A Basic Course in Real Analysis
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Lecture - 44
Fundamental Theorems of Integral Calculus

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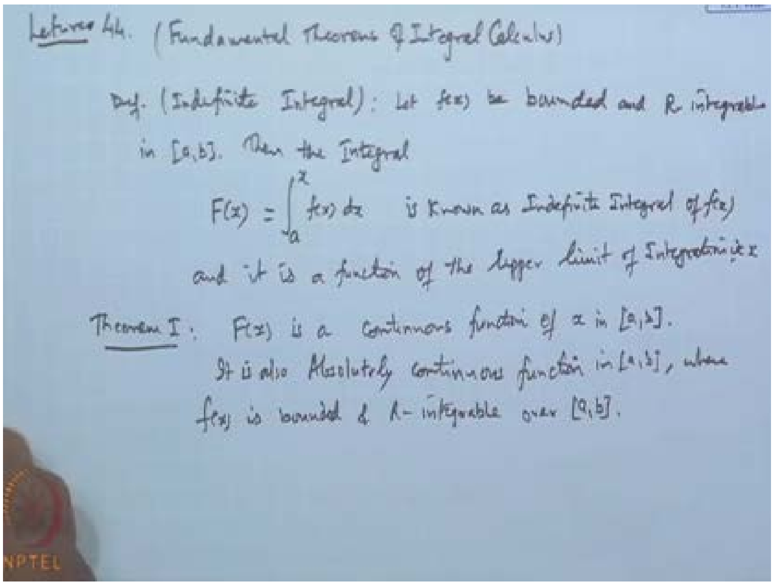

So, today we will discuss fundamental theorem integral calculus that requires the concept of an indefinite integral. So, before starting the fundamental theorem let us see the indefinite integral. Let suppose f x is be bounded and Riemann integrable in the interval say a, b - close interval a, b. Then the integral a to x f x dx, where a x is a point in between a, b is known as the indefinite integral of the function f x, and it is a function of the upper limit of integration; that is here upper limit x. So, we denote this thing as capital F x. So, it is conveniently to denote this expression F x as indefinite integral a to x f x dx, where F x we call it as an indefinite integral of the function f x, which is a function of the upper limit x.

So, x changes the corresponding F x will change. In fact, the behavior of the f x depends on this x. So, first result which is valid for the indefinite integral is that if the function F x is bounded and Riemann integrable in the interval x, then function F x will be a continuous function. So, F x is a continuous function of x in the closed interval a, b where it is also absolutely continuous function in the closed interval a, b; in the interval a, b (( )), where f x is bounded and Riemann integrable over the interval a, b.

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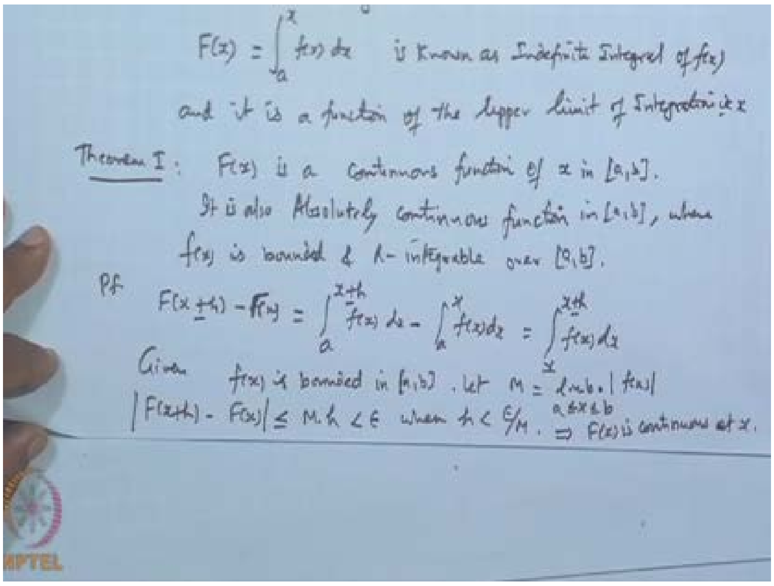

This is the proof. The proof is not that difficult. Let us consider F x plus h minus capital F x; capital F x plus h minus capital F x. This is equal to what a to x plus h f x dx minus a to x f x dx by definition and this will be equal to what; x to x plus h f of x dx because of the Riemann integrable properties of the Riemann integral functions we get this. But this m dash given, function f x is bounded and let in the closed interval a, b and let m be the least upper bound of the function f x mod f x over this interval a, b. Let it be this. Then the F x plus h minus F x this can be written as F x plus h minus F x is less than equal to capital M times into length of the interval say h. So, it can be made less than epsilon when h is less than epsilon by M.

It means the difference of this can be made less than epsilon when h is sufficiently small. So, when h tends to 0 limit of the F x plus h will go to F x. So, this shows f is continuous; F x is continuous from the right. Is it not? So, continuous from x plus from right, but the same result hold if I take x minus h then F x minus h; here the difference will come only this. So, what we get? We will get only thing is that this x plus minus h when you take the modulus of this sorry it is modulus of this, then there is no change. You will get m dash h and which gets to there. So, this is continuous at the point x in the interval a, b. So, this indefinite integral F x is a continuous function of x.

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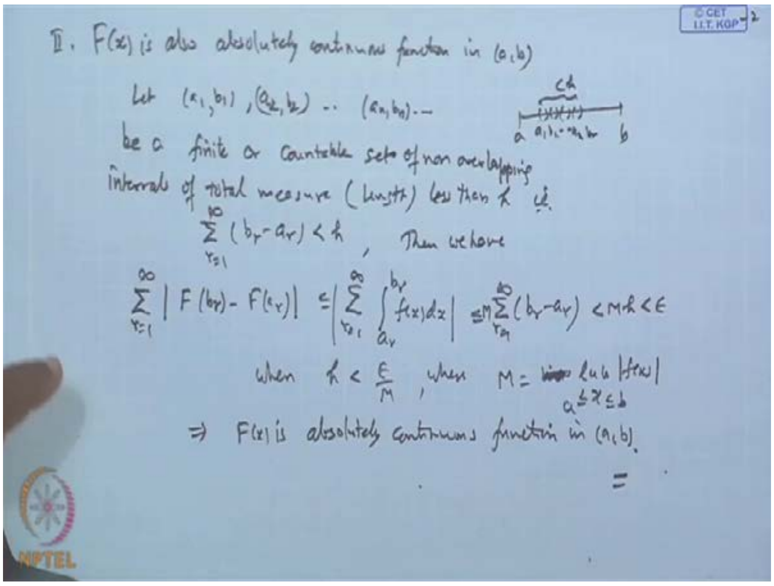

Second result is that second part is still absolutely continuous; F x is also absolutely continuous function. So F x second part of this; F x is also absolutely continuous function in the interval a, b. So, absolutely continuous function means this is the interval a, b. If I picked up the say intervals a 1, b 1; a 2, b; 2 a n, b n and so on such that the sum of these is less than h, then the images of this F of x at these point b i minus f of x i will be less than epsilon; that is what we wanted to show. So let us consider, let a 1, b 1; a 2, b 2; a n, b n and so on, these are the a finite or countable set finite or countable set of non overlapping intervals of total measure; that is the length is less than h, that is sigma of b r minus a r r is one to infinity is less than h.

Then we have the sigma of mod F of b r minus F of a r r is 1 to infinity. Now this will be less than equal to what integral of this part; that is integral sigma r is one to infinity and then integral a r to b r f of x dx and which can be written as less than equal to because mod of this is there, so modulus of this. So, this is less than equal to sigma r is 1 to infinity M times of this; mod of this is m. So, we can write m into mod of b r minus a r, but this is less than h. So, we get this is less than M h which is less than epsilon when h is less than epsilon by M, where M is the maximum or the least upper bound of the function mod f x over the interval x lying between this. So, this is what; therefore, this implies the function F x is absolutely continuous function in the interval a, b and that is proved here.

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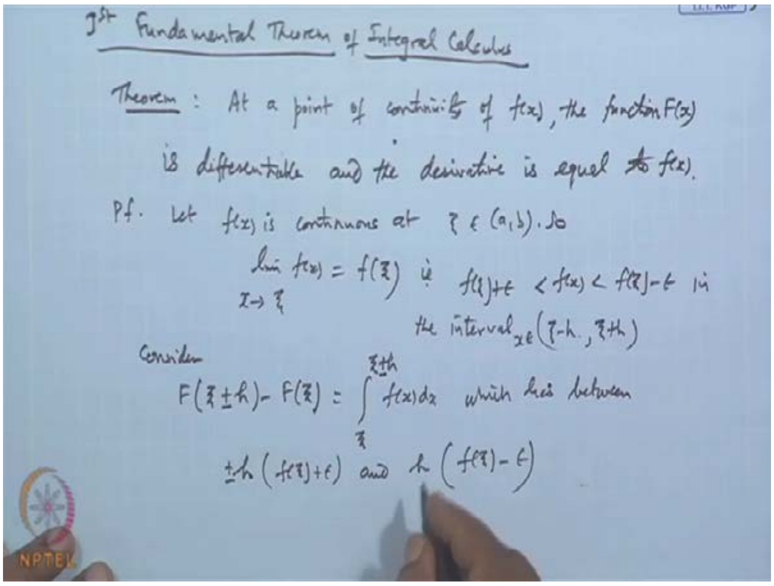

Now after getting, we get now the first fundamental theorems of calculus; so first fundamental theorem of integral calculus. The theorem says like the theorem say h at a point of continuity of the function f x, the function capital F x; that is the in definition integral capital F x is differentiable and the derivative is equal to f x. So, let see that proof: Now what is given is the function is continuous; let f x is continuous at a point xi belongs to the interval a, b suppose. So, limit of this function f x as x tends to xi is the same as f of xi or that is we can say, for any epsilon greater than 0 we can write the f x lies between f xi plus epsilon and f xi minus epsilon, by definition of this. So, this f x lies in this interval; in the interval xi minus h xi plus h; that is when x lies between this interval then f of x will lie in this interval, by definition of the limit.

Function is continuous for given epsilon greater than 0 we can identify the h. So that, the all x which lies in this interval the corresponding f x will lie in this interval. So, this is by definition. Now consider this f of xi plus h minus capital F of xi. Now this will be same as xi plus xi plus h by definition f x; just by definition of this indefinite integral we can use this, now this f x lie between these two bound. So obviously this interval will lie between them; so this integral which lies between what? plus h f xi plus epsilon and h f xi minus epsilon, because f x lie between these two bound. So, length of the interval is h. So, this value of this integral will lie between these two bound.

Similarly when you take minus sign you are getting this minus xi minus h. So, it will lie between the plus minus this interval; plus minus this, that is all. So, divide by this; therefore, when you take F of xi plus minus h minus capital F of xi divide by plus minus h. Then it lies between what? f xi plus epsilon f xi minus epsilon. So, limit of this when h tends to 0 is nothing but the f of xi and this limit is equal to what; the derivative F prime at a point xi. So if the function is continuous at this point, then the capital F x will be differentiable at that point and the derivative of the capital F at this point coincide with the functional value at the point xi; that is what we have shown.

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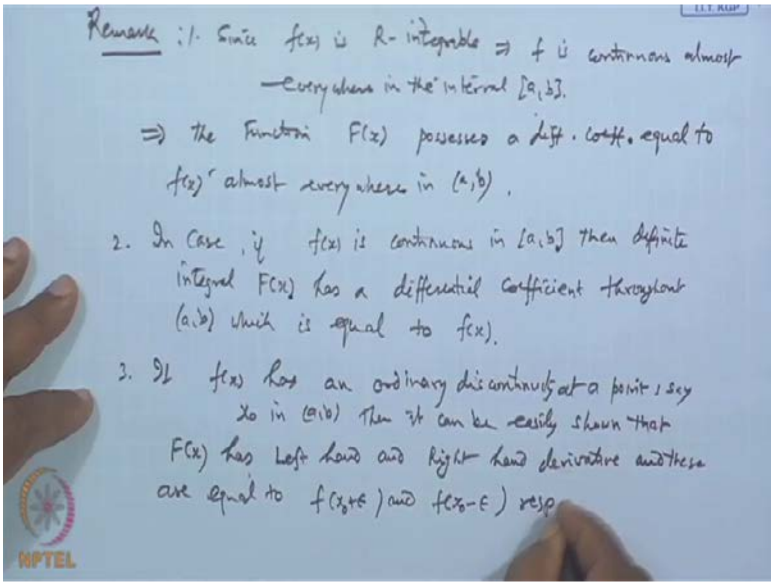

Now as a remark we can say, now xi is an arbitrary point xi. So, at any point of continuity the function F x differentiable and derivative is equal to f x. So, this can be shown. Now remark we can say, given that function f x is Riemann integrable; since f x is Riemann integrable, so the necessary sufficient condition for the Riemann integrable is the function must be continuous almost everywhere. So, this implies f is continuous almost everywhere in the interval a, b and f x is continuous almost. So, the corresponding indefinite integral capital F x will be differentiable almost everywhere in the interval a, b. So, this implies that the function capital F x possesses a differential coefficient equal to f x almost everywhere in the interval a, b; this is the first one.

Second is, in case if f x is continuous throughout the interval in the closed interval a, b then the definite integral capital F x has a differential coefficients throughout the interval a, b which is equal to f x. So these two remarks are very interesting, important in fact. The third remark, we can say at the point of ordinary discontinuity. If f x has an ordinary discontinuity at a point say x naught in the interval a, b, then ordinary discontinuity means that left hand limit and right hand limit exist but may not be equal. Then in that case, the function then it can be easily shown that the function capital F x has left hand and right hand derivatives, and these are equal to f x naught plus epsilon and f of x naught minus epsilon respectively. Now at the point of discontinuity of second kind of f x, it means the function is not defined at this point; the capital F x need not need not have derivative on either sides. So, this we get it.

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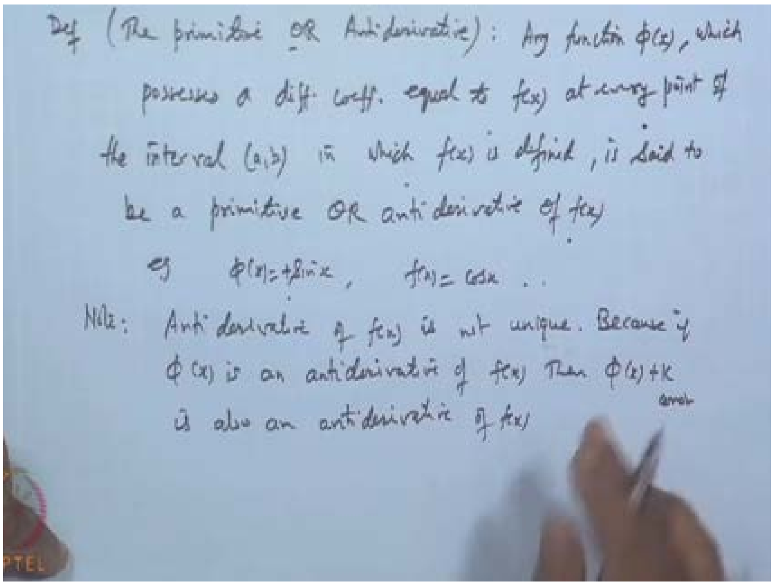

The concept of the primitive or it is also known as anti-derivative; primitive or anti-derivatives. Any function phi x which possesses a differential coefficients equal to f x at every point of the interval a, b in which f x is defined, is said to be a primitive or anti-derivative of f x. So what he says is, suppose f x is a function, a function phi x is such that, is differential coefficient coincide with the f x at every point. Then we say phi x is an anti-derivative of f x. For example, if I take the phi x is suppose sin x, then f x is say cos x. So minus sin x, this function is an anti-derivative of cos x. Is it not; because the derivative of this sin x, this is plus. So, derivative of this sin x coincide with f x. So, it is sin x will be anti-derivative of cos x like this.

So, similarly we can get another example for like this. Now, these anti-derivatives may not be unique. In fact, we have the anti-derivative of the function f x is not unique because if we. Because if phi x is an anti-derivative of f x, then phi x plus constant K is also an anti-derivative of f x; because derivative also coincide with phi x. So, it is not unique in nature. It is a function is anti-derivative then constant plus that function will also be anti-derivative.

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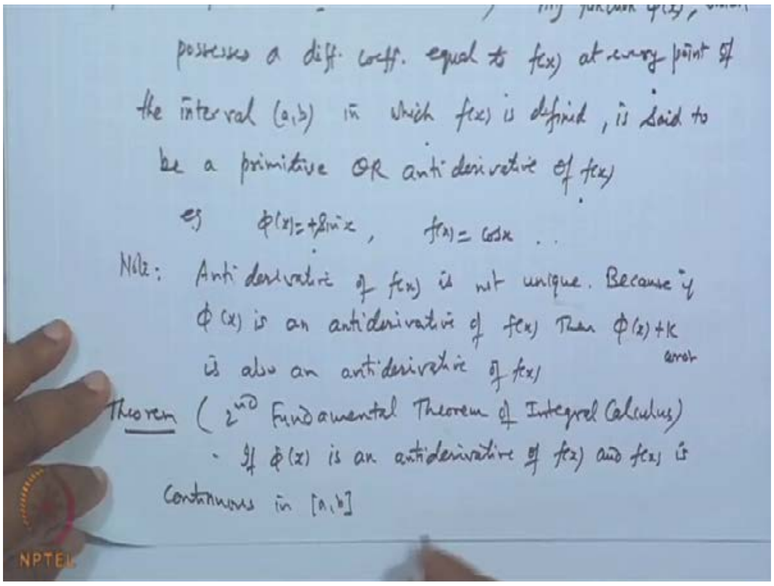

Now we have a best one there, we have the theorem which is known as the second fundamental theorem of integral calculus. What this theorem says is if phi x is an anti-derivative of f x and f x is continuous in the closed interval a, b.

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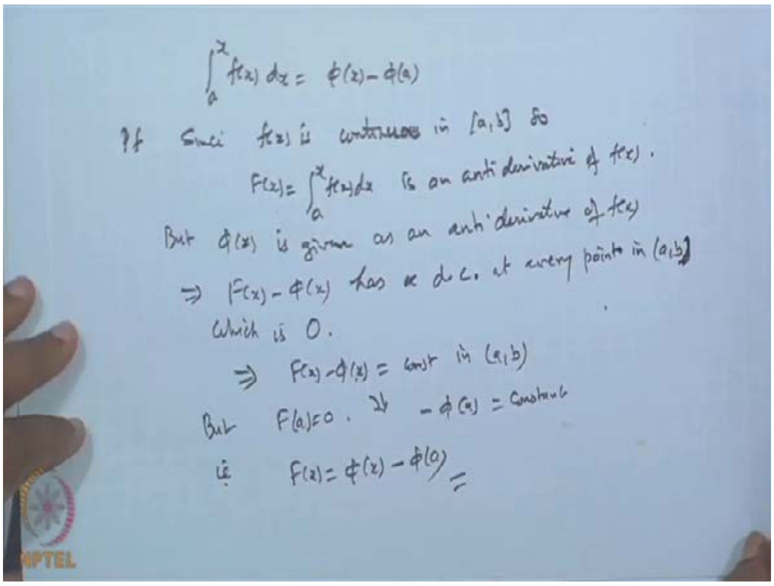

Then a to x f x dx is equal to phi of x minus phi of a. This is also known as the Newton's formula of course; so that we will see then then f x is continuous in this. Now let us see the proof of it. Now f x is continuous. Since given that f x is continuous in the interval a, b. So, the function capital F x which is a to x f x dx is an anti-derivative of f x. This we have already proved, the derivative of a prime x will coincide with the f x. So, it is an anti-derivative of this. Let us suppose phi is another anti-derivative but phi is already given; phi is given as an anti-derivative of f x. So what the result if phi x is anti-derivative, then phi x plus constant will also anti-derivative. It means the difference between phi x and f x must be a constant.

So this implies that F x minus phi x, this has a derivative differential coefficient at every point in the interval a b open interval of course which is 0. Why? Because the derivative of this is F x, derivative of this is also F x. So, this function has a differential coefficient at every point and the value is 0. Therefore, this implies that the function F x minus phi x must be a constant function in the interval a b. But what is F of a, f of a is 0. So, if I substitute here x equal to a. So, from here we implies that minus phi a is constant; that is F of x comes out to be phi x plus constant. So, constant means phi of a. So, that is proved the result. So, that is equal to F of x and this completes the proof.

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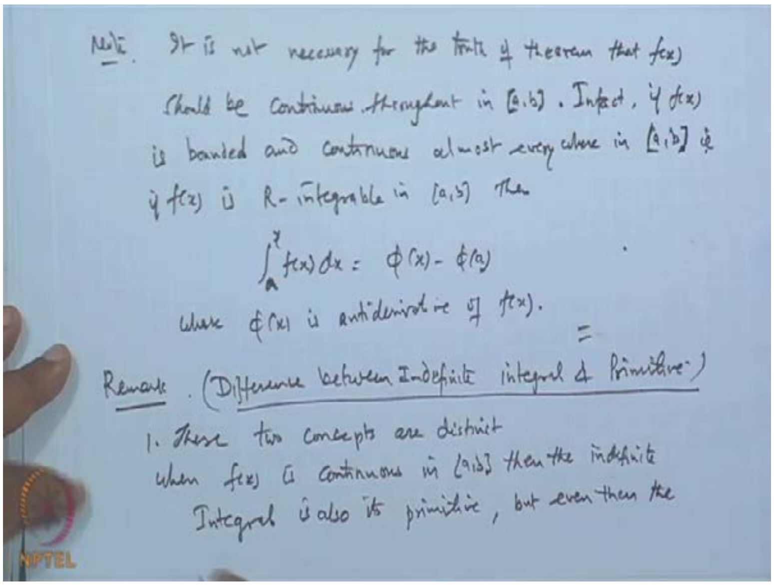

Remark or note: It is not necessary for the truth of the theorem that f x should be continuous. In fact, if it is a Riemann integrable function continuous throughout in the interval a, b. In fact, if function f x is bounded and continuous almost everywhere in the interval a, b; that is disclosed interval a, b; closed interval a, b; that is the function f x, if f x is Riemann integrable function in the closed interval a, b then the integral a to x f x dx is nothing but the phi x minus phi a, where phi x is the anti-derivative of f x. So, that is what.

Now, this theorem we call it as a second fundamental theorem. Now there are certain cases, remark: What is the difference between the primitive, anti-derivative and indefinite integral. The difference in fact, they are not the same. The difference between indefinite integral and primitive; because indefinite integral is also a primitive because its derivative coincide with this, and primitive by definition those functions whose derivative coincide with f x is the primitive. So, some may think that both are basically they are the same concept but this is not so they differ.

We will give an example where the function has indefinite integral, but primitive does not exist. Similarly the function has a primitive, but indefinite integral may not exist for it. So, we can give this only what is first remark is, first point, these two concepts are distinct, they are not same; two distinct concepts. In fact, the indefinite primitive are two distinct functions, when f x is continuous in the closed interval a, b then the indefinite integral is also its primitive. But even then the primitive is not the indefinite integral. In fact, primitive is not the indefinite integral.

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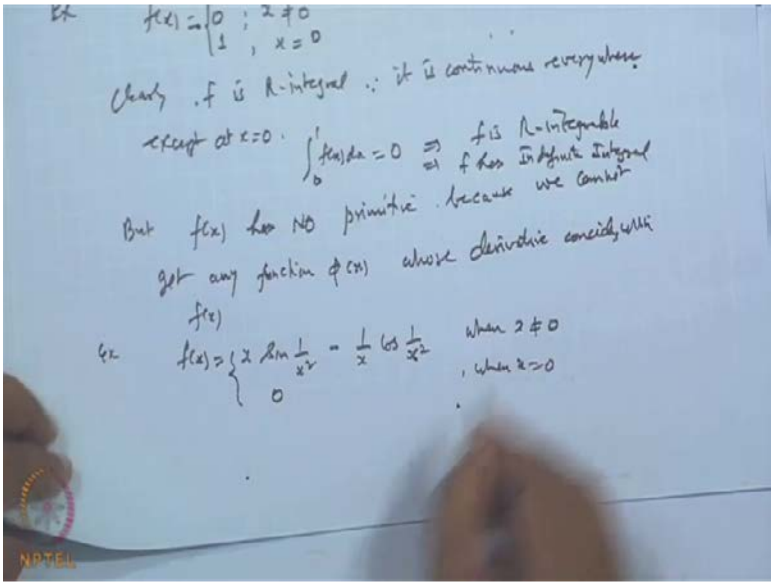

So, example let us see; f x has no primitive. Let us consider the function f x which is 0 when x is not equal to 0 and equal to 1 when x is equal to 0. Let us see now clearly f is Riemann integrable. Why? Because it is discontinuous only at one point because it is continuous everywhere except at the point x is 0 which has a measure 0. So, it is almost continuous function. Therefore this and the value of this integral 0 to 1 f x dx is 0; just I have taken the interval 0 1. But what is f x. So, f is integrable. So, this shows f is Riemann integrable function and value is 0. But f x has no primitive because we cannot get any function phi whose derivative coincides with f x.

Because if we take phi x to be constant, derivative will be 0; but at the point 0, the derivative is not coming to be 1 and if you want the 1; so phi x must be x. So again the phi x is x, then it must be derivative will be 1, but it cannot be 0 for other points. So, we cannot find the primitive. So, the function is Riemann integrable means it is indefinite integral exist. So, that is the f has a indefinite integral, but it does not have any primitive. Another example let us take. This function f x which is x sin one by x square minus 1 by x cos 1 by x square, when x is not equal to 0 and equal to say 0, when x is 0.

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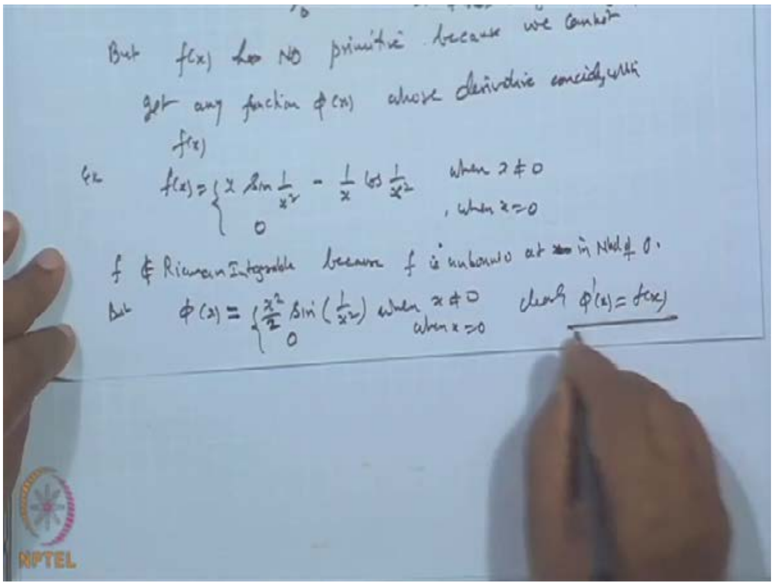

Now this function is not integrable. Why? f is not Riemann integrable; does not belong to Riemann integrable function. Why? Because f is unbounded at the point x is 0, nearby in the neighborhood of 0; neighborhood of 0 this goes very frequently up and down. So, it is not bounded function, f x is unbounded. So therefore, it cannot be Riemann integrable function. But the phi x exist; phi x if I write like this x square by 2 sin 1 by x square when x is not equal to 0 and equal to 0 when x is 0. Now if we take this phi then clearly the phi dash x coincide with f x. So phi is the primitive, but Riemann integrable does not exist. So, phi is primitive which is primitive of f x, but it is not Riemann integrable. So, this can show that we get it.

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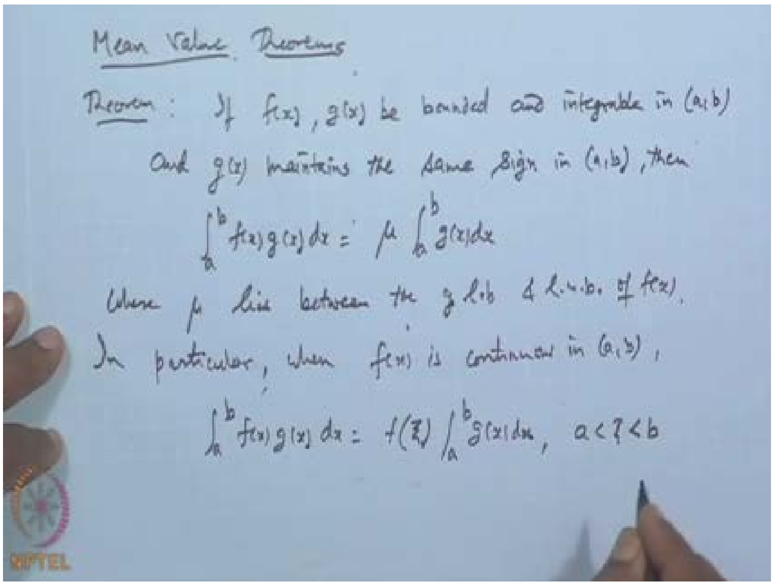

Now there are two mean value theorem which we will just state. Theorem: The first theorem says if f x g x be bounded and integrable in the interval a, b and g x maintains the same sign in the interval a, b. Then integral a to b f x g x dx is mu times a to b g x dx where mu lies between the greatest lower bound and least upper bound of the function f x. In particular when f x is continuous in the interval a, b then integral a to b f x g x dx equal to f of xi a to b g x dx where for some xi lying between a to b.

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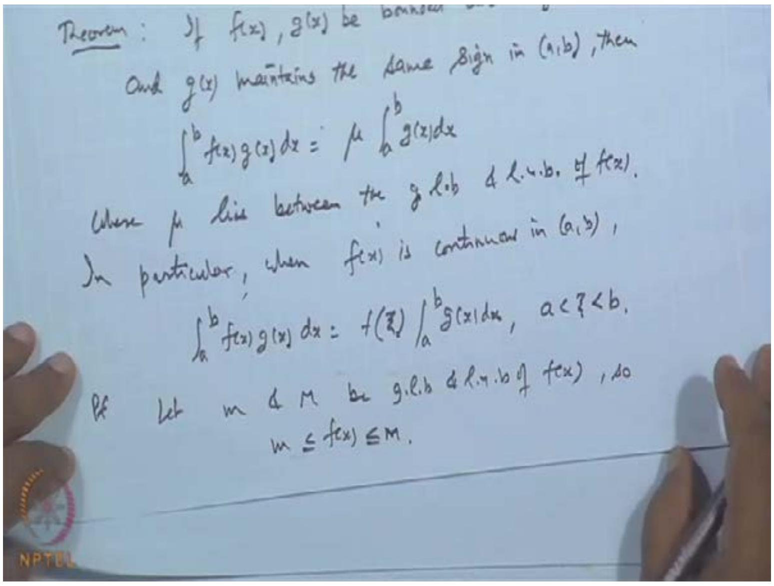

Let us see the proof of this. So what is given, f and g be a bounded function. Let f x and g x be a bounded function and integrable in the interval a, b and g x maintains the same sign. So, let us suppose let m and capital M be the greatest lower bound and least upper bound of the function f x. So, value of the f x will lie between m and capital M. This is clear. Since g x maintains the same sign, either it is positive or it is negative.

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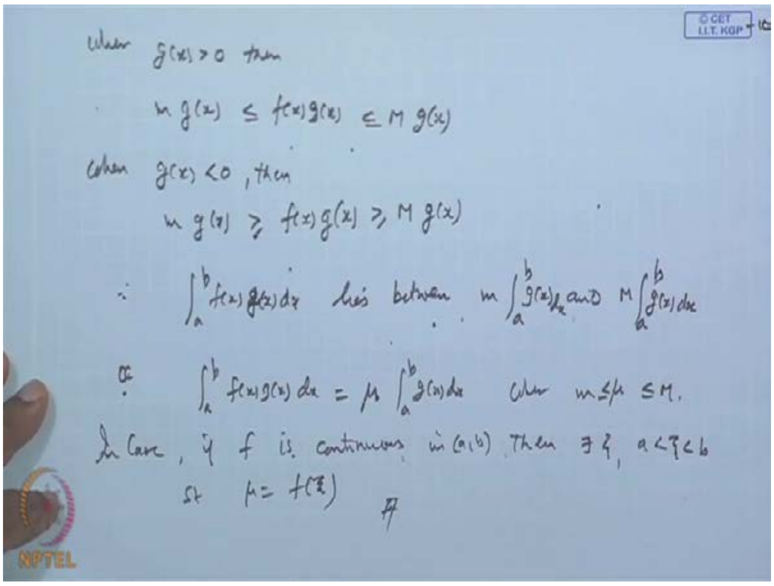

So, if g x is positive. So, when g x is greater than 0 then we have this multiply by g x both side, we get the same f x g x which is less than equal to capital M g x. And when g x is negative then the order will reverse. So, we get m of g x is greater than equal to f x g of x which is greater than equal to capital M g x like this. Therefore integral a to b f x g x dx, this integral; so this integral will lie between m integral a to b g x dx and capital M integral a to b g x dx. Either this will be the lower bound when this is positive or this will be lower bound when this is negative. So, it lies between these two, but f and g both are giving to be what; Riemann integrable functions, is it not. So, there will be some point; it means everywhere the integral a to b f x g x dx this Riemann integrable exist.

So there will be some point where the value of this, that is a to b f x g x dx will be some constant times mu into a to b g x dx. So, we get this integral where mu lies between m and capital M, clear. So, that is just by. Now in case if f is continuous then this mu can be replaced continuous in the interval a, b. Then there exist a xi in between this, such that mu becomes f of xi. So we get the result; that is proof. So, this is the first theorem, mean value theorem.

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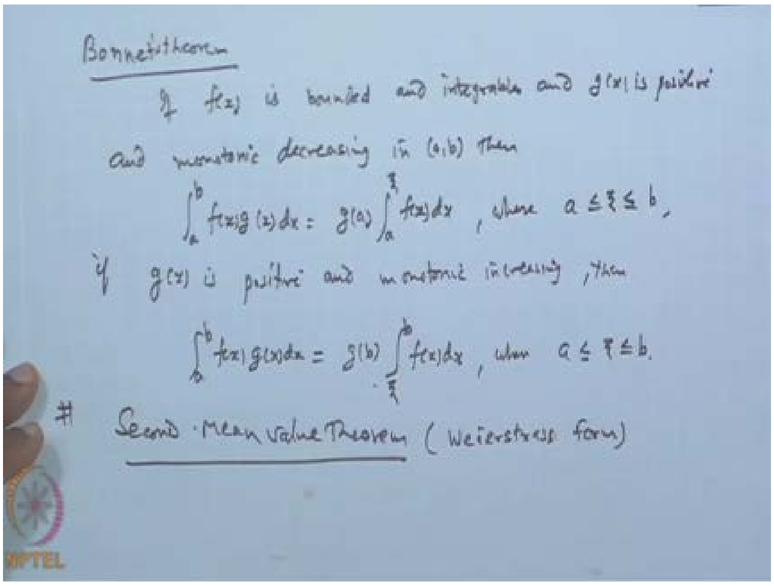

The second mean value theorem, we can get it from this result which is known as the Bonnet's theorem. We will not prove it, just a statement of the Bonnet's theorem says. If f x is bounded and integrable and g x is positive and monotonic decreasing in the interval a, b. Then integral a to b f x g x dx is g of a to xi f x dx where xi lies between a to b. Now if g x is positive and monotonic increasing, then integral a to b f x g x dx is equal to g b integral xi to b f x dx, where a is less than equal to xi less than equal to b. Now this theorem will be used in proving and introducing the mean value theorem; second mean value theorem.

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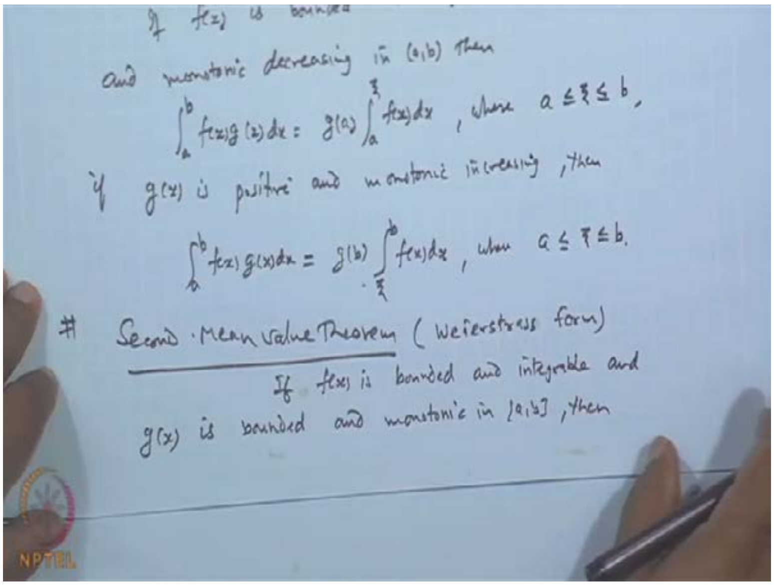

The second mean value theorem it is also known in the Weierstrass form. This is known as the Weierstrass form. What this theorem says is if f x is bounded and integrable and g x is bounded and monotonic in the closed interval a, b.

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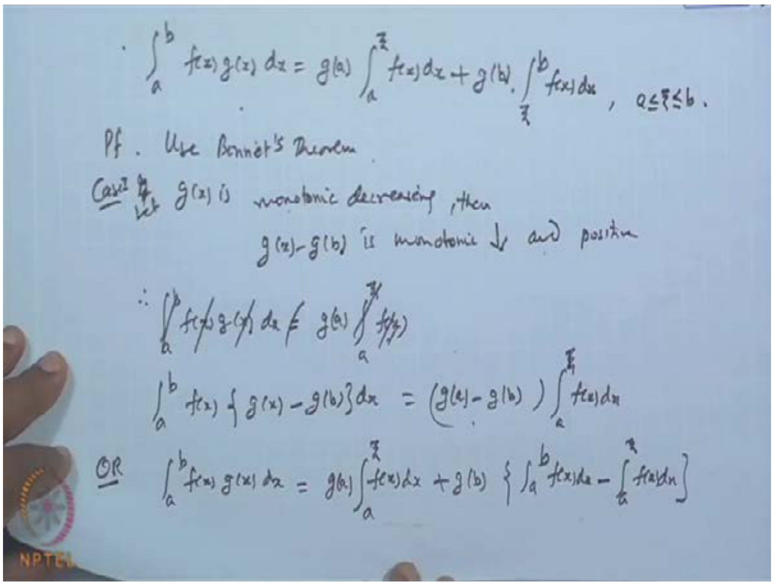

Then integral a to b f x g x dx is equal to g of a integral a to xi f x dx plus g of b integral xi to b f x dx where a is less than xi less than equal to b. The proof is, use Bonnet's theorem to prove this result. So what we get is, if our function g is monotonic decreasing, case one: Let g is monotonic decreasing function, then the g x minus g b will be positive is monotonic decreasing and positive because monotonic decreasing means it has a value at the point a is the highest and b is the lowest. So we are taking g x minus g b so is always be positive and monotonic and have the same sign of course.

Then therefore, integral a to b f x g x dx, this will be equal to g a integral a to xi f x. In fact, this is f of g x minus g b d a, this will be equal to better you apply this. So we can write like this f of g x, no first you write this one. Therefore a to b f x and then g x minus g b dx, because this is monotonic decreasing and positive; so by the Bonnet's first theorem part, this will be equal to g of a minus g of b integral a to xi f x dx because this was the Bonnet's theorem. Bonnet's theorem says if f is bounded integrable and g x positive and monotonic decreasing, then f of x g x dx is g of a into this. So, here this thing is monotonic decreasing and positive.

So, the value of this at the point a into at some point a integral of a xi f x dx. So, this will come. Now this can be written as integral a to b f x g x dx. This will be, just arrange the terms that's all; g of a integral a to xi f x dx this term first and then plus g b common and then inside this we get a to b f x dx minus a to xi f x dx this part, but this will be equal to what. This is equal to g of a a to xi f x dx as it is plus g of b and this can be written as integral xi to b f x dx, where xi lies between this and that is what we wanted to show. Is it not; that is the first part of this is f x is a monotonic, then we get g of a, this part plus g of b, this part. Now on this case this is monotonic increasing.

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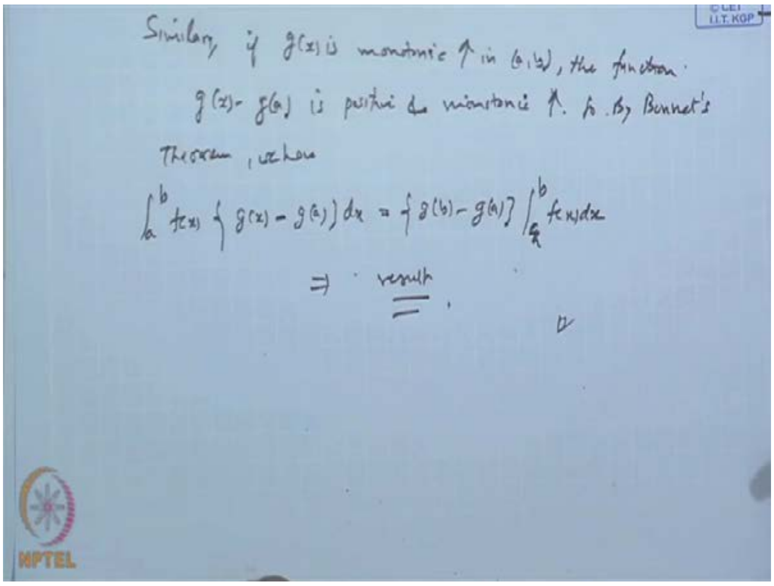

Similarly if g x is monotonic increasing in the integral a, b; the function g x minus g a is positive and monotonic increasing. So, by Bonnet's theorem again we have a to b f x g x minus g a dx is g b minus g a integral xi to b f x dx and which implies the result. So, we get this thing; that is all.

Thank you very much.

Thanks.