A Basic Course in Real Analysis
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Lecture - 41
Properties of Reimann Stieltjes Integral

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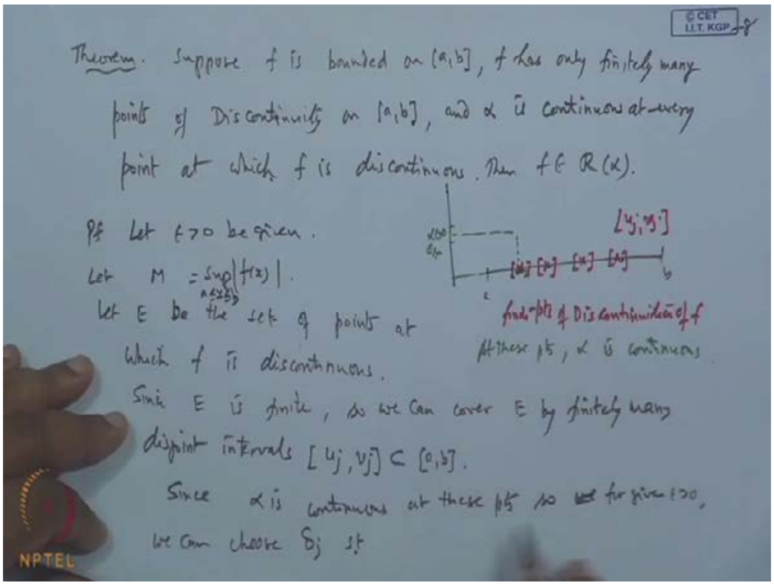

So, yesterday we were discussing this theorem and theorem was suppose, f is bounded on the closed interval a, b and f has only finitely many points of discontinuities on the closed interval a, b; and alpha is continuous at every point at which, f is discontinuous then f is a Riemann Stieltjes Integral. So, actually what he says is that if, f is given to be a bounded function, but it has a point of discontinuities over the interval a, b, and the number of the discontinuous points are finite. And alpha is such which is continuous at the point, where the f is having point of discontinuity, where f is discontinuous alpha is continuous. Then in that case the function f will be a point of the class R alpha; that is the Riemann Integrable, Stieltjes Integrable function with respect to alpha.

So, in particular when alpha is equal to x, then you can say every bounded function which has a finitely many discontinuous points over the interval a, b must be a Riemann Integral functions. And we have started the proof also, the proof were a what is given is the f is a bounded function. So, we can take the supremum value of f x over the interval a b as suppose m, this is over the interval a, b, this one and let epsilon be greater than given.

Now, suppose E is the collection of those points where the function has a discontinuously and this is finite. So, let these are the points where the function has a discontinuously and this is finite. So, let these are the points where the function has a point of discontinuity since they are a finite number. So, we can enclose them by means of a intervals u j, v j closed intervals like this and the length can be chosen very, very small sufficiently because these are single term points and all these subinterval is a part of the interval a b.

Now, since alpha is continuous at the point this point. So, by definition if we take any epsilon greater than 0, then the image of the alpha x when x belongs to this interval must lie within this epsilon neighbourhood. So, suppose there are n sub intervals end points where the function f is discontinuous, so we are getting n sub intervals of this type and if we choose the epsilon as epsilon by n. Then over each one we can say this is less than epsilon by n and total sum of the alpha will be less than this alpha v j minus u j sigma of this sum of the corresponding can be made less than epsilon, because alpha is continuous.

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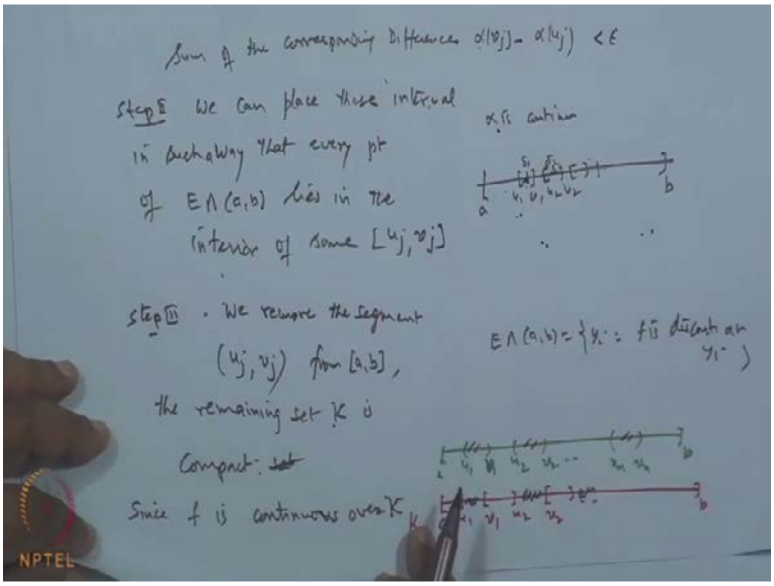

So, once it is alpha is continuous we can make this for given epsilon. We can choose the delta sufficiently small number say delta one, delta two, delta j, n such that whenever the point is there it will be lying between alpha, epsilon then epsilon plus, epsilon plus total is epsilon by n epsilon by n and total is less than epsilon, this we can do it with the help of continuity of alpha. So, this was the discussion.

Now, what we do in second case that we will place this intervals which we have chosen u j, v j in such a manner over the interval a, b; that the point of the discontinuity of f lies within this intervals, within these intervals. So, let us place this or you can say that if we take a point E intersection a b. The E intersection a b is set of those points which are common in E and a b. E is the set of those points where the function is discontinuous. So, basically we are choosing all the points where the function is discontinuity, within the subintervals u 1, v 1, u 2, v 2 all you can one in one of the point all the points will lie within this lie. So, lies in the interior of the clear. So, up to here we have discussed.

Now, what we do is in the step three, we remove we remove the open interval remove the segment u j, v j from the close interval a b. It means what we are doing is that this is our interval a b and here is the point say u 1, b 1, u 2, u 1, v 1, u 2, v 2 and like this suppose these are the points say like u n say v n, then what we do is we are removing this portion, this portion we are removing.

So, we are getting now after removing a into u 1, a u 1 this is 1 then we start with v 1, u 2 and like this, then we start with v 2 and continued this way and up to here say v. So, this set K which is which in which these things are not available these things are not available. So, this set K forms a compact set. So, once we remove the segment u j, v j from a then the remaining set K is compact, because it is a finite union of the compact closed and compact set intervals like this. So, it is a compact set compact is compact or compact set .Now, function f is given to be continuous since f is continuous over K, because the point of discontinuity we have already removed. So, f is continuous over K and K is compact set. So, every continuous function on a compact is uniformly continuous.

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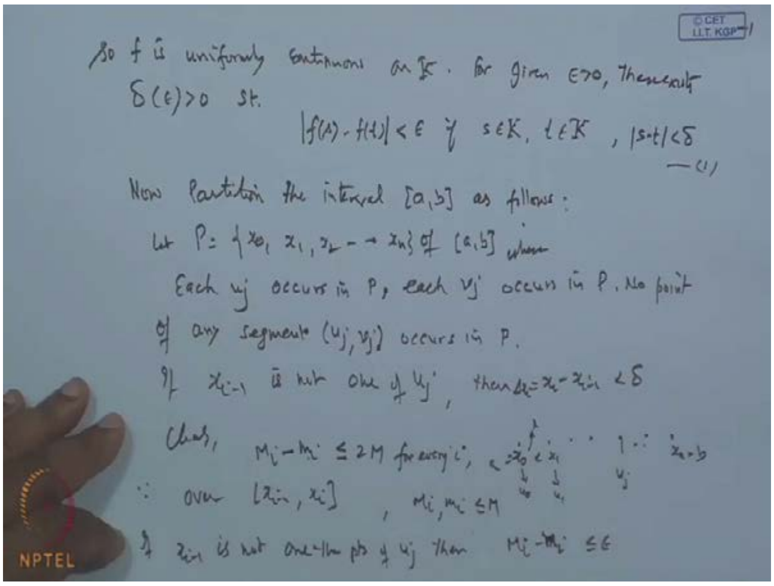

So, f will be. So, f is uniformly continuous is uniformly continuous function on K, uniform continuous function on K clear this one. So, by definition what is a f is uniform means, so for a given epsilon greater than 0. So, for given epsilon greater than 0 there exist, there exist a delta which depends only on epsilon not the point greater than 0, such that the mod of f s minus f t, this will remain less than epsilon if s belongs to K, t belongs to K and mod of s minus t is less than delta, by definition of the uniform continuity let it be 1.

Now, let us partition now partition, partition the interval a b as follows follows suppose p is the partition p is the let p is the partition x naught, x1, x 2, say x n of a, b; a, b; then we take such partition that each u, p is a partition where, where u j each u j occurs in P each b j each v j occurs in P, means this u 1, u 2, u n, v 1, v 2, v n these are the one of the points of the partitioning point of the interval a b, a b and no point and no point of any segment, any segment u j comma v j this open segment occurs in P.

This is the restriction we are putting we are choosing the partition in such a way that, the coronal points of this subintervals which are covering the point of discontinuities are basically coinciding with the partitioning point, but none of the point in between in this inside this interval are the partitioning point are the points of the partition this is one thing, second thing is in case if x i minus 1 suppose is not is not one of the u j because this might be possible number of the points are finite may be the number of points are not as sufficient as the partitioning point is there.

So, some of the exercise will remain untouched. So, if x i minus is not one of the u j then what we do is. Then we put the restriction a that that x i minus x i minus 1, that is the delta x i this should be less than delta this is our restriction. So, once you have partitioning this point it means now we get this. So, this is our x naught, x 1, x 2, x 3, x n and so on, say this is a this is b and then we are coinciding this say suppose u naught, u 1 and so on continue suppose here is u j and after this we are not getting anything, and the points in between u j are not there, are not there one thing.

So, let us see suppose we take any interval clearly over any subintervals M i minus small m i this will remain less than equal to 2 m why? Because what is our m is the supremum value we have chosen, this is m is the supremum value over interval. So, if I consider the x i minus x i minus 1 clearly because over the interval over x i minus to x i the minimum value of the function is m i supremum value is a. So, M i as well as small m i both are less than equal to m. So, M i minus m i is less than equal to 2 m this is true for each i for every i for every i this is true, and further if x i 1 is not one of the point if x i minus 1 is is not one of the point of u j unless this is not one of the point then the difference between M i minus small m i this difference can be made less than equal to epsilon, why? Because of this part because a function is function is continuous as well as uniformly continuous over a k. So, if the point x i minus x i minus 1 less than delta, if this is because we have put this is not one of the point then this difference is less than delta. So, because of the form this difference should be less than epsilon.

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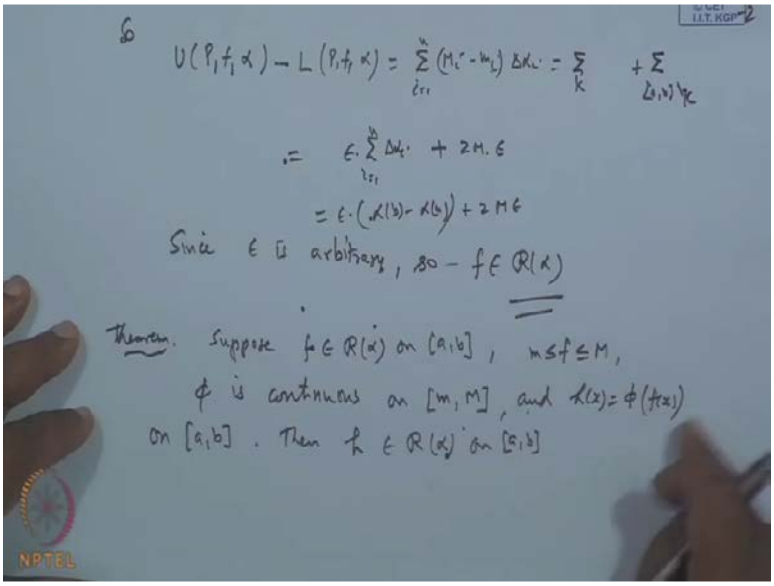

So, M i minus small m i will be less than epsilon. So, that for each one will be less than equal to epsilon hence, if we construct the proof upper sum minus lower sum. Upper sum of the function f with respect to alpha over the partition P minus lower sum of the function f with respect to alpha over the partition P, this is basically what sigma i equal to one to n, say m i minus small m i into delta alpha i, which can be brake up as sigma over k, plus sigma over a b minus k, because the dropped interval these which are dropped.

So, in this case is over K this is less than epsilon this is less than epsilon. So, we get epsilon into sigma of this part, but sigma of this delta alpha i is one to n is nothing but the alpha b minus alpha a. So, this will and this part over this m already we have function is bounded. So, M i plus m i is less than equal to 2 M and since sigma of alpha i alpha b minus this total is less than epsilon, this is because of the continuity this sum of this corresponding is less than epsilon. So, we get from here is that this is less than epsilon and this part will be epsilon alpha b minus alpha a plus 2 M epsilon but epsilon is arbitrary but epsilon is arbitrarily small number.

So, this shows that the right hand side will go to 0 and the left hand side. So, f will be the element of this. So, this is what proof. Now another results also which is interesting the result says suppose f is, f is in r alpha on the close interval a b Riemann Stieljes Integral with respect to alpha over the interval a b and f is bounded function, bounded by say M and capital M. Phi is continuous continuous on the close interval m and capital M this phi, then and h x is phi of f x on the close interval a b on the close interval a, b. Then this result says the h belongs to R alpha that is Riemann Stieljes Integral on. So, if function f is Riemann Integrable function and phi is a continuous function on the m. That is the where the function have a range set, range of the set f is m interval then the composition of this function, continuous image of a of a Riemann Stieljes Integral function will be a Riemann Stieljes Integral that is what he says.

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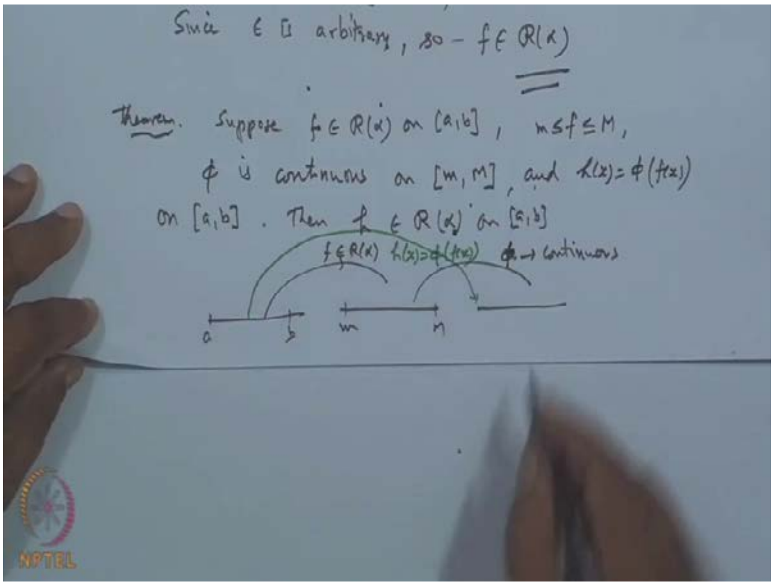

So, this is just like that here we are taking this function this is our interval a b, function f is given. So, here is our m and capital M, where the function attains the minimum and maximum value and over the function h is defined, h is defined. So, what is a when you combine the h phi composition position f x then it will transfer directly from here to here, which is h composition h x equal to phi composition f x, and if this is in r l this is this function phi or this phi sorry this is phi, phi and if this phi is continuous and this is an this is continuous this is in r l, R alpha then our h will be in R alpha that s what is. So, let’s see the proof of it.

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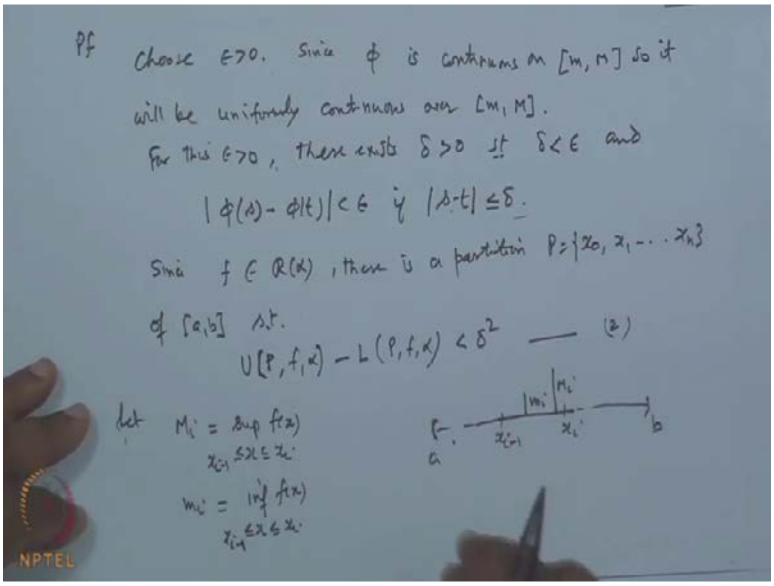

Choose epsilon greater than 0, now given that phi since phi is continuous on the closed interval m and capital M. So, this it will be uniformly continuous, uniformly continuous over this interval m and capital M, because every continuous function over a compact set is uniformly continuous. So, by definition epsilon we have already chosen. So, let us choose the delta. So, for this epsilon for this epsilon greater than 0, which is chosen here there exist, there exist a delta greater than 0 such that.

Now I am putting the restriction on this delta is smaller than epsilon, because this any delta then all the delta which are less putting delta to be less than epsilon this image will go there. So, let us put take the delta to be equal to epsilon and phi of s minus phi of t is less than epsilon if mod of s minus t is less than equal to delta. What he says is for the, because phi is uniformly continuous. So, for any epsilon greater than 0 there exist a delta, such that image of this any point in which satisfy this condition must fall within this range. Now suppose I take delta which is greater than epsilon then I can pick up an another delta which is smaller than this less than epsilon. So, the again this image will fall here. So, nothing we are not loosing anything, but it will be advantageous while proving the whole theorem.

Now, further since f is given to be an element in R alpha. So, by definition the result a necessary and sufficient condition for a function to be a Riemann Stieljes Integral on viewing the class of it, if there exist some partition for a given epsilon there exist some partition such that upper sum minus lower sum is less than this. So, let f belongs to this then there is a partition, there is a partition say P, x naught, x one, x two and x n of a b, f a b such that, such that the upper sum of the function f with respect to alpha minus lower sum of the function f with respect to alpha, over the same partition is less than say delta square clear.

So, this we are getting. So, let it be equation two, now over the subintervals we have let x i minus 1, x i plus this is the sub interval of this partition a b this is the partition a b this is now here the function attains this minimum value say m i and maximum value is suppose capital M i. So, let m i and capital M i, m i is the supremum of the function f x when x lies between x i minus one to x i while the M i small m i is the infremum value of the function f x when the x lies between x i minus one to x i clear let be this one.

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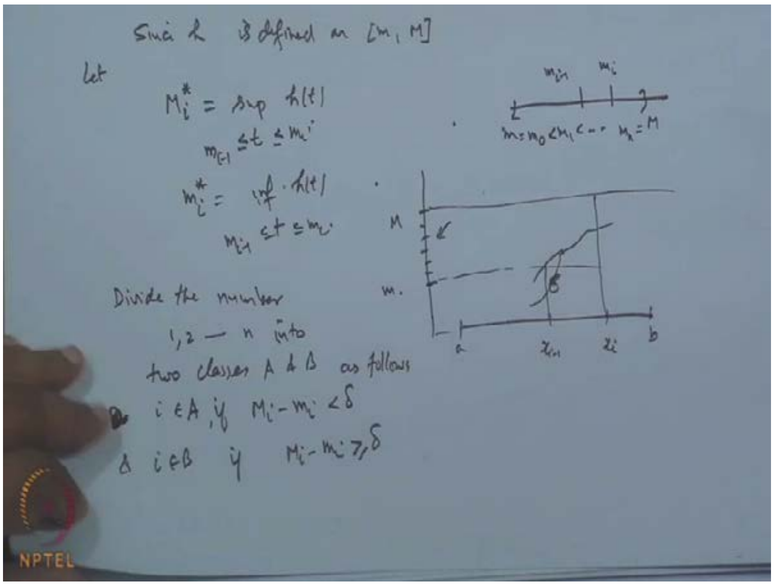

Now, function h is defined since h is defined, h is defined because what the result is the h is f is Riemann Integral, f is this and phi is continuous function phi is continuous on this and h is a function which is basically phi of h f x. So, what are the range of f h is defined on it. So, basically the h is defined on h is defined on m and capital M. So, in this case when you partitioning this m n say m and capital M and suppose I partition it say m naught less than m 1 and. So, on say here is m n less than m. So, if we picked up the m i minus one and m i.

Now in this case now you consider the upper and lower bound for the function h then we say we denote this as let us denote, let M i star is the supremum value of h t, I am using the t where t lies between m i minus one to M i and small m i star is the infremum value of h t when t lies between m i minus one to m i, is it not just i am taking this one now. So, this is our interval a b here I am taking one say subinterval say x i, x i minus 1 and here is say suppose image is.

So, this is our say m i m and here is capital M say this is our. So, basically this will be the all the functions will be somewhere here clear. In fact, this is wrong this one like this. So, here is now we are dividing here and then h is defined over this h is defined on this side now let us choose the suppose for i, divide the number divide the number one, two, three n into two classes, two classes.

First class is A, two class is A and B as follows. When i belongs to A for I belongs to A our choice of delta shows, i belongs to a means that is if i belongs to A for i belongs to A or a i belongs to a if or just i belongs to a if M i minus small m i is less than delta and i belongs to B, if M i minus small m i is greater than or less than delta. So, this will be like this.

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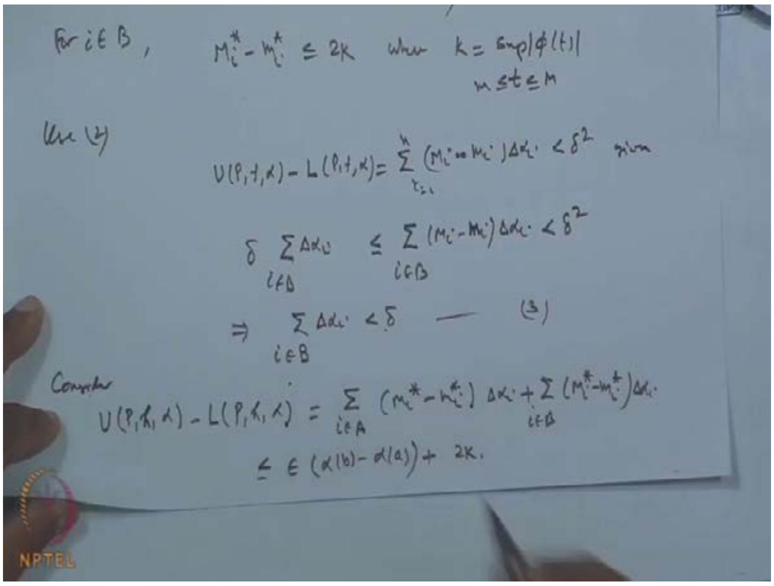

Now picked up now class one, so for i belongs to. So, for i belongs to A, what is our when i is in a M i minus small m i is delta. So, in this case what is the M i star minus m i star small m i star this will be because phi is what phi is uniformly continuous. So, if i belongs to A, it means it satisfies this condition, once it satisfy this condition then because the h of phi defined on what on this interval. So, is it satisfies this condition therefore, the difference of this M i star minus small m i star must be less than epsilon.

So, since our i belongs to this. So, our choice of choice of delta choice of delta shows M i minus small m i star is less than epsilon and this follows from, from this equation that is the relation I will say the relation is say x. So, it follows from the relation star, so nothing but this. So, first thing and if for i belongs to B, where this greater than or equal to delta, we have in this interval the M i star minus small m i star is less than equal to 2 K, where K stands for the supremum value of the phi when t lies between m and capital M.

So, supremum value and since it is greater than. So, use the condition two, condition two is this condition we have taken, that is m i minus delta square where is the delta square condition. So, this is condition two this is our condition two M i sigma of M i minus M i small m i delta alpha i is less than delta square. So, use the condition two. So, use two then in that case the U P f alpha minus L minus L P f alpha this is what. This is sigma M i minus sigma m i minus small m i delta alpha i, i is one to n, but if we take and this is given to be less than delta square this is given to be delta given

Now, if I take the only for sigma M i minus small m i delta alpha i over the set b only, then obviously it will remain less than delta square because this sum is smaller than this. So, it is less than delta square, but this will be x w equal to what over v this is greater than less than delta. So, it is delta sigma i belongs to B delta alpha i. So, what this imply this implies that sigma delta alpha i when i belongs to B is basically less than delta let it be three. Now, consider the partition now consider the upper sum of the function h with respect to alpha minus the lower sum of h with respect to alpha and this we can divide in two parts when i belongs to A, M i star minus small m i star delta alpha i plus when i belongs to B, M i star minus small m i star delta alpha i

Now, see this will be M i star minus a when i belongs to A we have already justified this is less than epsilon. So, this is basically less than epsilon and then delta alpha i will be remain less than the v sigma of delta alpha i nothing but less than alpha b minus alpha a. So, we get this part the second portion is this entire thing is less than 2 K because this is 2 K. So, it is less than 2 K and then over B the sigma is less than epsilon sigma is less than this delta alpha i when you choose what is i delta sigma is less than delta.

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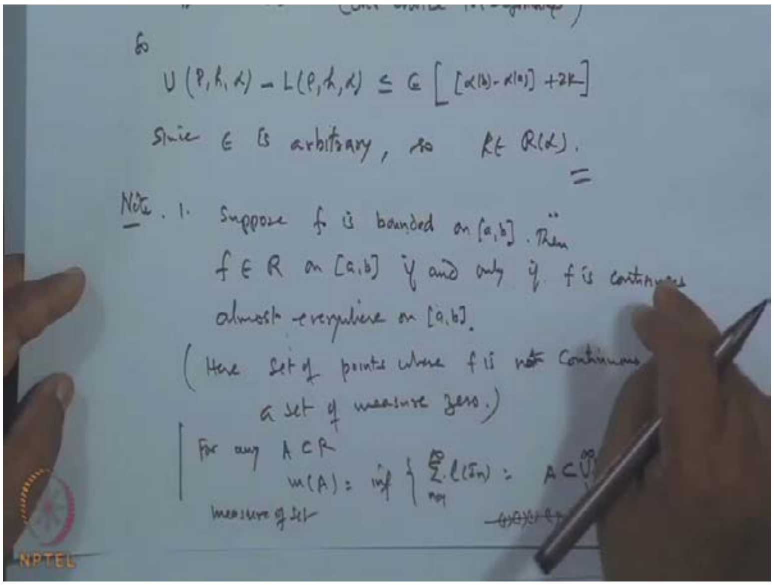

So, it is delta. So, once it delta, but delta is less than epsilon, this is our choice in the very beginning, beginning of this we have taken delta to be less than epsilon. So, we can choose the delta outside so we get P, h, alpha minus L, P, h, alpha is less than equal to epsilon alpha outside what you are getting is alpha b minus alpha a plus 2K, but epsilon is arbitrary since epsilon is arbitrarily.

So, this that satisfy the sufficient necessary condition of that theorem, therefore h must be in R alpha and that proves the results. So, that proves the results. Now, let’s come to, now as a we will not show that, but one note or result we will see then proof we will see, while going for the exercise we will do the proof for this the result is here. What this says is suppose f is bounded on the close interval a b then f is Riemann Integral on the close interval a b if , if and only if and only if, f is continuous almost everywhere on a b.

In fact, the concept of the almost everywhere this is a concept which is given in the measure theory measure integration, but what is the meaning of almost everywhere. Suppose a property holds everywhere except at some point and that set of those points form a measure 0. Then we say the property holds almost everywhere that is, so here we mean the set of points where f is not continuous, f is not continuous forms a set of measure 0.

Measure 0 it means the length of that set measure of that set is 0, because when it is a interval we say the measure of the set a is a interval interval is the b minus a, but if the set a contains the point of real numbers lying on the real line and they are scattered you cannot say that s a difference between the last point, and the first point is the length of the interval. So, it cannot be a measure it will be more value than whatever actually have.

So, what we do here that we find if in arbitrary set for an arbitrary set, for any set A which is a subset of R. The measure of the set a e define as measure of this is out of course, thus I want the measure of a set a, is the infremum value of sigma length of I n, n is one to infinity such that the countable union of I n covers A. So, what we do is we consider the intervals i 1, i 2, i n an open intervals an open intervals where these interval covers the point of the set A, then find out the length take the summation and then change the interval again take the infremum value.

So, if infremum exist we say it is a measure of the set A, where i n is an open interval. So, this is what we have, but we will not discuss in detail but so whenever the set is function is continuous except at the point which forms a set of measure 0, then the function must be a Riemann Integral function. So, because every continuous function is Riemann Integrable, but the function, which are even not continuous a bounded functions having a simple continuity point of discontinuity and then those numbers are forms a measure 0 it will be a Riemann Integrable function.

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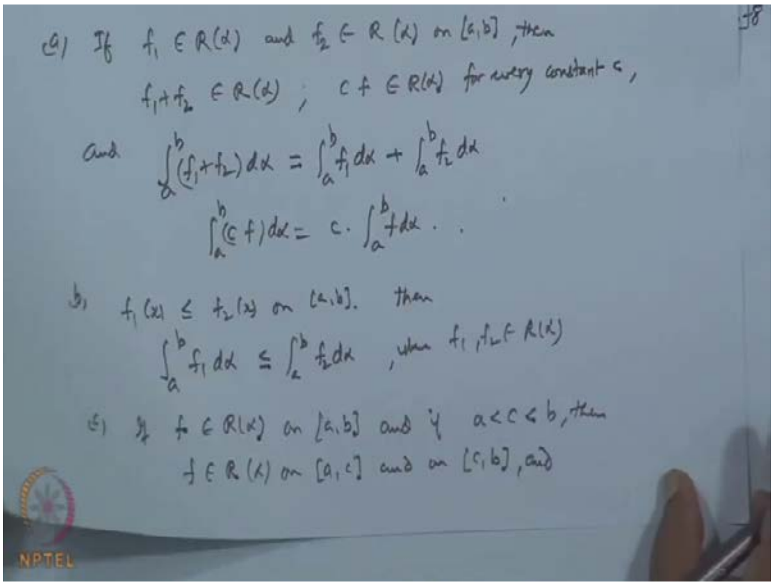

So, that is what is a now let’s see the some few properties of the Riemann Stieljes Integrals, we will list the proof property and proof one or two will be good enough now if f 1 is a Riemann Stieljes Integral with respect to alpha, f 2 is a Riemann Stieljes Integral with respect to alpha on the close interval a b, then the sum f 1 plus f 2 will also be Riemann Stieljes Integral with respect to alpha and c f will also be a Riemann Stieljes Integral with respect to alpha for every constant.

Constant c and the integral a to b, f 1 plus f 2 d alpha the Riemann Stieljes Integral, where sum of it will be sum of the integrals f 1 d alpha plus a to b, f 2 d alpha and c times f d alpha is c multiply by a b to b f d alpha. So, this is the first result. Second result says if suppose f 1 and f 2 are the two integral if f 1 and f 2 are the two functions such that f 1 x is less than equal to f 2 x on the close interval a b and they are also Riemann Stieljes Integral and then then integral a to b a to b, f 1 d alpha is less than equal to integral a to b, f 2 d alpha of course, where f 1 and f 2 both are in alpha, R alpha.

Third property is if f is if f, belongs to R alpha that is Riemann Stieljes Integral on the close interval a, b; and if and if a is less than c less than b then f is f is Riemann Stieljes Integral on the close interval a c, as well as on c b means if we divide a b into two parts a c to c b then over each subinterval function f will remain Riemann Stieljes Integral with respect to same alpha and the value of integral will be the sum of this two and integral

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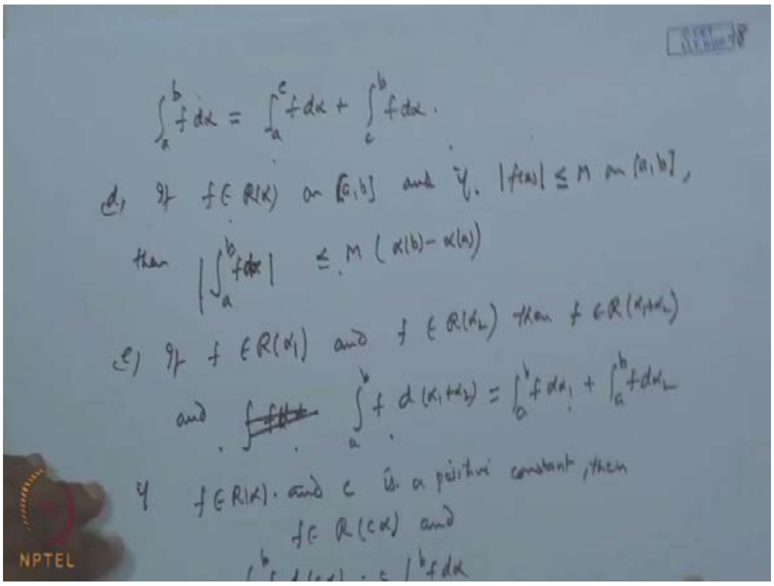

a to b f d alpha is the same as a to c f d alpha plus c to b f d alpha. Forth property is if f is a Riemann Stieljes Integral with respect to alpha on the close interval a b, close interval a b and if, the function f x is bounded by m on the close interval a, b; then integral a to b f d x f d alpha into f d alpha modulus of this modulus of this is less than equal to M times alpha b minus alpha a.

Modulus of this d alpha then e property, if f is if f is Reimann Stieljes Integral with respect to monotonic function alpha one and f is a Reimann Stieljes Integral with respect to function alpha means f is Riemann Stieljes Integral with respect to the two monotonic functions alpha 1 minus alpha 2 respectively are there then f will remain Riemann Stieljes Integral with respect to the sum of these alpha 1 plus alpha 2, because if the alpha 1 and alpha 2 are monotonic increasing the summation will also monotonic increasing if they are monotonic decreasing they will also be monotonic decreasing and hence and integral of f alpha 1 d alpha 1 integral f d alpha 1 plus alpha 2 over a to b is the same as a to b f d alpha 1 plus a to b f d alpha 2 and f.

If f is Riemann Stieljes Integral with respect to alpha and c and c is a positive constant positive constant then then f is Riemann stieljes integral with respect to c alpha and integral of this a to b, f d c alpha is c times a to b, f d alpha. The proof of this results property follows by using the definition itself, but however, we will see the proof of the first result only.

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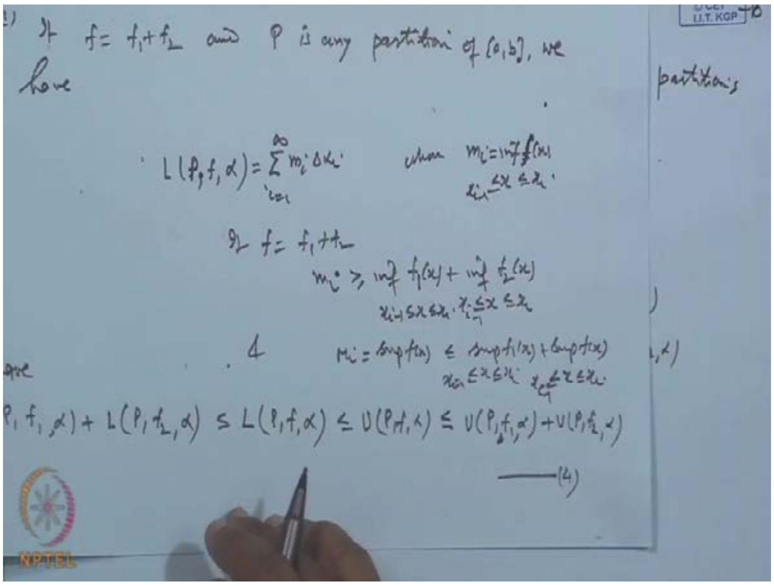

So, let's see the proof for first and rest will be. So, let f is f 1 plus f 2 and P is any partition. P is any partition of the interval a, b; any partition of a b then will we have, now let's see lower sum f lower sum of with respect to alpha over the partition P, is what is basically the sigma small m i delta alpha i is one to infinity. Where m i is the infremum value of the function f x, over x lying between x i minus 1 to x i. Now if i take the two function instead of f we take the f 1 and f 2 then what happen this and if f is the summation of if f is summation of f 1 and f 2 then the infremum value of m i this will be greater than equal to the infremum of the function f 1 x.

Over the same interval x i minus 1 less than x less than x i plus infremum value of f 2 x over the same interval x lying between x i minus one to x i and the supremum will attain and supremum of this f x will be that is capital m i will be less than equal to supremum of f 1 x over same x over this interval plus supremum of f x supremum of f x, when x lies between x i minus one to x i. So, supremum of the sum is less than equal to sum of the supremum and infremum will be greater.

So, that is what using this criteria we have now. The lower sum of P f 1 alpha plus lower sum of f 2 with respect to alpha, is less than equal to the lower sum of the function f with respect to because of this result, this result and which is a lower sum will always be less than the upper sum. So, we get this and then upper sum because of this it’s again less than equal to what? Upper sum of P, f 1 alpha plus upper sum of P, f 2 alpha this now apply this condition.

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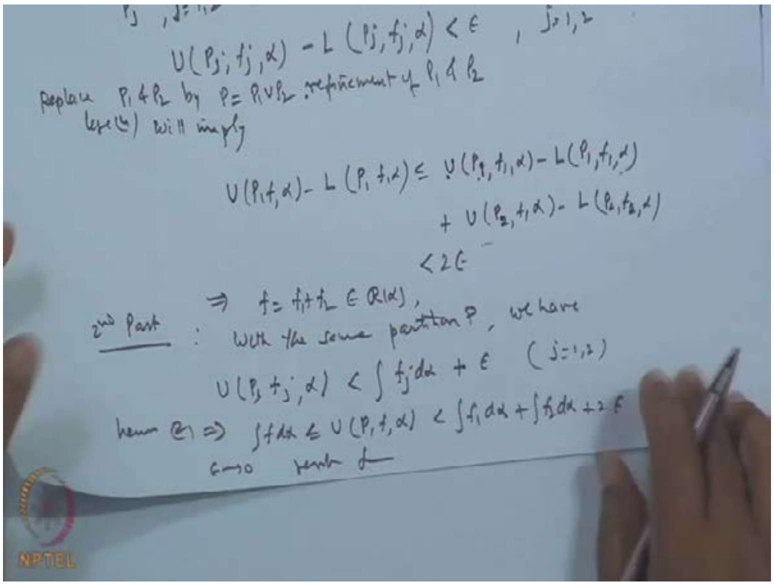

So, let it be say four, now if f alpha since if f belongs to or f 1 belongs to R alpha and f 2 also belongs to R alpha. So, for a given epsilon we can find there are the partition by necessary and sufficient conditions are the partitions P j where j is 1, 2, P 1 and P 2 respectively. Such that the upper sum of f j with respect to alpha over the partition P j minus lower sum of f j with respect to alpha over the is less than epsilon where j is 1, 2 this is by definition. So, use the now four use four. So, what you are getting is if we replace use four, replace P 1 and P 2 by P which is the union of this a refinement of P 1 and P 2 partition.

So, once we take this common partition then four will imply then four will imply will imply what? Upper sum minus the lower sum, what is the upper sum minus lower sum upper sum, P alpha minus upper sum minus lower sum is this, now upper sum in the first forth this is less than equal to this and lower sum is greater than. So, when you take the minus sign you will come you are coming like this. So, it is less than equal to upper sum of P 1 this is P 1 P 1 f 1 alpha minus lower sum P 1, f 1 alpha plus upper sum of P 2, I think this I did upper sum P, no this is upper sum of P P 1 is.

So, we are getting this part is P 1 alpha and then here we are getting this is less than equal to what upper sum of P 2, f 1 alpha minus lower sum of P 2, f f 2 alpha. From here just you get it like this what you are getting is the P j. So, if I take this why it is so because upper sum decreases. So, P of this is less than sum of this lower sum increases. So, we are getting this one and this is less than epsilon this is less than. So, this is less than 2 epsilon, so we are getting this is less than which proves which implies that our f which if f one plus f 2 must be that all.

So, we get. So, this time we have this one therefore, and now for the integral part we can say for second part we say with the same partition P with the same partition P, we have we have the upper sum P, f j, alpha is strictly less than f j, d alpha plus epsilon for j is 1, 2 and because this upper sum infimum value is this integral. So, we can get this thing. Hence the second implies this part hence will imply that integral f d alpha is less than equal to U, P, f alpha which is less than integral f 1 d alpha plus integral f 2 d alpha plus 2 epsilon and epsilon is arbitrary. So, the result follows.

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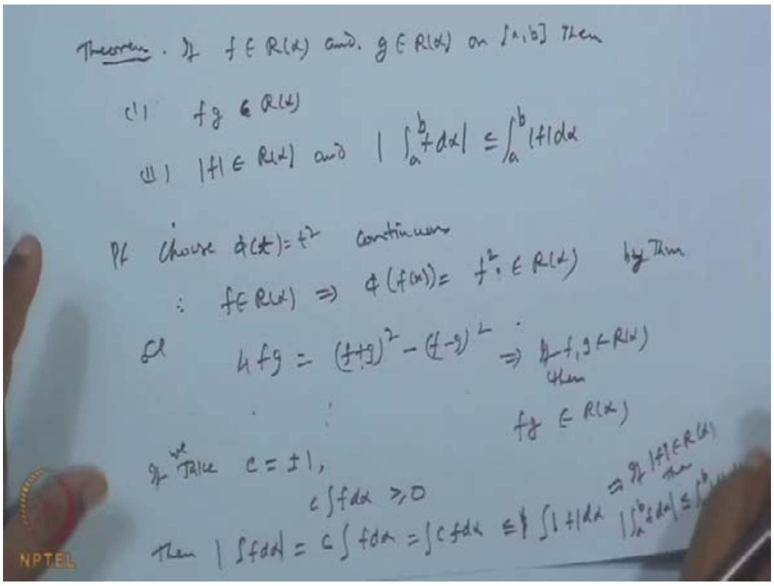

So, this now another property is also which we just mentioned, if f is in R l and g is also in R alpha on the interval a b, then the product of this is in R alpha then second part is if mod f belongs to R alpha and modulus of integral a to b, f d alpha is less than equal to integral a to b mod f d alpha and less than equal to. The proof is based on the if phi is a continuous function f is a Riemann Stieljes Integral. Then phi composition f will be a Riemann Integral, so depending on. So, choose phi x phi x of i t, s t square which is continuous and therefore, if f is R then it implies that phi of f x that is the f square must be i R by theorem previous. So, nothing similarly similarly what is four f g, this we can say f plus g square minus f minus g square.

Now this is integrable this is integrable belongs to. So, this belongs to. So, this implies if f and g both are in R alpha then the product f g will also be in R alpha because the constant times of this also in R alpha. So, this implies then further if we take c as the plus minus one. So, we get what c into integral f d alpha is greater than equal to 0. Suppose integral f d alpha is negative then take the c to be minus if is it positive then you take this one then modulus of integral f d alpha is equal to c times integral f d alpha, because this is non-negative and then this is integral c f d alpha is and this is less than equal to integral integral mod of f, d alpha because c alpha is less than equal to 0. So, what we says is, so if our. So, this imply that if mod f alpha belongs mod f belongs to R alpha then the result hold this result that is integral modulus of integral a to b f d alpha is less than or equal to integral a to b mod f d alpha holds and that is. Thank you very much.