, we can easily compare with the given integral with one of the example and then we say whether the given integral converges or diverges. But for this, there is certain necessary and sufficient condition; just like in a series we have a necessary and sufficient condition that series must be Cauchy. When the series is convergent, it has to be a Cauchy sequence and vice versa for the real and complex sequences, is it not. So similarly, here also we have certain results and that necessary and sufficient condition we will just discuss without proof.

(Refer Slide Time: 49:02)



So, convergence of infinite integral or improper integral; so, this is what we want that is necessary principle; necessary and sufficient condition. So, what we say is for the existence a necessary and sufficient condition for the existence of the improper infinite or we say improper also improper integral a to infinity f x dx where f x is bounded and integrable over any finite interval say a, x is that is that corresponding to a given epsilon, a value xi can be found such that modulus of integral xi to xi dash f of x dx mod of this is less than epsilon for all values of xi dash greater than xi. So, this is one. Similarly we can go for this second kind; this is improper integral of the first kind. Similarly for the second kind, we say when the function f x is unbounded at a point say b, the necessary and sufficient condition.

(Refer Slide Time: 52:32)



For the improper integral a to b f x dx to be convergent is that corresponding to given epsilon greater than 0 of course, a value epsilon dash can be found such that modulus of integral b minus epsilon b minus epsilon dash f x dx is less than epsilon for all epsilon dash r less than epsilon. So, this two states mean will be used as a general results and when you define others by we get; just we define the function one absolute convergence and then we will stop it. Absolute convergence of the integral means the infinite integral a to infinity f x dx is said to be absolutely convergent, when the integral a to infinity mod of f x dx converges. So, that is it. So, remark note is absolutely convergence improper integral will imply the convergence of improper integral; improper integral is absolutely convergent, it will convert and that is all.