A Basic Course in Real Analysis

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Lecture - 43
Definite and Indefinite Integral

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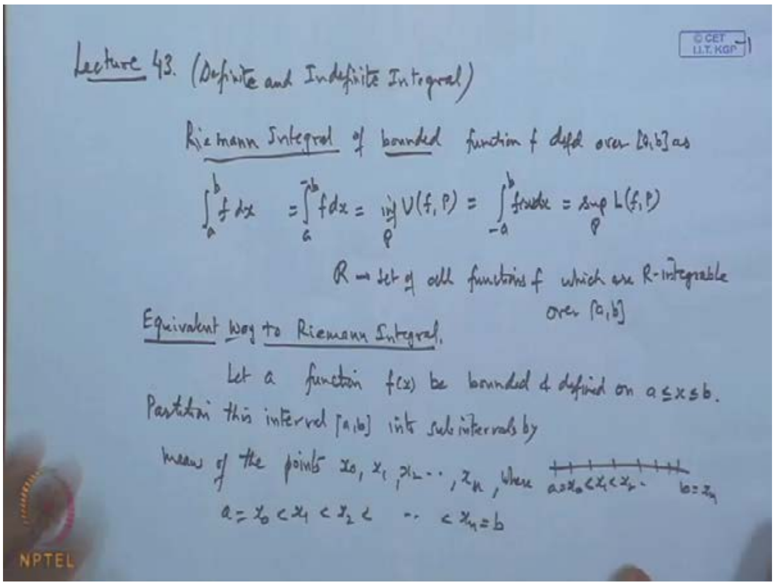

So, today we discuss the definite and indefinite integrals. We already discussed the Riemann integral of a function f, and we have defined the Riemann integral of a bounded function f, defined over the close interval a, b as the upper sum f dx which is equal to the infimum of the upper sum with the partition P and lower sum a bar b f d x dx or f x x dx which is the equal to the supremum value of the lower sum of f over the partition p. And supremum infimum is taken over all such partition p, and whenever these two very coincide. Then it is represented by a to b f dx and we say it is a Riemann integral of a function f over the interval a b, and the cross R is the set of all functions which are Riemann integrable functions f which are Riemann integrable Riemann integrable over the interval a, b; we denote by this. So, f will be an element for this class.

So, this is the way we have introduced the concept of the Riemann integral. Then equivalent way of introducing the concept of Riemann integral is as follows. What in a similar way what we do is, let us suppose a function let a function f x be defined, be bounded and defined on the interval a less than x less than b.

Then, partition this interval a, b into sub interval y introducing the point x1, x2, and x n is b. So, partitioning these interval into seven. So, partition this interval a b into sub intervals, intervals by means of, by means of the points x naught, x1, x2, x n, where a is x naught, less than x1, less than x 2, and less than x n is say b. So, we are getting this point. Let us suppose, let delta x i is x i minus x i minus 1 be the length of the interval of the sub interval x i minus 1 to x i, let us suppose this.

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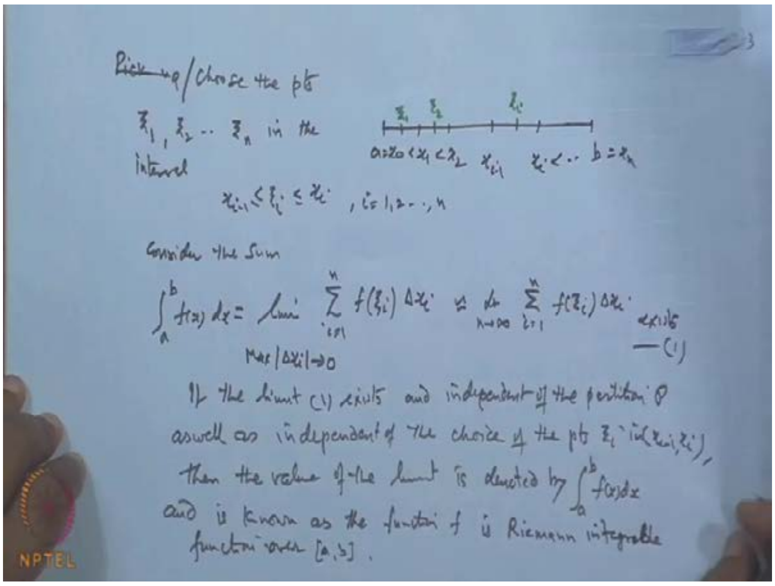

So, let us again take this one, a to b, a is x naught, here is x 1, say x 2, suppose this is x i minus one, here you say x i, and then like this and this way x n. Now, what we do is, we pick up the point; pick up or choose the points or choose the points instead of pick up, choose the points; take z one, z two, z say n, in the intervals, in the intervals say x i minus 1 less than z; and less than or equal to xi, here i is 1, 2 up to n. So, in this we are choosing the point z 1 here say here z 1 here z 2 this is say z i and like continue this.

Now, at this point find the functional value. So, now consider the sum sigma f of z i multiply by the length of the interval, in which this point z lies and i is 1 to n. This thing take the limit of this sum, when maximum of the length delta x i goes to 0 or this is equivalent to the same as limit as n tends to infinity i is 1 to n f of z i delta x i. If this limit exist, if this exist, then denote this limit as a to b f x d x and is called, and then we say; if this limit adjust; if the limit one, let it be one, exist and independent of, independent of the partition p, partition p, as well as independent of the choice of the points z i in the interval x i minus 1 to x i then the value of the limit, limit is denoted by integral a to b f x d x f x d x and is known as, is known as the, is known as that the function f is Riemann integrable function, integrable functions; the over the interval ab, over the interval ab.

So, this is equivalent way that the previous one. What we simply here we are assuming the function is bounded function. So, f of z cannot go to infinity, it will remains the less than bounded. So, whenever you choose the point z 1, z 2, z any arbitrary point, so basically the value of this, maximum value of this will be taken as this, is it not? So, it will, when the limit exists, means when this maximum delta z goes the points are so close to each other, that almost this entire line is covered and the curve will be something like .This, in case if the function is continuous function, then this is we call it as a definite integral and it represents the area bounded by this.

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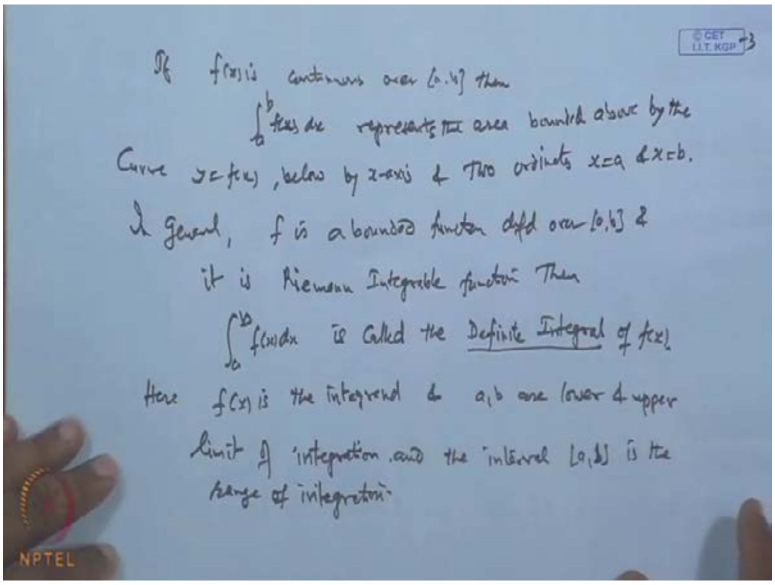

So, if the function f x is continuous over the interval a b, then integral a to b f x d x represents the area bounded above by the curve, by the curve, y is equal to f x below by x axis and two ordinates, x equal a and x equal to b. And we call it this as a definite integral, we call it a definite integral. But in general, in general, when f is a bounded function defined over the close interval a b and it is Riemann integrable function, and it is Riemann integrable functions, then integral a to b a to b f x d x, f x d x is known as f x d x integral remind in the interval is now called the definite integral of the function f x over this. So, this is basically it.

When it is continuous just you bring the area. But if it is bounded, then also it will give some value, finite value and we call it is a definite integral. Because it cannot go beyond this two limits, f is called a integral function. Here f is, f x is the integrant and a and b are lower and upper limits of integration, limit of integration and the interval a b is the range of, a b is the range of integration. So, we have studied already in class twelfth etc like integral a to b f x d x when f is continual and we call it as definite integral. So, here we are just extending the definition of definite integral for those functions which are Riemann integrable over the close interval a b.

So, this integral a to b f x d x and f belongs to a Riemann integral class of Riemann integral function will be termed as a definite integral. On the other hand, the function, the integral, the integral a to x f x d x or ft d t, f t d t., so that this limit should not be variable integration different where, where f x is or f t you can say f t is integrable, is Riemann integrable, is Riemann integrable, in the interval, in the interval a x, in the interval a x is called, is called the indefinite integral, indefinite integral of f x

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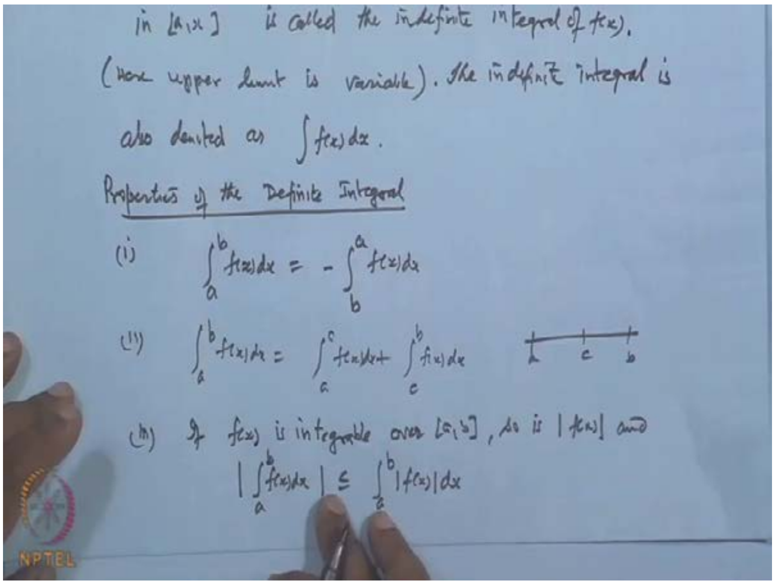

Here the upper limit is a variable one, here upper limit is variable. That is why we call it indefinite. This is also the indefinite integral, integrals is also written, is also denoted as integral f x d x without limits, without limit. So, there is. So, the only difference is when the integration is taken over some interval we are the lower and the upper limits are fixed then this integral and f is a Riemann integral function then this is called a definite integral when the upper limit is a variable one then we call it as a indefinite integral that is all. So, there are various properties of the definite and indefinite integrals. The property of definite integrals is already we have discussed in a general way, when we have discussed the properties of Riemann integral. So, when we take alpha x equal to x it reduce to the Riemann integral and a and b are fixed. So, it is a definite integral.

So, the property which are already drive for this Riemann integral will continue to hold good incase of the definite integral. So, we will just simply state those results without proof. So, properties of the definite integral: the first properties is that we know if a is greater than b, then we say a to b f x d x. But if we reverse the order of this lower and upper limit, then we get this thing and it follows from the definition that a to b f x d x the same minus times b to f x d x. Second property says if suppose the interval a b is given and there is a point c in between a b, then an f is Riemann integrable over the entire a b then f is also Riemann integrable over this a c and c b. An integral a to b x f x d x is the same as a to c f x d x plus c to b f x d x. And this also can be proved with the help of definition what we do is we partition the interval a b in such a way, so that one of the portioning point coincide the c.

And then you apply the definition of the Riemann integral, integral as a limit of the sum, we will automatically get this two things. So, nothing and third condition is if f x is integrable, f x is integrable over the interval a b, so is mod f x that is, if f is Riemann integrable mod of f is also Riemann integrable and the integral, and the modulus of the integral f x d x a to b is less than equal to integral a to b mod f x d x, mod f x d x. Again, this result follows from the definition because when we start with the definition and take the limit of the sum, then limit is a continuous function, modulus is also continuous function. So, you can choose take limiting modulus inside the limit and then summation sign you will get this result quickly. So, there is no much problem.

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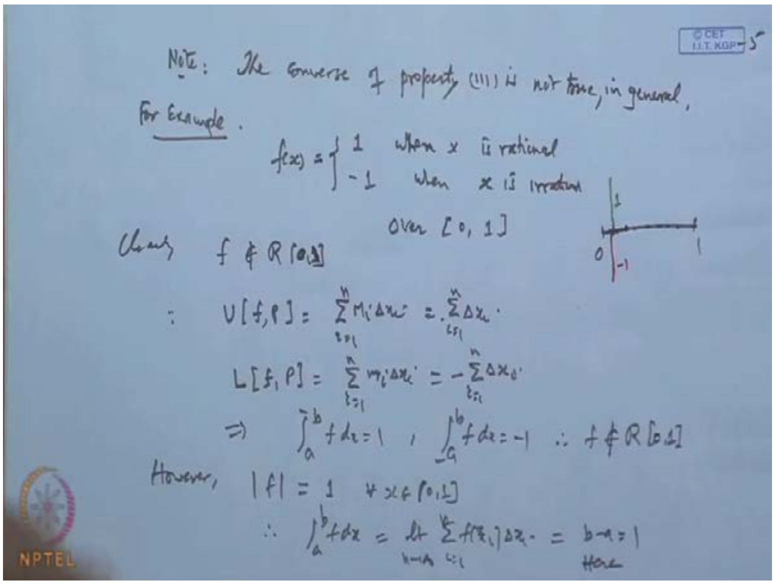

However, here is a remark. The converse of this note, the converse of property three is not true in general. It means a function is such that were the mod f will be Riemann integrable function, but f need not be a Riemann integrable function. For example, suppose for example, if we look a function f x as 1 when x, is when x is rational and minus 1 when x is irrational, over the interval say, 0 and 1 over this interval. So, over this interval, we are choosing the function f this is 0 1 interval at the rational points. These are the rational points, they are very close to each other, the value of the function is one and for the irrational points the value of the function is minus one, this minus one. So, value of the function is minus one like this. So, continue this way.

So, we get this function, now obviously this function is not Riemann integrable function. Clearly f does not belongs to Riemann integral function over the interval a b; a b means 0 one here 0 1. Why? Because what is the upper sum of this? Because upper sum of this function f with respect to any partition p is nothing but what, sigma the value of Mi into delta x i, i is 1 to n, is it not? Upper sum and this Mi will be when it is rational it is 1. So, maximum value will always be 1. So, it is equivalent to sigma i is one to n delta x i. While the lower sum of this, lower sum of this is nothing but what one to infinity small m i delta x i and that is equal to what m i is minus 1, so it is sigma i equal to 1 to n delta x i.

So, when you take the limit or take the infimum and supremum value of all such partitions, then what we get? So, this implies that upper sum of f d x is 1, while the lower sum of this f d x is minus one. So, they are not equal therefore, f cannot be a Riemann integral function over the interval 0 1. However, if you take mod of f then mod of f means always 1 for all x belongs to the interval 0 1. Therefore, the Riemann integral a to b f d x which is the limit of the f x and so on, limit of the sigma f of zi delta x i is 1 to n as n tends to infinity is nothing but the b minus a and here the b minus a will be equal to what is nothing but the 1. So, that is what. Here this is one because a is 0 b is one and we get this. So, here a is 0 b is one, b is one. So, this is. So, what I said that mod f is Riemann integral function, therefore, f mod of f belongs to R, but f is not in R. So, the converse is not true therefore, this.

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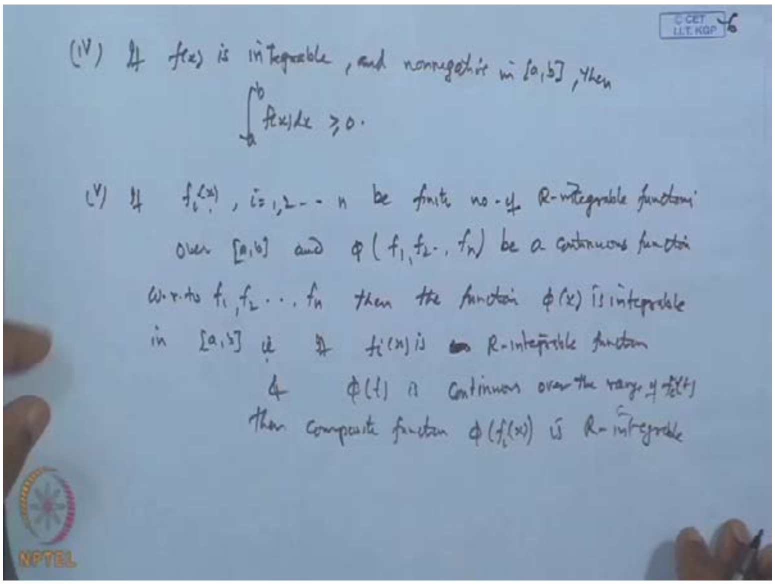

Now, fourth property is, if f x is integrable, f x is integrable and non-negative in the interval a b, then the integral a to b f x d x is greater than equal to 0. It also follows because its non-negative the sum will be sigma limit of the sigma f x i and sum f x i is positive, f of x i positive, delta x i is positive, sum be again non-negative, so limit will also be non-negative, so then nothing to prove it.

Now, fifth property, if f1, f2, fn; if f i is x i is equal to say 1 2, and say, n be integrable finite number of Riemann number of Riemann integrable functions over the interval a b, then and phi and phi of f1, f2, fn be a continuous function, be a continuous functions with regard to, with respect to f1, f2, f n with respect to these f1, f2, f n, then the function phi of x is integrable in the interval a b. That is, the meaning is if fi is continuous, fi is integrable function, fi is Riemann integrable function, and phi t is continuous over the range of fi at t, fi t, then the composite function phi of f t f i x is Riemann integral. And this also we have proved in general for the Riemann integral. So we know.

Similarly, the sum of this (( )) we can get it like this. Now, converse of this again, sorry, is a remark; note: if suppose f is Riemann integrable, phi is also Riemann integrable, then it's there no guarantee that phi of f will remain Riemann integrable. If f and phi, if f x and phi t are Riemann integrable functions, then phi of f, phi of f x need not be need not necessary, necessarily Riemann integrable. For example,

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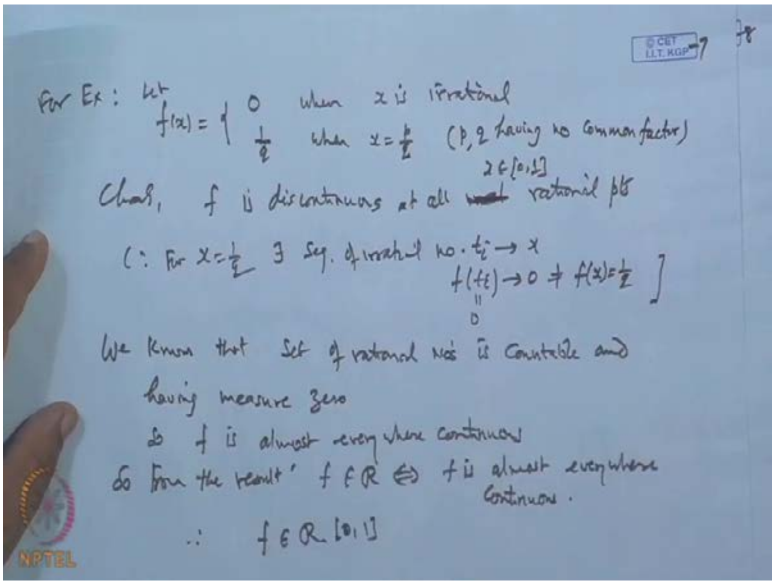

For example, let us take this problem f x is, let f x is defined as 0 when x is irrational and is defined to be 1 by q, when x is of the form p by q; p and q having no common factor. That is irrational number and f x is comes out to be this is a rational number. So clearly, if we look clearly, function f is discontinuous at all irrational points, sorry, all rational points, discontinuous at all rational points. Why? Because the reason is suppose, I take any rational points, suppose I take x equal to one by q, then corresponding to this, for this there exist a sequence of irrational numbers say t i which goes to x.

But what is the image of f ti. This f ti will always be 0. So, the limiting value will be 0, which is different from the f of x because this is coming to be 1 by q not 0. So, at every rational number is a point of discontinuity. But if we take the irrational number, then if we take a rational number and take sequence of the rational, then q tends to infinity it will go to 0. So, here you'll get the continuity. So, this discontinuous functions are the points of rational points and we know, we know that set of rational number, rational numbers is countable and having measure 0, measure 0. So, once it is countable in measure 0, so the function f is almost continuous function; almost everywhere continuous. Because the point where it discontinuous from a set of measure 0 and by definition result a function f is Riemann integrable if and only if it is almost everywhere continuous function.

So, by from the result that a function f belongs to Riemann integrable if and only if f is almost everywhere continuous. So, according to this we see therefore, our f belongs to the class R is Riemann integral function over the interval say, am choosing the interval any x is x naught and set 0 1 I am taking, no problem, 0 1. Now, let us take this over x belongs to here we say, x belongs to 0 1, clear? x belongs to 0 one interval. p q (( )) q is not 0 of course, otherwise will problem.

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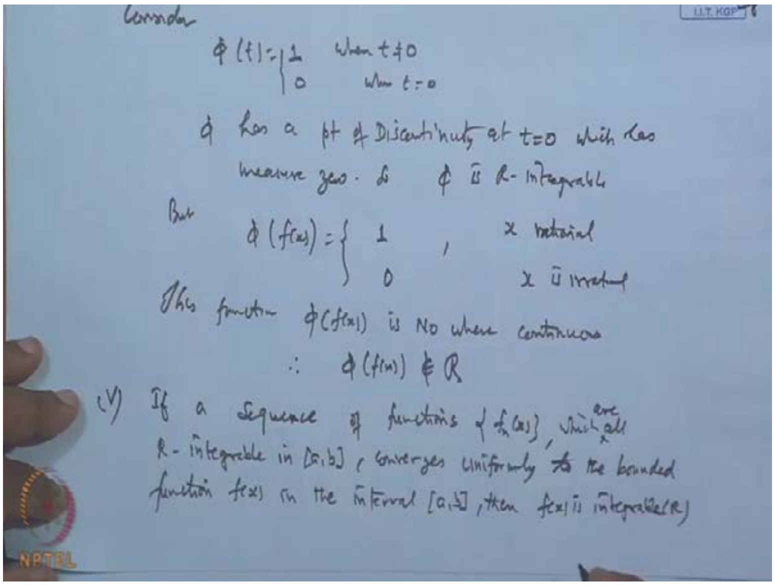

Now, if we look the function pi, let us consider the function phi, consider phi as the function, here phi t is 1 when t is non 0 and equal to 0 when t is 0 over the same range of these (( )). So, obviously phi has a point of discontinuity, point of discontinuity at t equal to 0 because limit point right hand limit is 1, but the functional value is 0. So, it is a point of which is again has a measure 0, which has measure 0 because single term set has a measure 0. So, phi is integrable. So, f is Riemann integrable; phi is Riemann integrable. Therefore, what is the phi f of t, but phi of f x if I take, then this value will be what when x is, when x is rational, the value of f x is non 0. So, according to this it will be the 1 and when x is irrational the t is 0. So, value of the function is 0. So, it is 0.

So, at the rational points, at the rational point, it is rational number, is not its value is 1 when it is 0 when its irrational. So, this function, this function phi f x is no where continuous; in fact it is discontinuous function throughout this interval, because if we take rational numbers, again there is a sequence of rational 0 0, which does not coincide with this if we take a rational sequence one. So, it is throughout the interval it is discontinuous function. Therefore, this therefore, a phi f x cannot be a Riemann integral function, because the necessary and sufficient condition for a function to the Riemann integral is it must be almost everywhere continuous function; but it’s not almost everywhere, it is everywhere discontinuous. So, it is not.

So, this shows that the continuity of the function phi is must. If one is, f is Riemann integrable and phi is continuous, then the composite function will remain Riemann integrable. But if phi is also Riemann integrable, then you cannot say whether the phi of fx will be a Riemann integrable function. That is, composition of the two Riemann integr able function need not be Riemann integrable function. That is what is showing.

Now, another fifth property is, if a sequence of, if a sequence of functions, if a sequence of functions fnx, where which are of, which are all, which are, which are all Riemann integral functions or integral in the interval a b, converges uniformly, converging uniformly to the bounded function to the bounded function fx in the interval, in the interval f on the function in the interval say a b, then f x is integrable, that is Riemann integrable and the integral a to b f x d x is nothing but the limit as n tends to infinity integral a to b f n x d x, that is all.

So, what I says is, if a sequence f n is given to be a integrable functions of say, Riemann integral function and it converges uniformly to f, when you say f n converges uniformly to x over the interval means f n x minus f x is less than f signer for all x. It in depend, the f signer delta does not depend that x, x is irrespect. So, converge uniformly through all n is greater than n naught, the converge uniformly to function f x in the interval a b. So, in f x minus f n is less than f for n is greater than equal to m then f x is integrable and the and limit is the of this is the integral of this.

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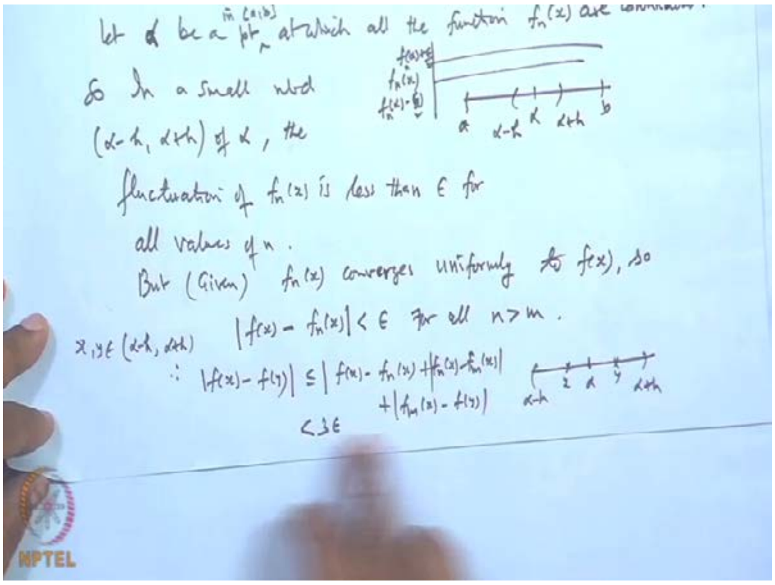

So, let us see the proof of this. Now, since f n x these are the element of the Riemann integral functions. So, by necessary and sufficient condition in of the result of the theorem that f belongs to r if and only if it is almost everywhere continuous, implies that each f n x is almost everywhere continuous function, almost everywhere continuous; almost everywhere continuous over the interval a b, this by definition. So, it means x the point where it is discontinuous from the set of measure 0. So, let us picked up the point, let a be a point at which all the functions f n x are continuous.

Let us take, let a point alpha. Let a alpha be a, because a we have already used alpha be a point in the intervals at which all the functions are continuous, are continuous. So this... So this is the a b and here is alpha, where all functions are continuous. So, we can find out a small neighborhood of say alpha minus h to alpha plus h where the fluctuation of the functions, fluctuation of the functions f and x, f and x will lie between f signer and minus f signer. That is f of alpha, f of alpha minus h, f signer, f of alpha plus h, each one of it. So, f n the fluctuation of this f n x is less than f signer, that is f n x minus alpha f n x alpha plus f signer. So, there exist. So, this is small.

So, in a small neighborhood, small neighborhood alpha minus h alpha plus h of alpha; the fluctuation of f n x is less than, less than f signer for all values of n. This fluctuation, suppose for this f signer by 2, f signer by 2, the fluctuation will remain less than f signer. So, this what one (( )) there. But f n (( )), but given sequence f n x is converges, is convergent or converges uniformly, uniformly, converges uniformly to the function f x. So, by definition. So, modulus of f x minus f n x this will remain less than f signer for all n greater than m, converges to them and of course, every x in this. Let us say therefore, therefore the fluctuation of f x therefore, if we take this interval alpha minus h plus h and here is alpha.

So, suppose. So, suppose I take a point x and y then what will be the mod of f x minus f y; this is x and y work belongs to the interval alpha minus h alpha plus h. So, if we take this thing which is less than equal to f x minus f n x, plus f n x, minus f m x, plus f m x, minus f y. Now, this is less than f signer, this is less than, this is less than. So, total is less than 3 f signer. So, fluctuation of the function f x in the neighborhood of this is less than 3 f signer. Therefore, f x is continuous and so alpha is a point of continuous.

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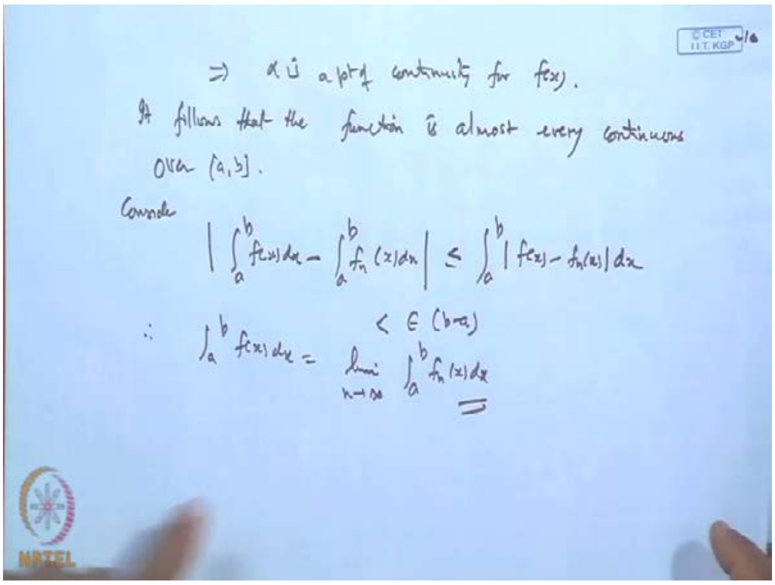

So, this shows that alpha is a point of continuous, continuity for the function f x for the function f x, because the fluctuation is this. And this shows that f x is it follows, but alpha is already it follows, because we can picked up any point where the all the functions are continuous. So, those continuity point the those point at which the f n x are continuous, will also at this point the function f x remains continuous. So, it means the function f x is almost if it follows that the function f x, which is the limiting function, is the function f x is almost everywhere continuous over the interval a b. Once it continues then let us find the this integral. Consider the integral a to b f x d x minus integral a to b f n x d x.

Now, this will be less than equal to, what, f signer because you combine this f x minus f n. So, it is less than equal to integral a to b f x minus, f n x d x less than equal to, but this is f signer. So, it is less than f signer into b minus a. And hence integral a to b f x d x is the limit of this as n tends to infinity integral a to b f n x d x. So, this proves. So, the uniform converges of the function a if f n converges the uniformly to effects, then f x will also be Riemann integrable; provided all f x Riemann integrable functions and this. So, this proves the result.

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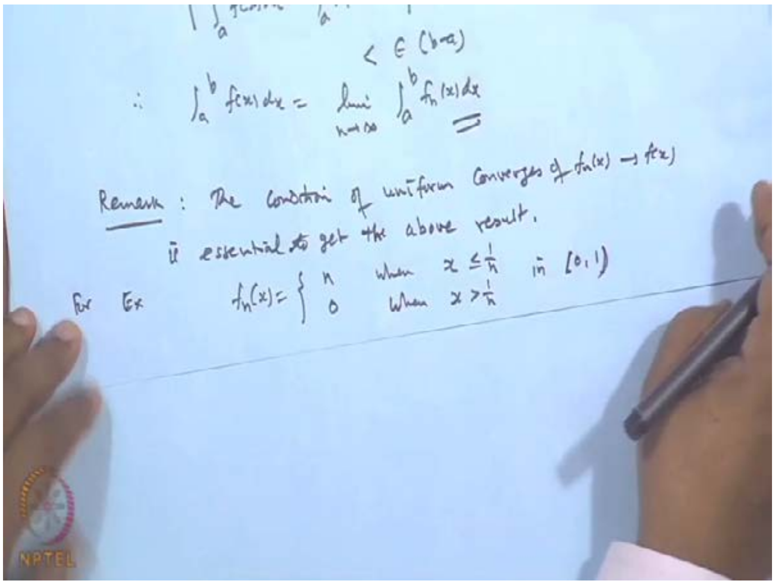

Now again here, we put a remark that condition of uniform converges is must, the condition the uniform converges, the condition of uniform converges of the sequence f n x to f x cannot be (( )) is essential, is essential to get the above result, to get the above result. Because if we drop this condition the above result may not hold good. For example, suppose I take the sequence f n x, f n x is defined as n when x is less than equal to one by n and equal to 0 when equal to 0 when elsewhere, when x is greater than one by n, it is 0 in the interval 0 1, x belongs to the interval 0 1, let it be 0 1.

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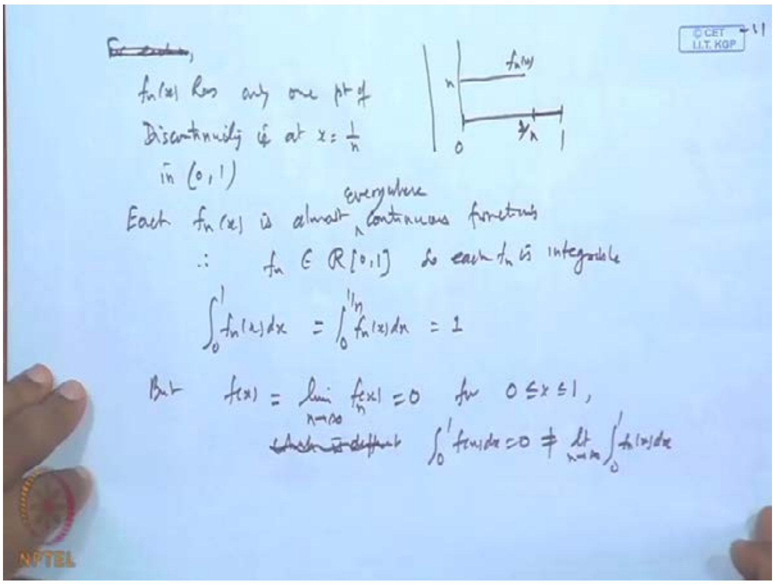

So, what we see here is the function f n this is a interval 0 1, the f 1 x, f 1 is less than equal to 1, f 1 functions is less than 1 and so. f 2 is say suppose n is this, 1 by n is this one by n is this. So, what is this function is the function f n is such the whole x when lying between 0 and 1 the value of the f n x is n, this is n, this is n. So, here it f n x is n and as soon as it crosses it comes out to be here 0. So, f n x is defined like this. So, for each n, f n x has only 1 point of discontinuity, f n x is only (( )), f n x is only one point of discontinuity, continuity that is at x equal to 1 by n in the interval 0 1. Because prior to this the value of the function is one, after right hand value of function limit value is 0. So, basically both are not coinciding. So, it is a point of. So, each f n x is almost continuous, almost everywhere continuous functions, almost everywhere continuous. Therefore, f n x each f n belongs to the Riemann integral class of 0 1.

So, integrable, so it is integrable. So, each of f n is integrable function. So, in each is each f n is integrable and value of this integral 0 to 1 f n x d x this is equal to 0 to 1 by n f n x d x and f n x 1 by n is n, so basically it is 1. But what is f x? f x is limit of this f n x as x tends to infinity. Now, limit of this f n x when n is sufficiently large it means this point is coinciding almost 0. So, the limiting value will be 0, because after this the value will be 0. So, it is zero. So, the limiting value of this is 0 for 0 less than x less than equal to one. But which is different from, which is different from this one- 0 to 1, 0 to 1, 0 to 1. So, all therefore, 0 to 1 f x d x is 0, which differs from the limiting value of 0 to 1 f n x d x. Because the limit of this is 1, so limit of this one and here it is 0. So, we get contradicts. And this is because the sequence are not converges uniformly through the function a. Because when you take the f n x and f n minus f n it is not less than f signer. So, it depends on the points here f signer therefore, this one. Now, we go for some properties of indefinite integrals also.

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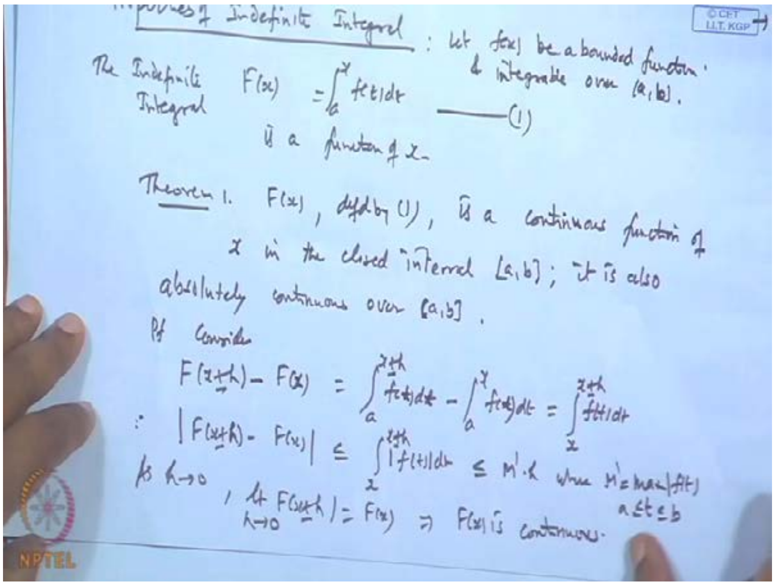

So, like few properties of the indefinite integrals. When f x is bounded, then this integral a to x f t d t, this is, this integral let us denote by capital f because as x changes the value function f x changes. So, when let f x be a bounded function, be a bounded function, let f x be a bounded function and integrable over the interval say a b. The indefinite integral, the indefinite integral, indefinite integral f x which a to x is obviously, is a function of x, function of x because it changes its value with respect to the upper limit.

So, let us denote this value by f x. Now, we have one result. The first theorem says if f x is this, let it be one. The function f x which is defined as one defined by one, defined by one is a continuous, is a continuous function of x, is a continuous function of x in the close interval a b, in the close interval a b. It is also absolutely continuous, it is also absolutely continuous function, continuous over the interval a b. So, how to prove it? Let us consider f of x plus h, minus f of x, minus f of x. So, this is equal to a to x plus h f x d x minus a to x f f t dt f t and f t dt f t dt. So, this will be. So, this is x f t d t. Now, this one equal to integral x to x plus h f t dt, now let f base with the least upper bounds for this. So, what we do is, so the modulus of this, so mod of f x plus h, minus f x is less than equal to integral x to x plus h modulus of f t dt, which is less than equal to say M dash into h where M dash is the maximum of the mod f t when t ranges from this interval the maximum value of the function over the interval a b.

Now, as h tends to 0, the right hand side goes to 0. So, the limit of this f x plus h as h tends to 0 this coincide with f x, f x. Clear? Similarly, if we start with the minus sign we will get the minus sign here, we will get the minus sign here, we get the minus sign here, we get the minus sign here, and it means left and right limit both coincide with f x the value of function. So, this implies the function f x is continuous, which we wanted. So, in a closure and x is a point arbiter point in this. So, it is continuous in this.

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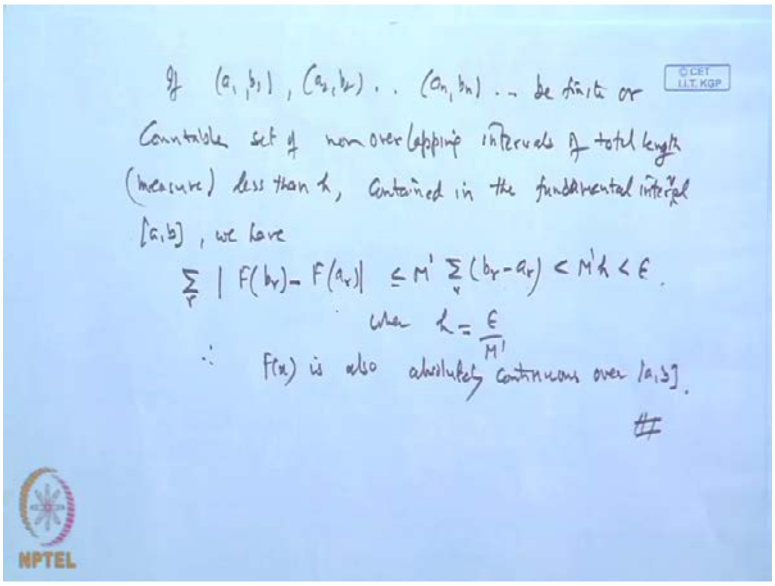

Now, if I take the intervals sufficiently small intervals a one b one etcetera, which suppose we take if a1, b1, a2, a2, b2 and say a n b n and so on be finite or countable or countable set of non over lapping, over lapping, non over lapping intervals of total length measure or length, total length that is called the measure also total length less than h, less than h contained in, contained in the fundamental intervals, fundamental intervals, interval a b, contained in the fundamental interval a b, then we have sigma mod of f b r minus f of a r mod of this is less than equal to M dash sigma b r minus a r over r and this length is less than h which is less than f signer when we choose h to be say f signer by M dash.

So, what we say here that for a given f signer greater than 0, we can find an h such that whenever the points are inside this sigma b r a r the sum of this finite or countable interval length is less than h. Then this sum is less than f signer. So, this shows the function f x, this is also a function f x is absolutely continuous function, absolutely continuous over the interval a b. So, this will be. So, this here we will stop it and next time we will continue with the same.

Thank you very much.