A Basic Course in Real Analysis
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Lecture - 42
Properties of Reimann Stieltjes Integral (Contd.)

So, we will continue our previous lecture that is the properties of Riemann Stieltjes Integrals; and in this lecture, we will particular deal with the step functions, and we will show that whenever alpha becomes a step function, alpha is taken as a step function then Riemann Stieltjes Integral reduces to a either a finite sum or may be infinite sum of the terms, series, in the form of the series. So, one can get the value of the integral in the form of the series. Similarly if alpha is a differentiable functions, then it can be reduced, Reimann Stieltjes Integral reduce to a ordinary integral, just a definite integrals and one can get the value easily for that.

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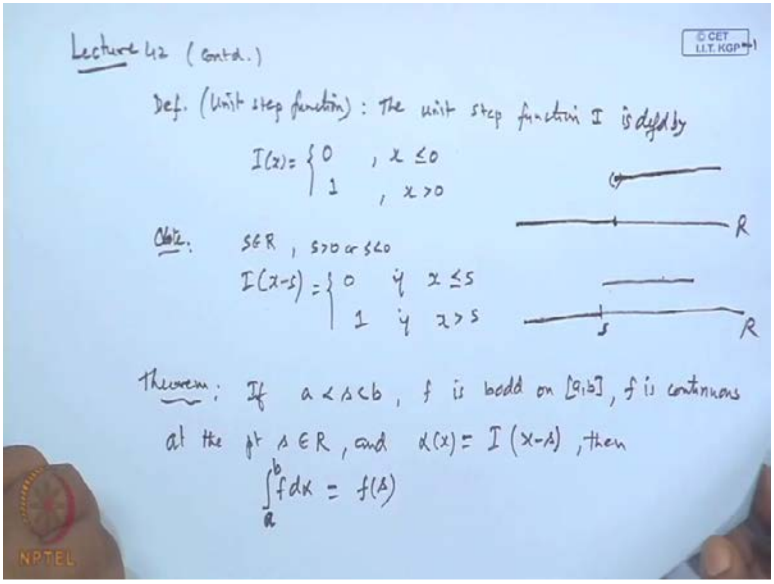

So, prior to this before starting we will give the definition of a unit step function. We define the unit step function, the unit step function I is defined by I x is 0, when x is less than equal to 0, and 1 when x is greater than 0. So, basically this entire real line; so, I x means when the x, real number x is less than equal to 0 the corresponding value of I x will be this and when it is greater than 0, the corresponding value of I x will be 1, this point is not included here, it is included here. So, this is our step function, if we call it as a unit step function; if we as a note, we can say if suppose I take s be any real number, in the some interval say a b, and when we say x minus s then the meaning of this is 0, if x is less than equal to s and 1 if x is greater than s. So, this will be starting if s is say positive or may be negative or s is negative. Suppose, I take s here; then all the, for all the real number which are less than or equal to s, the image of under I will be 0 and while, it will be 1 when it is greater than this.

So, this is the real line R; so this all about the step functions. Now, we will make use of this step functions and first we will show that if our f is continuous and also alpha is a step function, then in that case the integral can be computed just by computing the value of the function at the point where the function is continuous. So, the result is this, the result says if suppose s is a number lying between a and b, a b is a given interval, and f is bounded over the closed interval, bounded on the closed and bounded interval a b, f is also continuous, f is continuous on at the point, at the point s which is in R lying between a and b; and suppose alpha x is the unit step function x minus s alpha. Then the results says the Reimann Stieltjes Integral of the function f with respect to alpha from a to b over the interval a b is nothing but the value of the function at a point s where it is continuous. So, that way we can easily get the value of the Reimann Stieltjes Integral.

If I know this property that alpha is a unit step functions with respect to say s where alpha x is I x minus s and f is bounded and continuous function defined over the interval a b. So, let us see the proof of this. In order to prove this result let us use the since f is given to be a, f is continuous and bounded also. So, we can function f over the interval a b will attain maximum and minimum value because the interval a b is closed and function is continuous. So, it will attain is maximum and minimum value therefore, we can find out the maximum and minimum Riemann sum for this function f.

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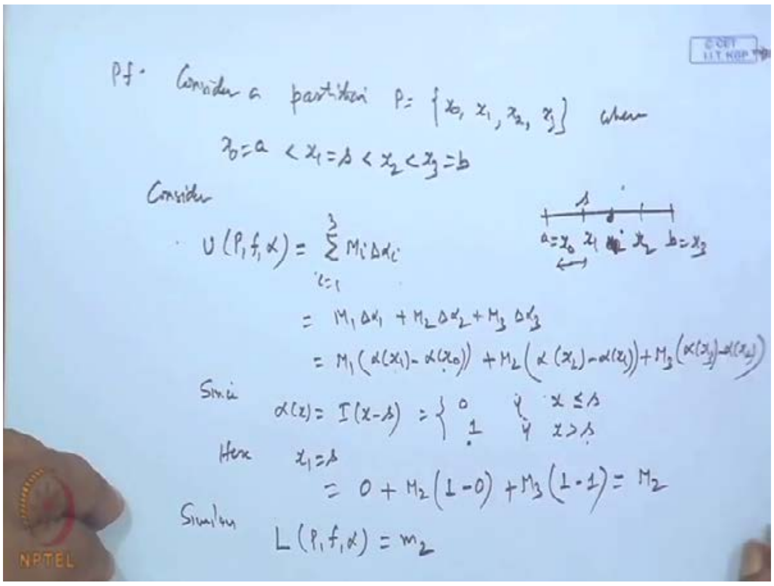

So, let us take consider a partition p having the point suppose x naught x 1 x 2 x 3. It may be more point, but just for simplicity I am taken and result can be extended when there are so many other points involve in between a and b, where x naught I am taking as a which a which is less than x 1 is suppose s and less than x 2 which is less than x 3 is suppose b. So, this is the interval a b, we have partitioned this interval by choosing this partition x naught x 1 x 2 and x 3 is b. So, this is our say x 3. So, let it not be this here this will be x 2, clear? This will be x 2. So, x 1 is s. This is our point s, we have taken this one.

Now, consider this upper sum. Upper sum of the function f with respect to alpha over this partition p. So, this sum is sigma M i delta alpha i, i is 1 to 3 and that is the meaning of this is M 1 delta alpha 1, M 2 delta alpha 2, M 3 delta alpha 3 and if I further expand it delta alpha 1 means it is defined over this interval. So, the choosing as alpha x 1 minus alpha x naught, this is our then M 2 alpha x 2 minus alpha x 1, then plus M 3 alpha x 3 minus alpha x 2, this is by definition. Now, function since our alpha x is given to be the I, x minus s is it not. So, this is the value is 0 if x is less than equal to s and 1 if x is strictly greater than s. Now, here x 1 is s. So, it means when the value of alpha at the point x 1 will be 0, value of alpha at the point x naught is 0, so, first term will be 0, the second term will be M 2, alpha of x 2 since x 2 is greater than s.

So, alpha of x 2 will be 1 and then alpha of x 1 because it by definition it will 0 and then M 3 alpha x 3 and alpha x 2 will all be 1. So, basically you are getting M 2. So, upper sum will always be M 2. Similarly, we can say the lower sum of the function f with respect to alpha will be the small m 2 like this.

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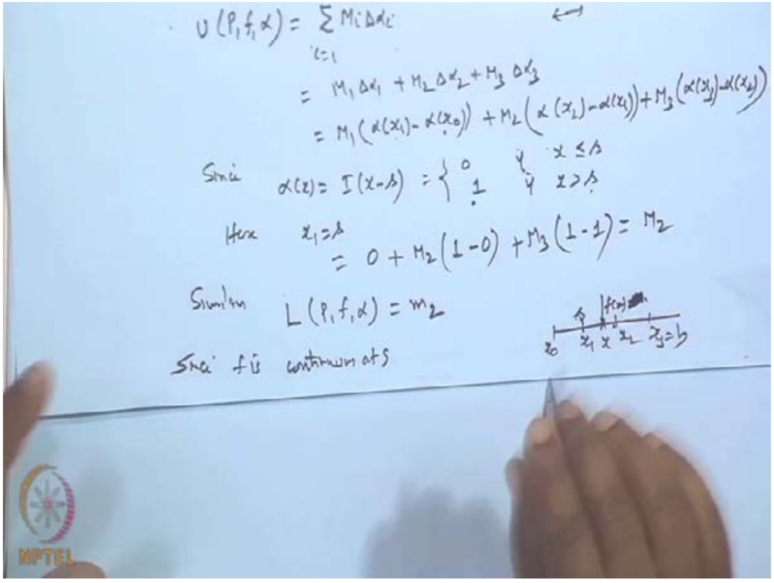

Now, since f is continuous at s. So, it means this is our x naught, x 1 which is s, x 2 x 3 which is b. So, if I take any arbitrary point here say x, the value of f x in this value is nothing but what, this is always the maximum value of this will be because its m 2, m 2 is the maximum value over this interval. So, it will be the maximum value of this f x, will be m 2. So, it will close to this.

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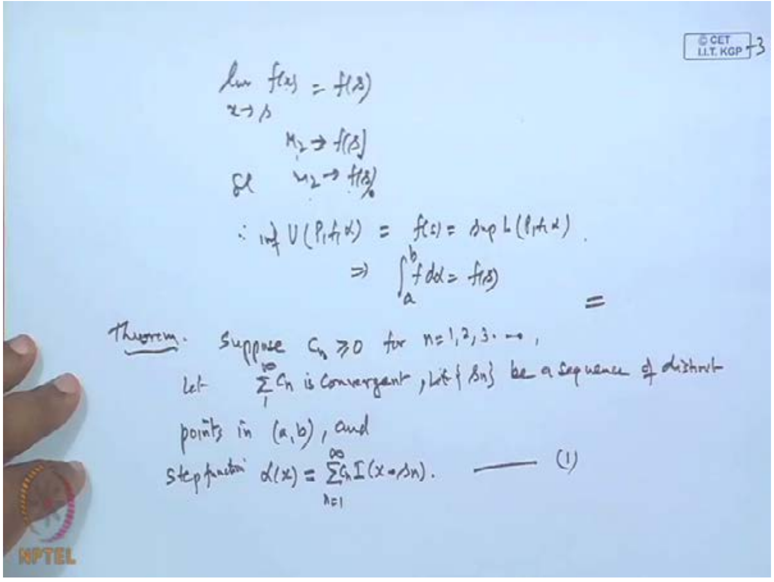

So, function f is continuous at s therefore, limit of the function f x when x tends to s will be f s, but the function f is such, f is such which is defined over this interval x 1 to x 2. The function f x is our alpha x into this. So, when you take the value, the value will be M 2 and this will come out to be f of s. So, what M 2 sorry M 2 will go to f of s and M 1 similarly small m 2 will also approach to f of s because of the continuity. So, both M 2 and small m 2 will go to f s and small f s sorry small m 2 will also go to small f s, but this is the largest value and smallest value, but as x approaches to s this will go to f s.

So, M 2 will approach to similarly small m 2 will approach to f s. So, this shows that both upper sum of the function when the limiting value of this is a limit of this x tends to s is our f s and that is the infimum of this and which is equal to supremum of the lower sum of f with respect to alpha. So, this shows integral a to b f d alpha will be f s because always this limiting value is coming to be f s. So, upper sum when you take the infimum value it is f s, when you take the supremum value of the lower sum it is also f s. So, it is integrable and integral comes out to be this. So, this proves the result. The second results also in connection with this step functions, this will give the result that if alpha is a step function then Reimann Integrable Integral, Stieltjes integral will be reduced to the infinite series or a finite series depending on this.

So, suppose C n is greater than equal to 0 for n is equal to 1 2 3 and so on and let sigma C n 1 to infinity is convergent. Let this converges, is convergent, let this is convergent the sequence s n is a sequence of distinct point, let sequence s n be a sequence of distinct point, distinct points in the interval a b. In the open interval a b and alpha x is the step function n equal to 1 to infinity, this we call step function, alpha x is this C n I of x minus s n. Let it be number 1.

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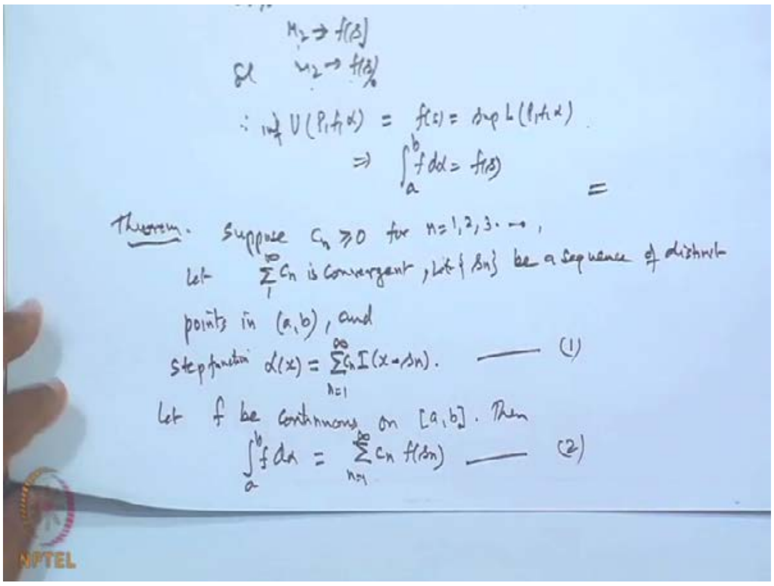

And let f be continuous on the closed interval a b, then the Riemann Stieltjes Integral of the function f with respect to alpha over a b is equal to the summation of the series 1 to infinity C n f of s n. So, Riemann Stieltjes Integral can be calculated in terms of the series if alpha is given to be a step function. So, that is very important result we get.

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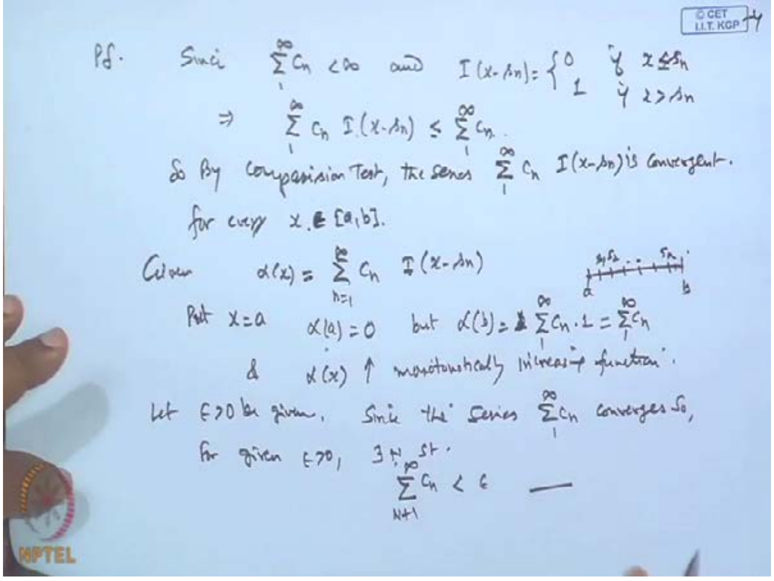

So, proof of this. First thing is that since sigma of C n 1 to infinity is convergent and I of x minus s n is 0 if x is less than equal to s n and 1 if x is greater than s n. So, this implies that the series 1 to infinity C n, C n of say I of x minus s n will always be dominated by the series sigma C n 1 to infinity because this term, this will help, this will reduce when all the terms for all x which are less than s n the terms I of this will be 0. So, the series will have a lesser term and all the positive terms. So, it is less than equal to C n and, but this is given to be convergent.

So, this is given. So, by comparison test the series sigma 1 to infinity C n I of x minus s n is convergent, that is the first thing which we get it and convergent for every x, for every x belongs to the interval say a b. Here, of course we can chose the real line also for every x. Now, let us take the alpha. What is alpha? Alpha is given, alpha x as sigma n is 1 to infinity C n I of x minus s n.

So, what is the value of (( )). So, if we put x equal to a, x lying a b. So, let it be in the interval a b. So, x is suppose a then what will be the alpha a? This is our interval a b and here these are the s 1 s 2 s n, these are the points where s 1 s 2 s n they belongs to the open interval remember because these are the points of distinct. So, say s 1 s 2 s n and so on, these are the points. So, a is strictly less than s 1, a is strictly less than s 2 and so on.

So, when a is strictly less than 2 s n for each n therefore, I of x minus s n will be 0. So, it will be 0, but what is the alpha of b? When you take x of b all the s n are less than equal to b, are basically less than b. Is it not? So, it will be completely the value of this will be 1. So, what we get from here that alpha a is 0, alpha b is 1 and alpha x is an increasing function, is a monotonically increasing function, that alpha s 1. When you take x equal to x 1 then it is less than x equal to x 2 and (( )). So, this is a monotonically increasing function. So, alpha is a monotonic function, first thing and this is and alpha b.

So, what is sorry this I of x n is 1. So, basically 1 into sigma here I am sorry this will be I of you can write this thing as sigma C n 1 to infinity and this will be 1. So, you are getting sigma C n, sigma C n sorry this is 1, we are getting this. So, one thing is clear that this will be monotonically increasing function. Now, let epsilon greater than 0 be given. The series since the series sigma C n 1 to infinity converges.

So, the remainder term of the series is less than. So, we can choose. So, for given epsilon greater than 0 there exists an n such that the sum of the series n plus 1 to infinity C n will remain less than epsilon, let it be because by definition of the remainder term of the series is convergent. So, this is less than epsilon.

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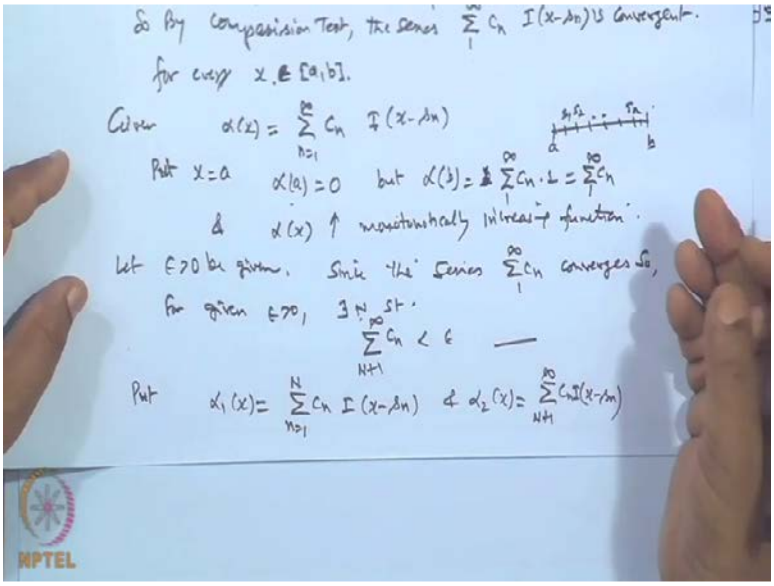

Now, this will be, let us take put alpha 1 x be the sum of this series of first n terms C n I of x minus s n and alpha 2 x, I am taking the sum of the series where the terms are taken from n plus 1 onward of the series C n I of x minus s n. Let it be this two. Now, since I of x minus n because of this becomes a step function. So, sum of the step function, in case of the step function, previous theorem we have say it is a Reimann Integrable functions.

So, the sum of this Reimann Integrable functions we can write the sum of the, this therefore, we can say f is alpha d alpha 1 sorry step function. So, integral f d alpha becomes the, by the, if alpha is a step function then integral f alpha x is nothing, but the f s, is it not that what we have seen.

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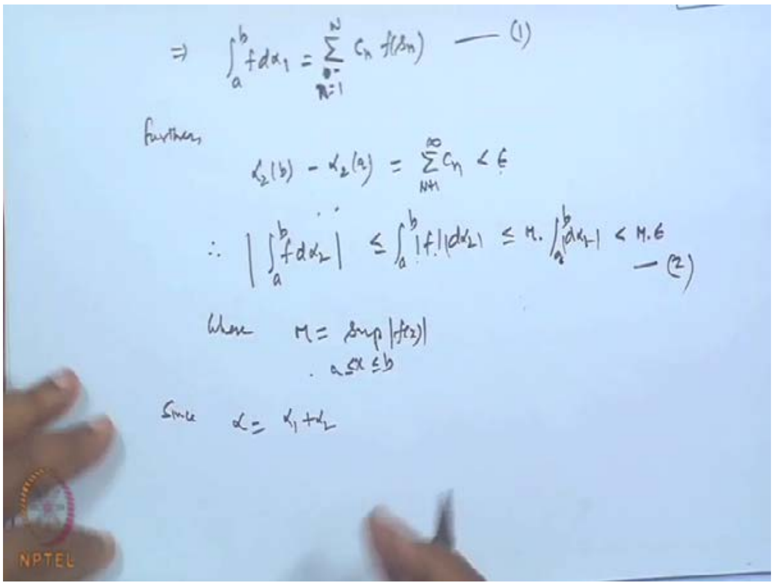

So here, so we get from here is. So, this implies the integral a to b f d alpha 1 is nothing, but the sigma i is equal to 1 to n, 1 to n, i is equal to 1 to n C n, n is equal to 1 to n, n equal to 1 to n C n f of s n. So, this will be here alpha 1 and this is number 1, because it will follow from the previous result, this result is there which we have proved earlier the result is this. Is it not, according to this alpha is given to be continuous, f is continuous and this one. So, already f is continuous given and bounded and alpha x is this. So, value will be this f s.

So, here if you take this sum, if you take the integral of this f of f of alpha 1 x and d x then we get immediately this can be written as sigma into this form. So, there is nothing into explain. Now, further what is our alpha 2 b minus alpha 2 a? What is alpha 2? The alpha 2 is defined as this, this is alpha 2 x. So, if I replace x by a then alpha 2 x will be 0 and alpha 2 b will be sigma of C n. So, basically this is equal to the sigma n plus 1 to infinity C n, but which is given to be less than epsilon because of the series is convergent. So, remainder will be less than epsilon hence therefore, therefore, the integral a to b f d alpha 2, with respect to alpha 2 the modulus of this is less than equal to a to b mod of f d alpha 2 or mod of d alpha 2 and then d alpha 2 again this is bounded.

So, it is less than equal to m times and m times integral a to b d alpha 2 mod of this and mod of this and this value nothing, but the alpha 2 b minus alpha a which is less than epsilon. So, this is less than m into epsilon. So, let it be the equation 2. So, over alpha 1 we are getting this thing, over alpha 2 we are getting this thing. So, where m is the supremum value of the function f x over the interval a b, that is what. Now, since our alpha is nothing but the alpha 1 plus alpha 2 because when you take the alpha x, alpha x is the sum of the series 1 to infinity, I have break up into two parts 1 to n and to n plus 1. So, alpha is alpha 1 plus alpha 2 x. So, we get from here is then.

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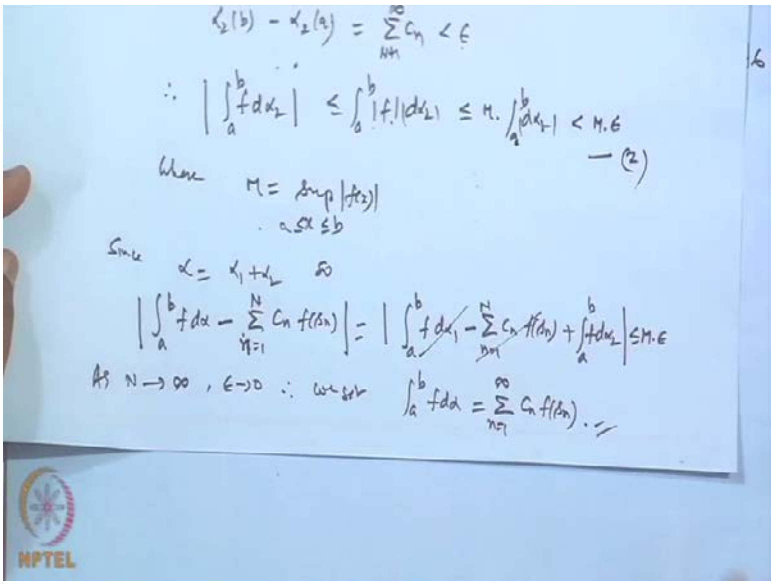

So, integral a to b f d alpha minus sigma i is equal to 1 to n, n is equal to 1 to n and then C n f of s n mod of this. Now, this will be, this can be written as what? It is nothing but the mod integral a to b, f d alpha 1 minus sigma n is 1 to n C n f of s n plus integral a to b f d alpha 2, because this d alpha is alpha 1 plus alpha 2. We can write this. Now, this part is already given to be less than epsilon, this part is less than basically this entire thing alpha 1 is the same as this. So, this part is 0, we get this cancel only this part is left and this part, we have shown this will less than m into epsilon.

So, it is less than equal to m into epsilon, but epsilon is arbitrary. So, when as n tends to infinity epsilon will go to 0. So, epsilon will go to 0 therefore, we get integral a to b f d alpha is nothing, but the sigma n is 1 to infinity C n f of s n and that completes the results, clear. So, what this result says is that in case of the function alpha is monotonically is a step function then our Reimann Stieltjes Integral reduces to a infinite series or finite sum, clear.

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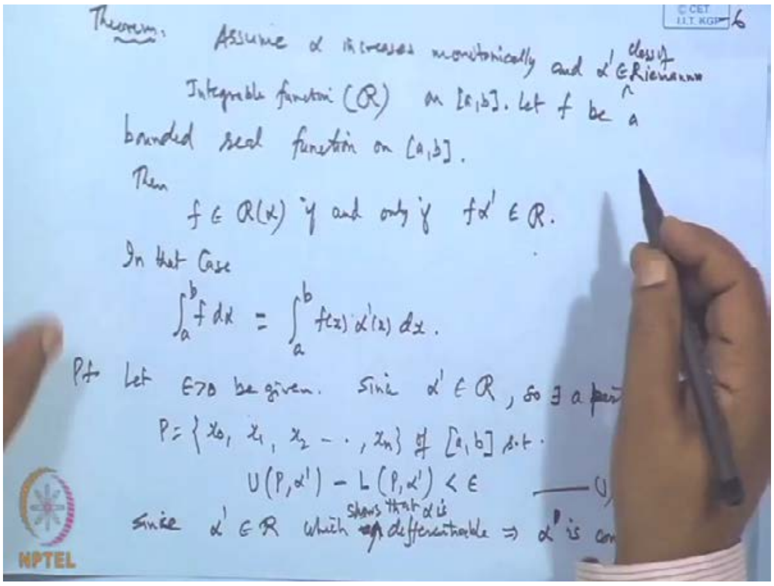

The next result is also interesting which shows the Reimann Integrable functions, integral can be calculated as if it is a definite integral under certain restriction. So, assume alpha increases monotonically and alpha dash is a Reimann Integral function on is a Reimann, this we say is a Reimann Integrable function, belongs to Reimann Integrable function, we denoted by this. So, alpha dash belongs to the class of Reimann Integrable function, class of Reimann Integrable function this on a b. Let f be a bounded real function defined on the closed interval a b. Then the result says f is in Reimann Stieltjes Integral with respect to alpha if and only if the f into alpha dash, this product is Reimann Integrable function is belongs to the class of Reimann Integrable functions on a b and in that case the integral a to b f d alpha Reimann Stieltjes Integral of the function f is the same as a to b, a to b f x into alpha dash x d x as if it is just a definite integral when the function is like this Reimann. So, this same test. Let us see the proof of this, let epsilon greater than 0 be given. Now, given that since alpha dash belongs to the Reimann integrable functions over the interval a b. So, by definition there is a partition.

So, there exists a partition, there is a partition p say x naught x 1 x 2 x n of a b, such that the upper sum of the function alpha dash minus lower sum of the function alpha dash with respect to the partition be is less than by necessary and sufficient condition for a function to be the Reimann Integral is the difference between upper sum minus lower sum will remain less than, there will be a partition where the difference, fluctuation of the function will remain less than epsilon. This is the fluctuation of the function. So, let it be 1.

Now, since alpha dash is given to be, since alpha dash means is a differentiable function. Alpha dash belongs to R, belongs to R which is differentiable, belongs to R means exists. So, it is differentiable. So, it implies that alpha is continuous, alpha dash is continuous because if every continuous, if the function is defined it has to be continuous. So, alpha dash will be. So, that is a continuous function, alpha will be a continuous function otherwise we cannot talk about the derivative. Since alpha dash exists, it means which shows, which shows that alpha is differentiable. So, alpha must be continuous that is the main from here, because this is given alpha dash alpha, dash means derivative of alpha. So, this exists; so it must, alpha must be a continuous otherwise the derivative cannot be, if it is discontinuous cannot be talk about that differentiability.

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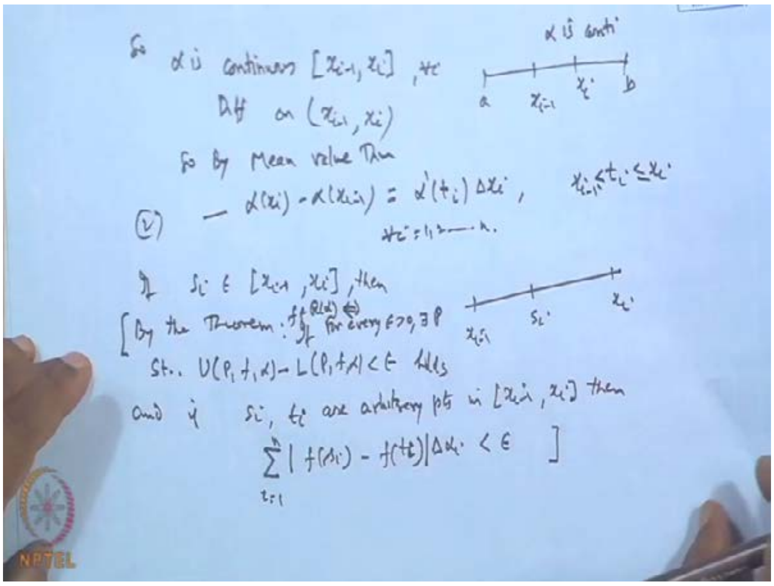

So, alpha is continuous. So, alpha is continuous throughout the interval a b. So, this is the interval a b and partition we are choosing the partition say x minus 1 and x i this is the partition. So, alpha is continuous on the partition. So, alpha is continuous on the closed interval x i minus 1 and x i for each i, it is differentiable on the open interval x i minus 1 x i. So, by mean value theorem, Lagrange's mean value theorem we say alpha x i minus alpha x i minus 1 must be equal to the derivative of the function alpha dash at a point say t i delta x i where the t i lies between x i minus 1 to x i, is it not.

So, there exists a point, now this may be closed also. So, t i value for some there exists a some point t i where this will exist by mean value theorem. So, let it be this say number 2 for each i and this is true for each i 1 to n, true. Now, what is given is that our alpha is an increasing monotonic function, alpha dash is this and f be a bounded real function, then (( )) this. So, let us take the point suppose this is x i minus 1 here is x i and t i is somewhere.

So, let us choose the point s i, choose a point s i belongs to x i minus 1 to x i, then let us apply that result, what this result is which we have proved earlier the result is this one. Result says if and by the result by the theorem which we have proved earlier the theorem is, what theorem is if the partition if u p alpha f alpha minus l p f alpha is less than epsilon. If for every epsilon there exists a partition, if for every epsilon greater than 0 there exist a partition p such that this holds, such that this holds then and if s i and t i are arbitrary points, are arbitrary points in the interval x i minus 1 to x i then this sum sigma i is equal to 1 to n mod of f s i minus f t i into delta alpha i is less than epsilon, this result we have already proved. So, using this result because our function alpha i is given to be continuous function bounded sorry given satisfying this condition. So, for any partition p we can get this. So, what we get apply the condition alpha dash because alpha dash. So, one more thing is that this shows that if alpha is a function f must be in R if and only if. So, this shows that this one, this is f in R, f belongs to R alpha if and only if this result is true. So, what is there alpha dash is in R alpha dash is in R. So, we can get this result quickly.

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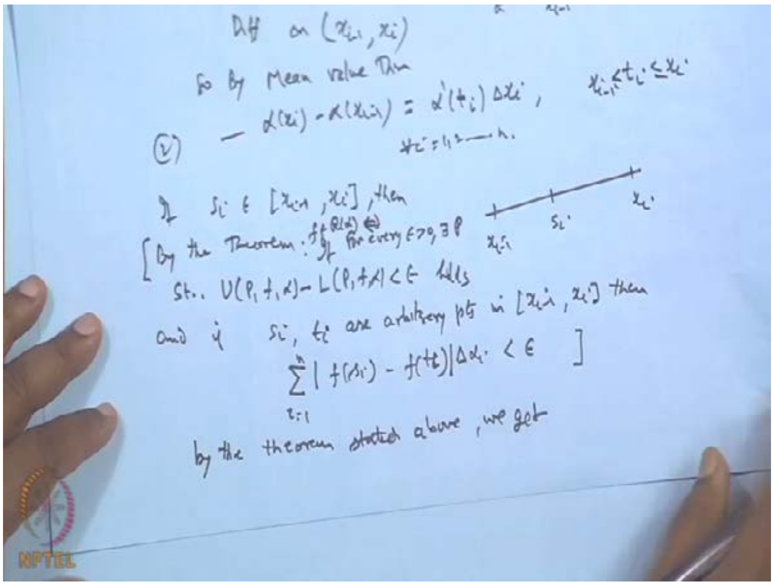

So, by the result if s i be any point of this then by this theorem, by the theorem stated above we get immediately sigma i is equal to 1 to n alpha dash s i minus alpha dash t i into delta x i is less than epsilon and because alpha dash is in R alpha, is in R on a b this is given and thus that is why this result is valid.

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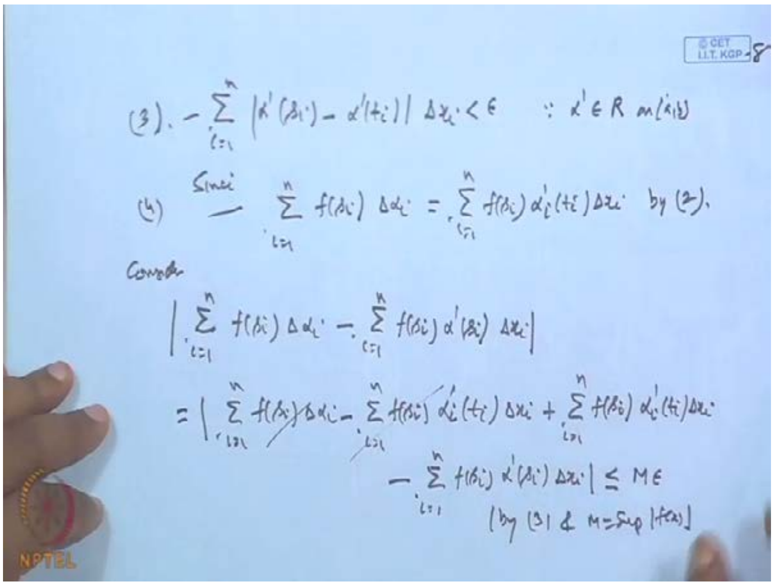

So, let it be condition 3. Now, since our this sum sigma i is equal to 1 to n f of s i delta alpha i, this is nothing but what sigma i equal to 1 to n f of s i, delta alpha i we have already proved it in the second this is our delta alpha i, is it not. So, this delta alpha i which is equal to say delta alpha i is nothing but alpha dash t i delta x i. So, by using the second by using the second we can say this is equal to alpha dash i t i delta x i by second. Then consider this sigma i is 1 to n f of s i delta alpha i minus sigma i is 1 to n f of s i alpha dash s i into delta x i. Consider this thing now, we have already shown this part is it not the sigma f i delta i is this thing. So, here is t i. So, what I say, I subtract this thing t i and at t i. So, if I subtract ti and (( )). So, let it be 4. So, using 4 consider this.

Now, this can be written as at small sigma i is 1 to n f of s i delta alpha i minus sigma i is 1 to n f of s i alpha i dash t i delta x i plus sigma i is 1 to n f of s i alpha i dash t i delta x i minus sigma i is 1 to n f of s i alpha dash s i delta x i, just a (( )). Now, this part is a exactly same as this. So, this will go to 0. Now, this is there, but because of this we get the earlier one result alpha dash i minus this because of the 3. So, apply the 3 condition. So, this is less than epsilon and f is bounded. So, can you not say this is less than equal to M epsilon by third and M is supremum value of mod f x because of this. So, in particular we can say this. So, let it be. So, what we get is.

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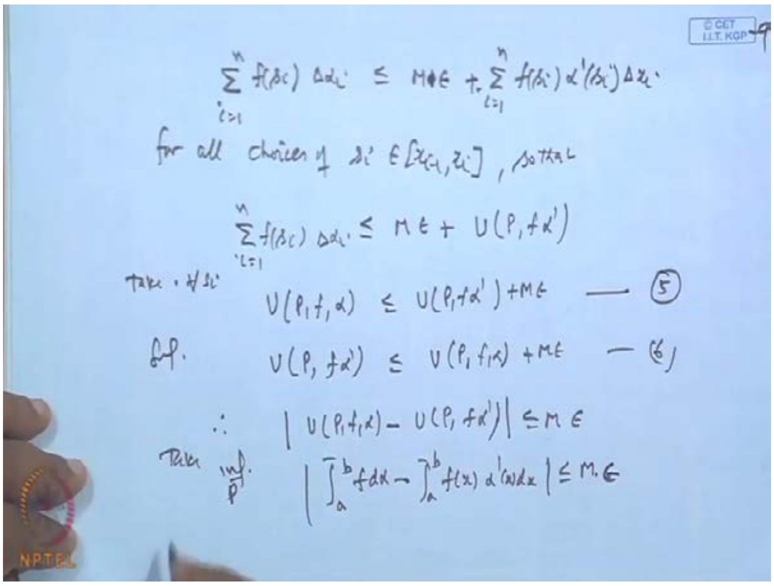

So, we get basically this result sigma i is equal to 1 to n f of s i delta alpha i is less than equal to M plus sorry M into epsilon M epsilon plus sigma i is 1 to n f of s i f of s i alpha dash that is what he is saying, is it not f of s i, f of s i alpha dash i f of dash this thing, this part and f of s i can taken outside. So, what you are getting is f of s i delta alpha i minus this part. So, minus this part is less than epsilon. So, this thing is less than equal to this plus this. So, f of s i alpha dash s i delta x i, this is true; so which we get from here. So, we get this thing now this is true for what? For every i and s. So, if I replace this by the upper sum of this. So, for all choices of s i which is in x i minus 1 to x i, this result is holds. So, you take the upper bound for this. So, once you take the upper bound. So, that we get what this is less than equal to M epsilon plus the upper bound of the function f alpha dash because this is the f alpha dash function. So, replace this by maximum value.

So, you are getting the upper bound for this and which is less than equal to i is greater than equal to 1 to n f of s i delta alpha i. Now take the upper bound for this. So, now, take the upper bound left hand side take upper replace by this maximum value. So, you are getting this is true for every s i. So, we get the upper bound of this f alpha is less than equal to upper bound of f alpha dash plus M epsilon, let it be say fifth.

Now, in a similar way we can also use this thing if I take this is less than epsilon. So, this part is less than equal to this greater than equal to this minus epsilon. So, from similarly you can say similarly same argument we can say that upper sum of p f alpha dash is less than equal to upper sum of p f alpha plus M epsilon in a similar way from the same inequity. Therefore, the difference of this upper sum of p f alpha minus upper sum of p f alpha dash alpha dash is less than equal to M epsilon. Now, if we take this is. So, what you get? Take the infimum value of this, take the infimum. So, take infimum over partition p. So, this will lead to the integral a to b upper sum f d alpha, this will lead to the integral a to b bar f x alpha dash x d x and this entire thing is less than or equal to M epsilon.

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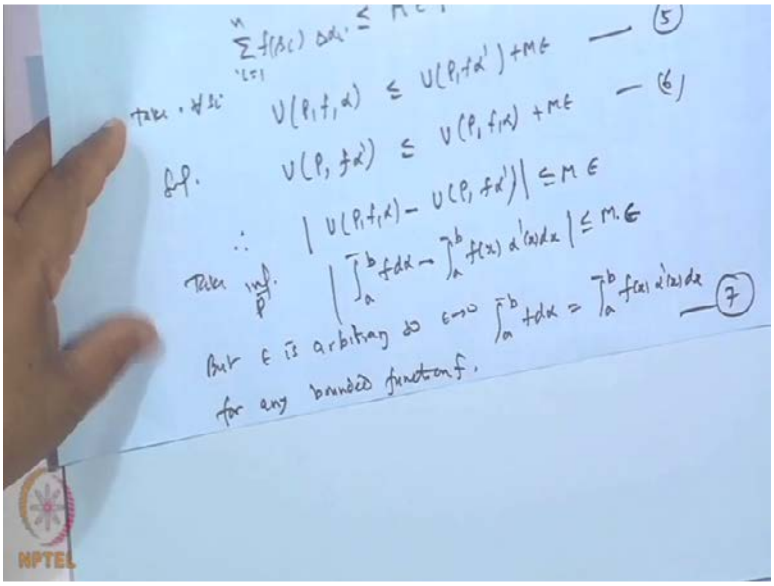

But epsilon is arbitrary, small number. So, as epsilon goes to 0 we get integral a to b bar f d alpha is nothing, but the a to b upper sum of this upper sum of f x alpha dash x d x. Let 7. Similarly, for any bounded function this is true for any bounded function. Similarly, we can prove the lower integral.

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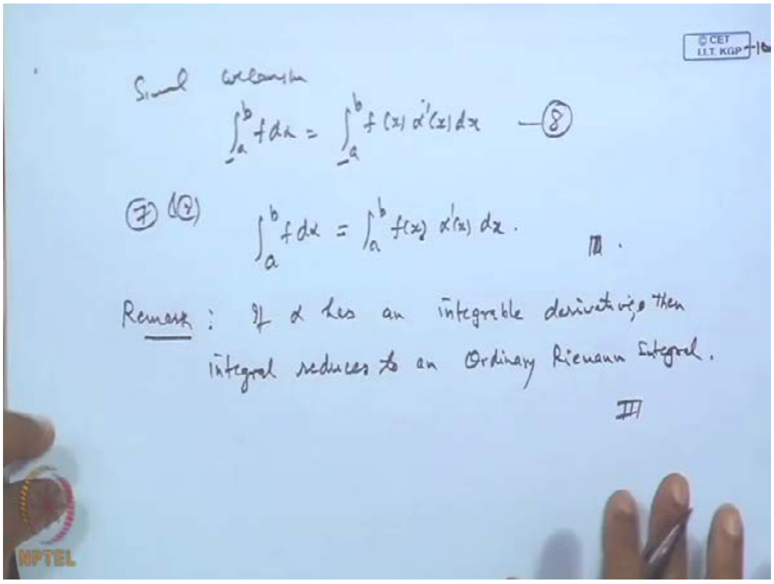

Similarly, we can show that a bar f say a bar of this part that is a bar of f d alpha is the same as a lower bar b f x alpha dash x d x. So, 8 and 7 and 8 implies if f is given what is given is that function alpha dash is in this and f is given to be a Riemann Stieltjes

Integral then this left hand side are equal therefore, these two are equal and we get integral a to b f d alpha is the same as a to b f x f x alpha dash x d x, d x. This proves the result complete. So, this shows that remark, you can say remark is that if what if alpha has an integrable derivatives then the integral reduce then integral reduces integral reduces to n ordinary Reimann Integral, hence this can be easily completed, so this is what we are getting. So, this almost we have completed. Now, we will give a slight, just a concept of what is our measure. We have discussed already earlier let us see some slightly in detail what is the measure and what do you mean by the almost every real functions and like this.

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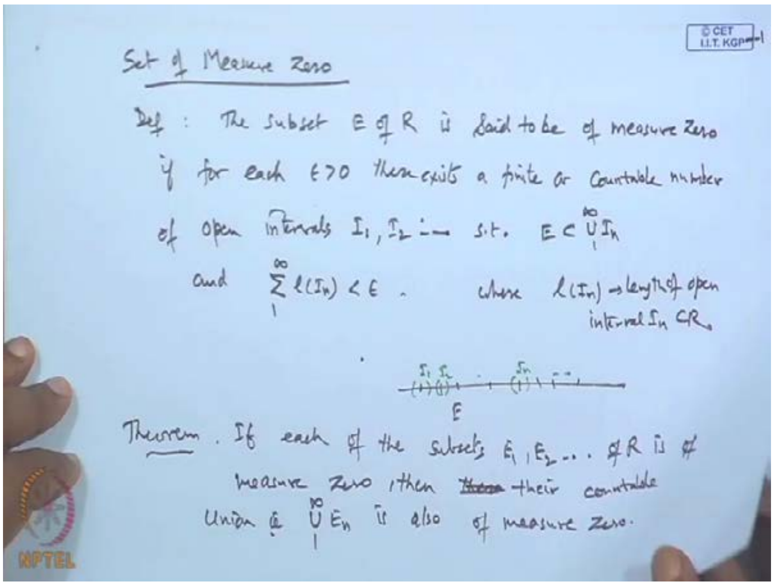

So, let me just say few concepts set of measure 0. Let us see. We define that the subset E of R real line, subset E of R is said to be of measure 0 if for each epsilon greater than 0 there exists a finite or a countable number of open intervals I 1 I 2 and so on such that there countable union covers E and the length of this, length of I n 1 to infinity is less than epsilon where l denotes the length of the interval, length of open interval I n of R. What you mean by if suppose a subset E is there, suppose these are the points in E, these are the set E and if we are able, this set E is said to be a measure 0 if b encloses this points by means of an open interval I 1 I 2 say I n and so on such that length of these open intervals is less than epsilon such that countable union this covers C means all the points of belongs to the countable union of I n, but the sum of their length cannot, is not exceeding by epsilon then we say the set E has a measure 0 or is said to be of measure 0.

Now, thus what we can say is (( )), there is one result which will be useful. The result says if each of the subsets E 1 E 2 and so on of R is of measure 0 then their countable union, then their countable union, that is union of E n n is 1 to infinity is also of measure 0 and this is very easy to prove.

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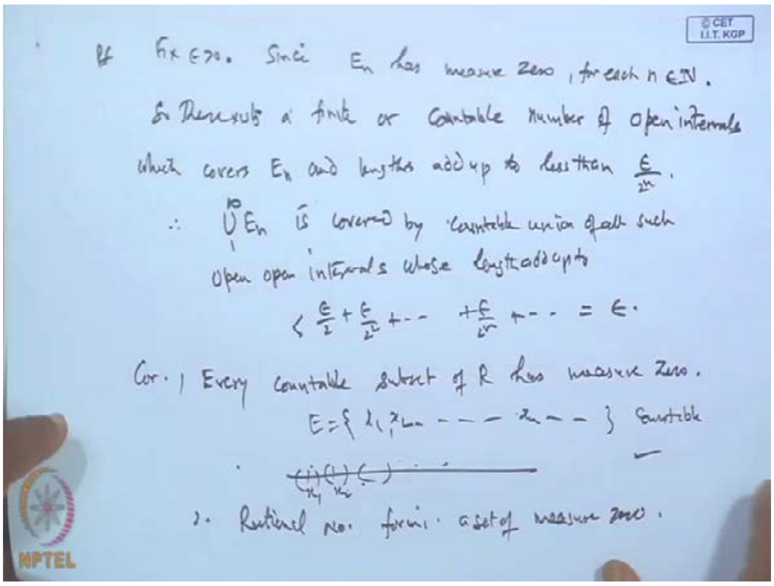

What suppose I fix up proof of this simple, suppose I fix up epsilon greater than 0. Since E n has a measure 0, has a measure 0. So, by definition there are the (( )) for each n belongs to say I, I is the natural number or some n n natural number then since E n is there. So, there exists. So, there exists a finite or countable collections countable number of open intervals which cover E n and whose length add up is less than epsilon, length add up to less than say epsilon over 2 n. Therefore, the countable union E n therefore, countable union of E n 1 to infinity is covered by countable union of these intervals, of all such open intervals of interval whose length add up to what epsilon by 2 epsilon say first term epsilon by 2 square epsilon by 2 n and so on.

So, if I add this become less than epsilon. So, up to less than this number hence it is there. So, this proves. As a corollary we can say every countable subset of measure R has measure 0. Every countable subset because why the reason is suppose E is a countable sets, we can arrange in the form of the sequence like this. This is a countable set. So, each one x 1 x 2 x n we can enclose it by means of a countable number of intervals and sum of these will be each sum is less than epsilon. So, countable union of this sum will also be less than epsilon. So, this is countable. Rational numbers forms a set of measure 0 because rational numbers are countable measure 0.

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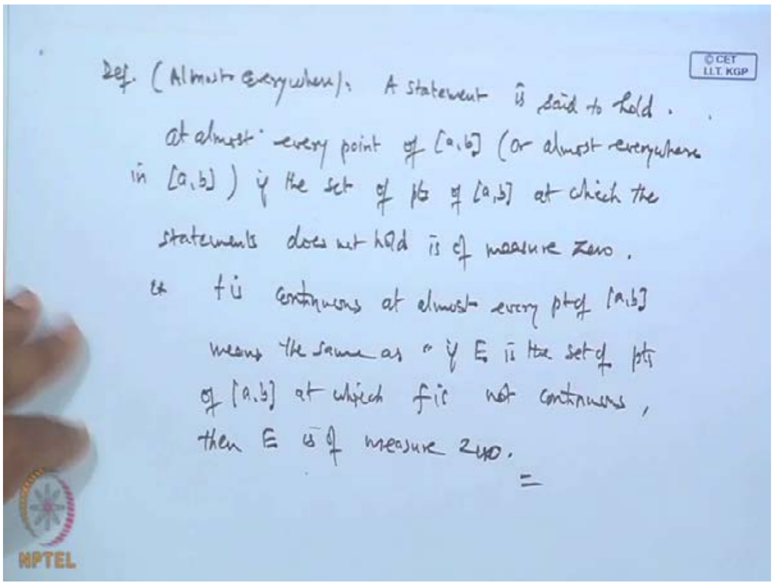

So, this shows this much. Now this is last one which is very important, almost everywhere. A statement is almost everywhere. A statement is said to hold at almost every point of the interval a b, a b or almost everywhere in the interval a b, if the set of points of a b, at which the statements does not hold is of measure 0. So, that is the definition. So, thus we say for example, if we say f is continuous at most at almost every point of the interval a b means the same as if E is the set of points of a b at which f is not continuous then E is of measure 0. So, that is what is an, f is continuous almost everyone is. Now, based on this we have a very result important theorem.

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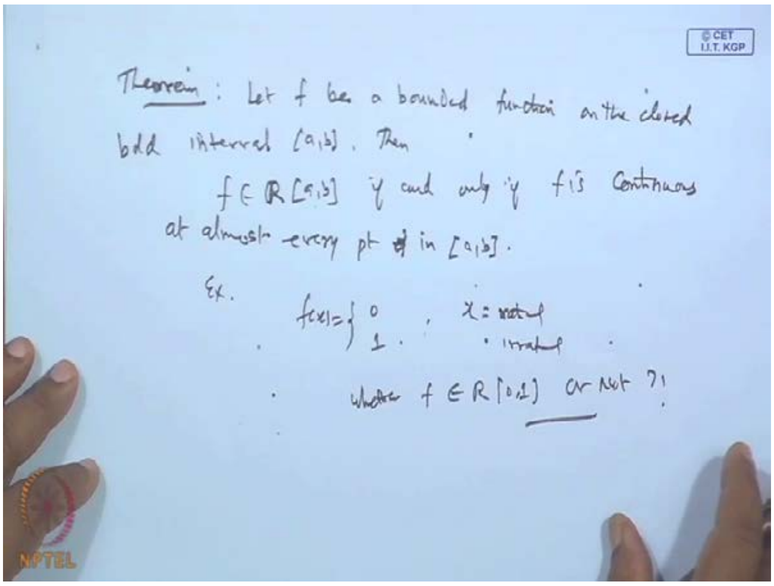

And the proof is of course, we are just neglecting, because this proof based on already we have proved this earlier. The theorem says let f be a bounded function on the closed interval, closed bounded interval say a b, then f is Reimann Integral over interval a b, f is Reimann Integrable functions over the interval a b, if and only if f is continuous at almost every point of point in the interval a b, almost every point in the interval a b, then we say the function is continuous for this. For examples, let us take this function; suppose I define f x as 0 and 1, when x is say irrational and x is rational and this is irrational point. This is bounded function. Now, let us see what type of, we will discuss whether question is whether f is Reimann Integrable functions over 0 1 or not. So, this question we will discuss next when you go for the tutorials, similarly.

Thank you very much.