A Basic Course in Real Analysis
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Lecture - 40
Existence of Riemann Stieltjes Integral

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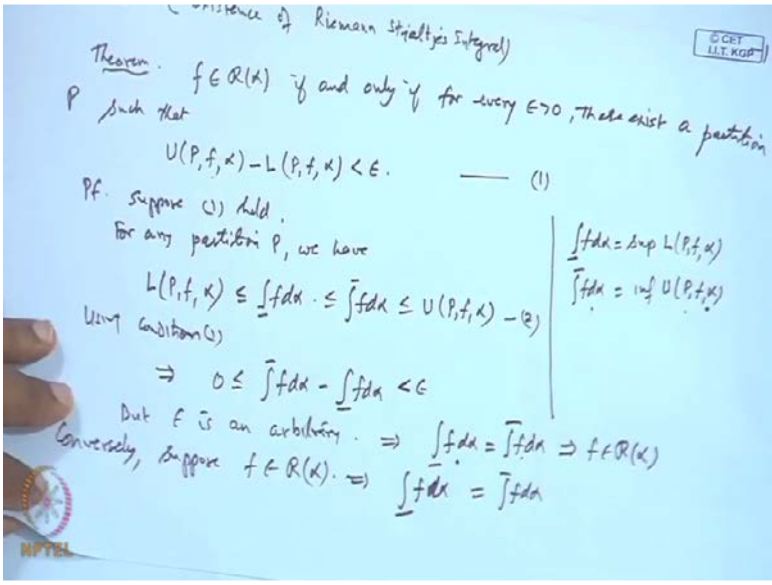

So, today we will discuss the result which we have stated yesterday, and this was the existence of the Riemann Integral, Riemann Stieltjes Integral and is a particular case are Riemann Integral. R alpha we have taken the class of all Riemann Stieltjes Integrals. So, if f belongs to R alpha means f is an element of the class R alpha, that is a Riemann Stieltjes Integral integrable functions, if and only if for every epsilon greater than 0 there exists a partition, there exists a partition.

There exists a partition P, there exists a partition P such that such that the upper sum of the function f with respect to alpha minus the lower sum of the function a with respect to alpha over this partition is less than epsilon. So, for every epsilon greater than there exists a partition which this condition holds. So, this condition is a if and only if condition let it be put it the condition as say 1. So, let us see the proof. Suppose, the condition holds, suppose 1 hold, that is this some for some partition for a given epsilon we can identify a partition. So, that this result upper sum minus lower sum is less than epsilon. Now, for any partition P we know or we have the lower sum of the function f with respect to the alpha, where alpha is a monotonic function defined over the interval a, b. So, this is less than equal to the lower sum of the function, lower integral of the function f which is less than equal to the upper integral of the upper Riemann Stieltjes Integral of the function f which is less than equal to the upper sum of the function f alpha. This we have already discussed, because how did you find the lower integral and upper integral.

Of the lower Riemann Stieltjes Integral of this is defined as the supremum value of the lower sum f with respect to the alpha where the upper Riemann Integral Stieltjes Integral of the function f is defined at the infimum of the upper sum of f with respect to alpha. So, if I remove infimum and supremum then this quantity will always be greater than the upper sum will always be greater than or equal to the upper integral, while lower sum will always be less than equal to the lower integral. So, for any partition P this result holds.

Now, using the condition 1 condition 1 what the condition 1 says that there exists a partition P such that the difference between the upper sum and lower sum is less than epsilon, this is given. So, since this result 2 is true for any partition. So, in particular this particular partition the upper sum minus lower sum will remain less than epsilon. Therefore, using this we see here that 0 is less than equal to upper integral Riemann Stieltjes Integral minus the lower Riemann Stieltjes Integral, this remain less than epsilon because this difference is less than epsilon so this and this is non negative because the upper sum and lower sum is defined, the maximum value m i and delta alpha and so on. Where alpha is a monotonic function. So, we are getting for that. So, we get this less than epsilon, but epsilon is arbitrary number but epsilon is an arbitrary number.

So, once it is arbitrary it can choose any, for any epsilon however, is small this result hold. So, this shows that lower integral and the upper integral, Riemann Stieltjes Integral coincide and once they coincide then this implies the f belongs to the class R alpha, the Riemann Stieltjes Integrable function. So, this is what. Conversely, suppose f belongs to the class of all Riemann Stieltjes Integrable functions then we have to show the condition 1 holds. Now, once f belongs to Riemann Stieltjes Integral it means upper sum and lower sum coincide and equal to this. So, what we get from here the upper sum is the infimum value of this and lower sum is the infimum lower integral, upper integral is infimum value and lower integral is the supremum value of this.

So, suppose I take this f belongs to R means that lower integral and the upper integral both coincide. First thing they are equal. Now, use this thing. If I remove the infimum then what happen is because this is the infimum value. So, this will remain less than is this value. So, if I take a numbers slightly higher than this then there will exist a partition P 2 where this value will plus epsilon will be greater than this number. Similarly, here if I remove this supremum because this is the largest value. So, this is greater than equal to this number. So, if I take a number slightly lower than this lower than this then there exists a partition where this number is greater than the number this minus some number epsilon by 2.

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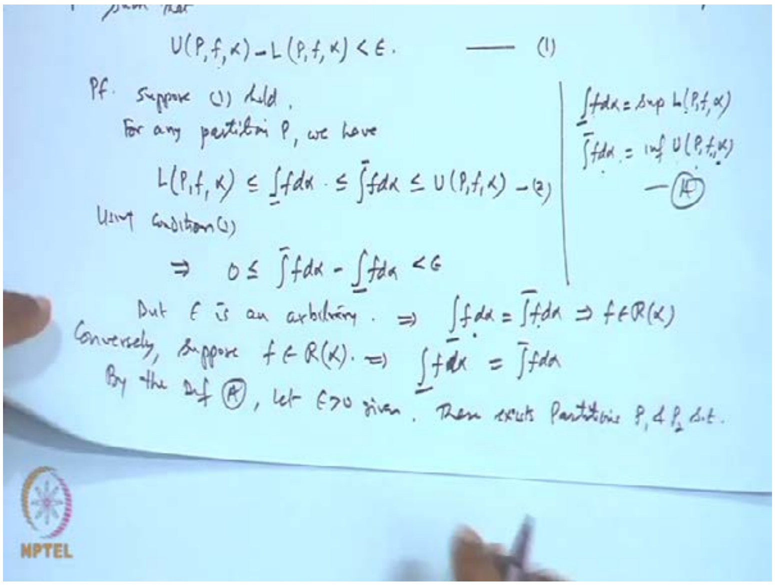

So, there exists by the definition definition I am calling definition A, definition. We can say that let epsilon greater than 0 is given. So, with this epsilon there exists or there exists partitions partitions say P 1 and P 2.

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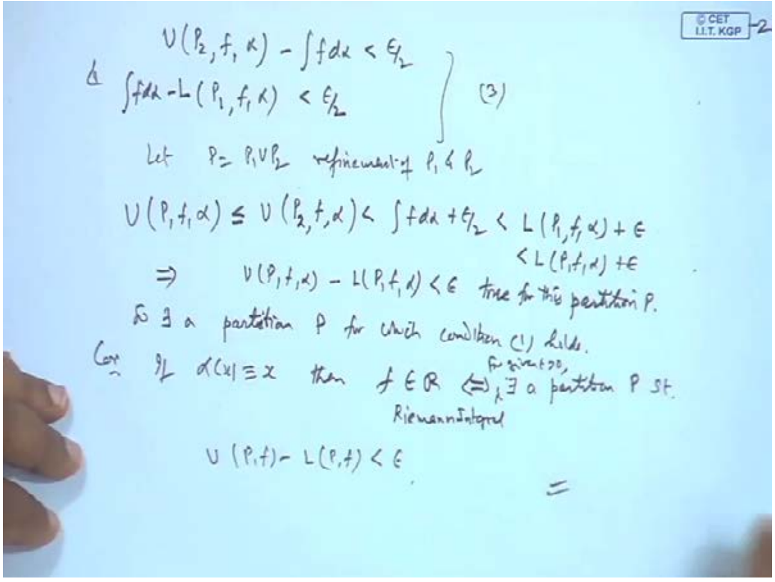

Such that such that the upper sum of the function f with respect to alpha over the partition P 2 minus integral f d alpha is less than epsilon by 2 and lower sum of this integral integral f d alpha minus lower sum of f with respect to alpha over partition P 1 is less than epsilon by 2.

This we can get it this way. So, let it be equation say third. Now, if we take the common partition let P is the P 1 union P 2. So, if we take the refinement of this, this is the refinement of P 1 and P 2. So, if this results holds for this then for the refinement also we can get this general result. What we get? Say, upper sum since P is the refinement of P 1 P 2. So, upper sum decreases then lower sum increases, this we have already discussed. So, start with this. Upper sum of the function f with respect to alpha over the partition P which is the refinement of P 1 P 2 since upper sum decreases. So, this is the less than equal to upper sum of the function f with alpha over the partition P 2 further using the 3 the upper sum of this less than integral f d alpha plus epsilon by 2, is it not? Now, using the second part of this 3 integral f d alpha is less than further lower sum of p 1 f alpha plus epsilon by 2. So, this is less than epsilon plus epsilon.

So, what we get it? From here we get this that U P f alpha minus L L one more thing because this lower partition, lower sum increases for the refinement. So, this is further less than L P f alpha plus epsilon. So, this minus this f alpha is less than epsilon at least this is true for this partition P. So, there exists a partition P. So, there exists a partition P for which condition 1 holds and that prove the results. So, this.

Now, this results gives a guarantee or criteria for a function f to be Riemann Stieltjes Integrable function, if the upper sum minus the lower sum is less than epsilon for some partition P. In particular when alpha x equal to x then you can get the corresponding corollary, as corollary. If alpha x is identically x then f belongs to R the Riemann Stieltjes Integral, Riemann sorry Riemann integral if and only if if and only if there exists a partition P such, for given epsilon for given epsilon greater than 0, there exists a partition P such that upper sum of the function P f minus the lower sum of the function with respect to partition P is less than epsilon and this is only part for the existence result for the Riemann Integral Function, integral function. So, this is so. Proof is okay, as a corollary you can get. Now, from this result we can drive few more conditions where you can say is a sometimes sufficient condition, sometimes necessary condition for the function to be in Riemann Stieltjes Integral or in particular Riemann Integral.

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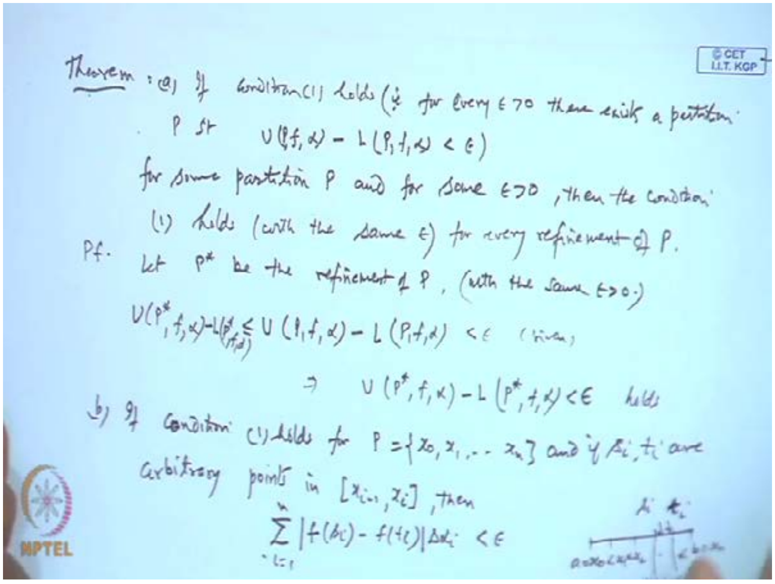

So, we have this theorem. The theorem says in the three parts a, if 1 holds, if condition 1 holds, 1 holds means the condition that is there exists a, that is for every epsilon greater than 0 there exists there exists a partition P. Partition P such that upper sum of the function f with respect to alpha minus lower sum of the function f with respect to alpha is less than epsilon. So, if this condition holds for some this condition holds for some partition P and some epsilon holds for some partition P and for this is 1 and for some epsilon greater than 0. Then the condition 1 holds with the same epsilon, with the same epsilon for every refinement of P of the partition P. Means the condition will say that there exists some partition, for every epsilon there exists a partition.

Now, what he says is if suppose this result hold for some partition for this epsilon then for the same particular epsilon this result will hold for any refinement, if we take a refinement of P, say P star then this result will continue to hold good. The proof of course is simple. Let me see the proof a. In case of let P star be the refinement of P with the same epsilon, with the same epsilon greater than 0. This epsilon we are having the partition which is a refinement of P, epsilon I am not changing refinement of P, but when P star is Riemann Integrable function of P the upper sum decrease and lower sum increases. So, upper sum of this is greater than equal to the upper sum of this partition of f with respect to alpha over the partition P star and the lower sum of this is increasing, this decreases, increases. So, minus of this will decrease.

So, L of P star f alpha, but this is less than epsilon, it is given. So, this implies that upper sum of the function f with respect to alpha over the partition P star minus the lower sum of the function f with respect to alpha over the partition P star is less than epsilon. Where, the P star is the refinement of this two. So, holds. So, this proves them. Second result says if 1 holds if the condition, if condition 1 holds, holds for the partition P say x naught x 1 x 2 say x n and if and if s i and t i is suppose are the arbitrary points, are arbitrary points in the interval x i minus 1 x i then the sigma of this f of s i minus f of t i under modulus sign, multiply by delta alpha i is less than epsilon i is 1 to n, 1 to n this is true.

So, it means if this is the partition of a b interval and if this is the point say x i minus 1, this is the point say x i and if we picked up the two point s i and t i and t i in this sub interval x i minus 1 to x i then the functional value at the point s i minus t i sigma of this multiply by delta alpha i is less than epsilon, that what he says. That if the function, if the condition 1 holds that is if there exists for a epsilon greater than 0 if there exists a partition such that the this upper sum minus lower sum is less than epsilon then it will also imply that this condition will hold. The reason is again very simple, the proof is given like this.

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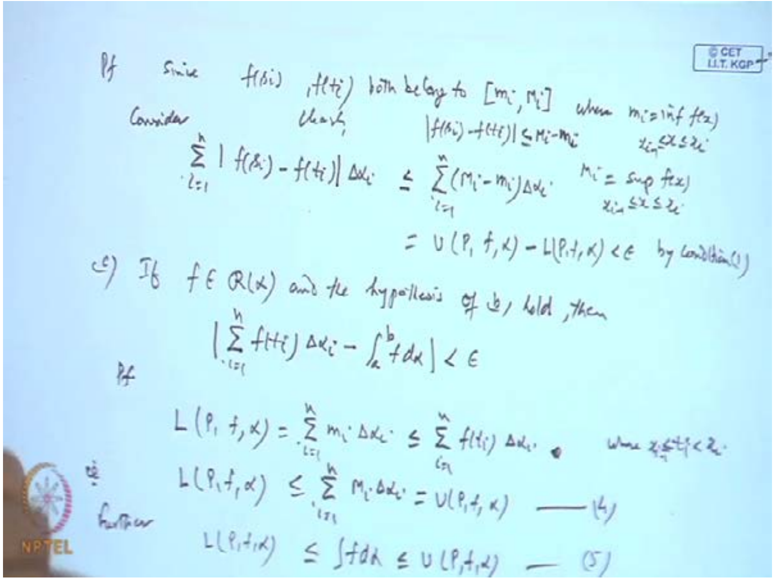

What is our f f of s i and (( )). Since, f of s i and f of t i both are, both belongs to the interval m i into M capital M i. Where what is m i and capital Mi, where M i is the infimum value of the function f x when x less than greater than equal to x i minus 1 and less than equal to x i and capital M i is the supremum value of the function f x when x lies in the interval x i minus 1 to x i. So, both these values are in the interval m i and capital M i. So, if we start with this say upper sum and the lower sum. What is the upper sum? Upper sum is f of x i minus lower sum. So, what we get it, f of s i sigma.

So, consider this thing, this i is 1 to n mod of s i minus f of t i. Consider this multiply by delta f i. Now, f i and s i is one of the value, is value lying between this. So, both are lying between m i and capital M i. So, clearly a f s i f s i and minus f t i mod of this will remain less than or equal to M i minus small m I, minus small m i is it not. So, we can say this is less than equal to less than equal to sigma M i minus small m i is 1 to n delta alpha i, but this is nothing but what? The difference of the upper sum minus lower sum with respect to the partition P, upper sum minus lower sum of the function f with respect to alpha over the partition P.

 And this already given by condition 1, because condition 1 holds, so this is true by condition 1, so this is. Therefore, this sum will be less than this. So, this proves the, proves. Now, third portion which he says if suppose if f is Riemann Stieltjes Integral with respect to alpha and the hypothesis of b and the hypothesis hypothesis of b that is sigma of this thing is less than epsilon for partition P x naught x 1 x 2 and s i t i are the point in between these x i minus 1 and x i. Then this condition holds, this hypothesis is given, hold, this hypothesis hold then sigma of, sigma of f t i delta alpha i minus integral a to b f d alpha is less than epsilon is less than epsilon that is all, i is 1 to n 1 to n.

So, proof is let me see. The proof we will start with again similar what start with the lower sum of the function f with respect to alpha over the partition P, this is equal to sigma i is 1 to n small m i delta alpha i, but m i is the infimum value of this function. So, obviously, this will remain less than equal to the value of the function f t i where t i lies between where t i lies between x i minus 1 to x i. These are the, this is the point.

So, i is to n and then delta alpha i, but again this value is less than equal to maximum value. So, it will be less than equal to less than equal to sigma i is 1 to n M of i delta alpha i because it is the supremum value of this function and which is nothing but what upper sum of the function f with respect to alpha over the partition P. So, this is the one result. So, this shows that is the lower sum of f with respect to alpha this, satisfy this condition let it be 4. Further, the upper lower sum of this because f is given to be Riemann Integrable Function. So, the supremum value this will coincide the integral f d alpha. So, if I remove supremum it is less than equal to this and again the infimum value of the upper sum is this. So, this is less than equal to upper sum of f with respect to alpha. This is, this holds for any partition P, is it not? So, if I take the 4 and 5 together then what we get? If you take the 4 and 5.

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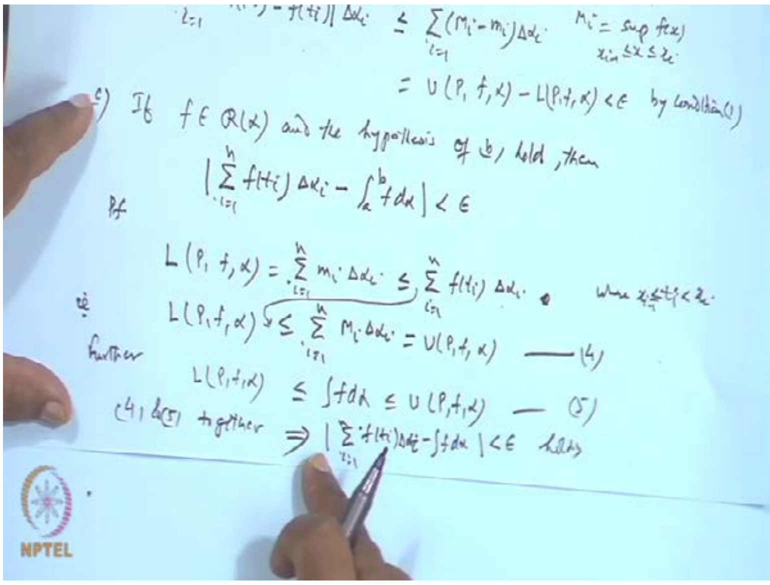

So, 4 and 5 together imply this difference if I take, this difference I take then what happen is the difference is coming to be 0. In fact, tending to 0 is very, very small because U P alpha minus L alpha is less than epsilon. The condition in the hypothesis 2, hypothesis 2 holds means the condition 1 holds, 1 holds means difference of U minus L is less than epsilon. So, difference of this minus this is less than epsilon in (( )) implies i is 1 to n 1 to n then f of t i take this difference, M I, this I will say here this part you can take it here and then again this is less than. So, we need this part. So, less than f of t i minus minus integral f d alpha, modulus of this that is sigma of this under this, modulus of this is less than epsilon holds.

Now, this result also suggest the way to define the Riemann Integral or Riemann Stieltjes Integral because we have discussed the definition of the Riemann Integral and Riemann Stieltjes Integral a by constructing the lower sum and the upper sum and then finding the supremum and infimum value and if the lower integral and upper integral coincide then we say f is Riemann Integrable Function, Riemann Stieltjes and when alpha x equal to x then it is the Riemann Integrable function. Now, in most of the books earlier if you see they do not take it to the upper sum and lower sum. What they, they start with a function f defined over the interval a b and then functions we assume to be a continuous function, bounded function sometime and then what we they do is they partition the interval. Find out the point in the sub interval, take the value of f of t i.

And then multiply this by the delta alpha i. So, the delta alpha i is left is delta alpha i, this is left. So, multiply delta alpha i and taking the limit when n is sufficiently large. It means when the number of the partitions are more infinity unfold. So, when the points are very close to each other then this integral limit exist we call it this integral of the every alpha and existence and we say the integral exist. So, this is the way of defining the Riemann Integral or Riemann Stieltjes Integral. So, however, the both way equivalent way of defining, but here this is a more better way because we are explicitly explicitly you are constructing the sum.

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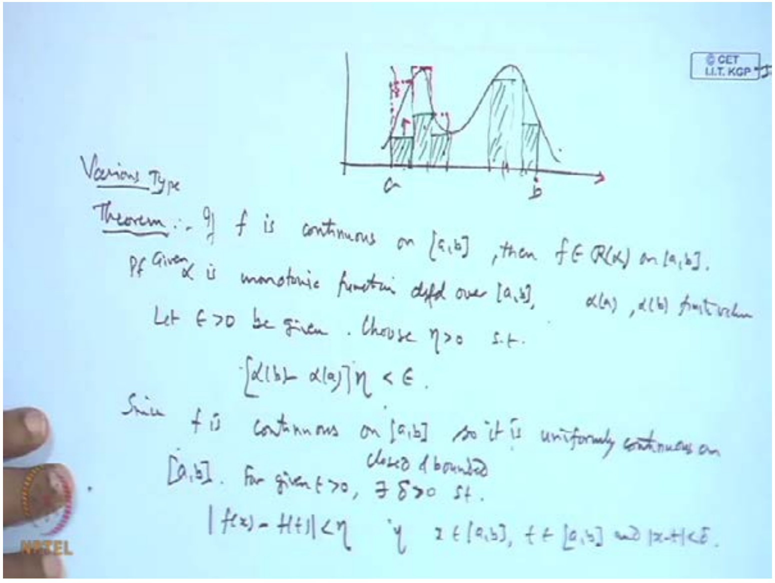

And then seeing the things because this will go like if suppose I have a graph of the function say like this, suppose this is our a this is our b and let us partition it partition it. So, this will be our partition and like this. Now, when you are choosing the value then what you are doing is you are constructing this sum in this interval you are taking the infimum value of the function. So, this is the infimum value of the function over this interval. So, you are taking this.

Then infimum value in this interval you are taking this one. Infimum value on this and total sum you are choosing and then changing the partition over this and basically after this changing etcetera we will get this limit when the supremum and infimum coincide then we say the function is integrable and what is the upper sum? Upper sum in this case will be this much. When you take second upper integral, this is the lower integral will give and upper integral upper sum will give this part.

So, here also this is, in this interval the upper sum is this. So, we can take this one like this and continue this one. So, when you choose the limiting value they basically the upper lower upper sum decreasing and lower sum will increase and they will coincide with this total area bounded by this curve. If the function is continuous it represents the area bounded by the curve y equal to f x and the ordinate x equal to n equal to v with x is of x. In case of the bounded function we say it is a enhancement of the definite integral and this we call it as a Riemann Integral for this and when we take up alpha x as a monotonic function in and then choosing the in values of x i n x i minus at the point with respect to alpha then you get the Riemann Stieltjes Integral. So, that is the difference between these three concepts.

However, the way in which it is defined is basically gives the other definition whatever the in terms of the upper sum, lower sum or directly also. So, as a limit of the sum, but since limit of the sum is difficult to compute the any integral because the partition is more important, when you take the limit of the sum of the right hand side because it becomes the infinite series. So, once you get infinite series then what happens is the sum of the infinite series will be difficult. Unless you choose a properly partitioning point. And looking the proper partitioning point with respect to function is not so easy. So, that is why we try to avoid that part limit of the sum to compute the Riemann Stieltjes Integral or Riemann integral, we take up upper sum and lower sum and taking the infimum and supremum value. So, that is what is. So, this is the one existence here.

Now, let us see few functions which are always (( )) functions like a continuous function. They are always a Riemann Integral function and some other case. So, let us take the various types of the integral, Riemann Integrable function, various type, various types. So, first theorem is if f is if f is continuous f is continuous on a closed interval a b a b then f is Riemann Stieltjes Integrable with respect to alpha on a b.

So, let us see the proof. alpha is given to be a monotonic functions. So, once it is alpha is given to be monotone, alpha is monotonic function either increasing or decreasing defined over the interval a b. So, alpha a alpha b these are finite values finite values alpha and finite value and now alpha b minus alpha a is a fixed number. So, what is left epsilon greater than 0. Let epsilon greater than 0 be given. Now, since alpha b minus alpha a is a finite one. So, we can identify some eta such that. So, choose eta greater than 0 such that alpha b minus alpha a multiply by eta is less than epsilon. This is possible because these are the fixed value, finite values. So, we can choose the eta to be epsilon over this number and since it is a monotonic either increasing or decreasing. So, it cannot be alpha a equal to alpha b, unless it is a constant function which we are not taking. So, with that this one less than epsilon. Further since f is continuous on the closed and compact, closed and bounded interval, this is a closed and bounded. So, every continuous function on a closed and bounded set or in a compact set is uniformly continuous.

So, this shows since f is (( )). So, it is uniformly continuous on the interval a b, this one. So, apply the definition of the uniformly continuity. So, there exists. So, for given epsilon given epsilon the same epsilon greater than 0 there exists a delta, there exists a delta positive, it does not depend on the points such that the value of this functional value f x minus f t remain less than eta for all x n t x n t in the interval, if x belongs to the interval a b, t belongs to the interval a b and such that mod of x minus t is less than delta. This is by definition of the uniform continuity of the function.

So, since f is uniformly continuous, it means if we pick up any two point in the interval a b which satisfy this condition that x minus t is less than delta that is in the delta neighborhood of the point t or any, then images will fall in the eta neighborhood of f t. That is this eta this say I think it is less than epsilon, this is less than eta will go. Now, choose the partition P.

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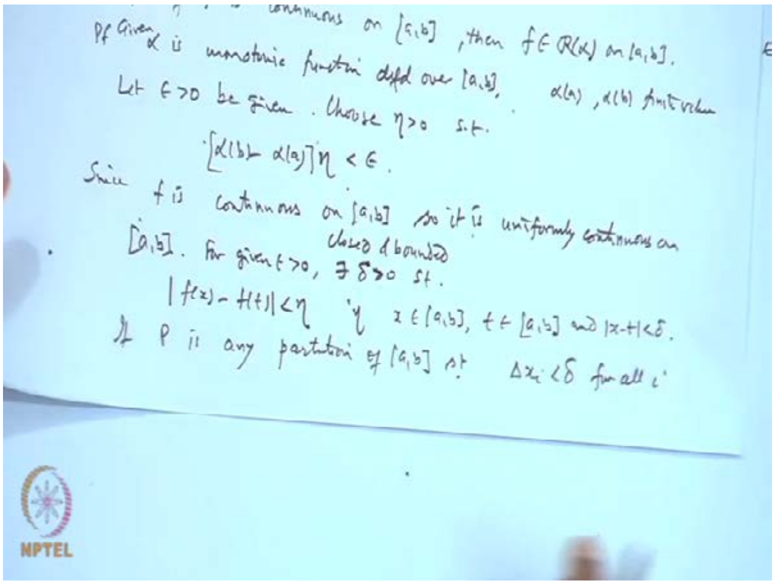

Now if P is any partition any partition of the interval a b such that such that delta x i is less than delta, delta x i is less than delta for all i for all i.

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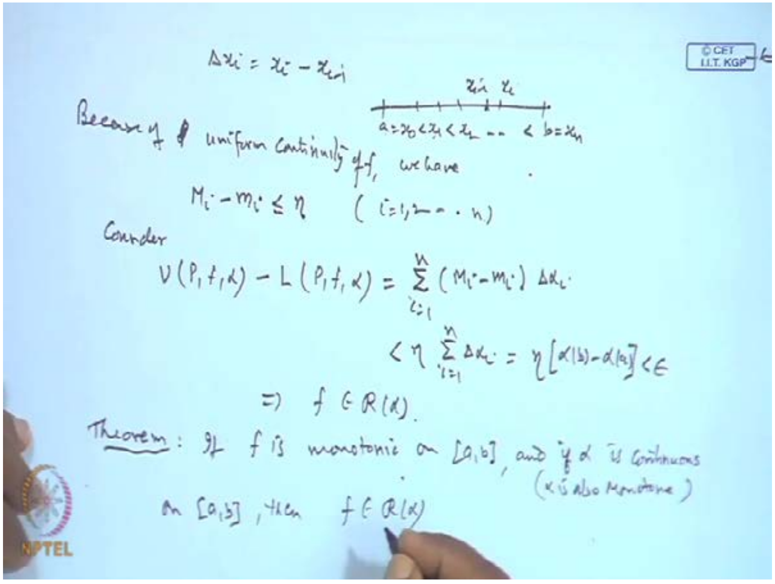

It means we are taking the partition a b as x naught less than x 1 less than x 2 and less than x n equal to v. So, here is the point say x i minus 1 x i. So, this delta x i is x i minus x i minus 1. Now, this delta x i is there because we are choosing the partition less than. So, this is positive, if the length of this is less than delta for all i for all i value. Now, if we picked up any point inside this delta x i, then what happens the images functional value at the point x i minus m i this difference will remain less than eta because of this. Because any point x y which is less than lying between this interval lying between this interval x i minus x i minus 1 or t i, if this is corresponding images will be there. So, the maximum value and the minimum value of the function will also satisfy this condition. If the point is here in the closed interval x i minus 1 and x i and function attains the maximum and minimum value, then they have to follow this result.

So, because of the. So, because of because of f uniform continuous uniform continuous because of uniform continuity of f, we get, we have M i minus small m i is less than or equal to eta for i equal to 1 to up to n. Now, consider this. We want this, we want the function is a Riemann Stieltjes Integral. It means we wanted to use that result that if the upper sum minus lower sum is less than epsilon for some partition. For a given epsilon there exists some partition such that upper sum minus lower sum is less than epsilon then the function f must be a Riemann Stieltjes Integral. So, let us consider, choose the partition P first and then consider the upper sum of the function f with respect to alpha over this partition P minus lower sum of the function f with respect to this partition. Now, this is by definition is nothing but i equal to 1 to n M i capital M i minus small m i delta alpha i, is it okay.

Now, what is this? This is already given to be less than eta. So, eta and sigma i equal to 1 to n delta alpha i, this is nothing but what? This is the value of alpha at the point i is 1. So, delta alpha 1 means alpha x 1 minus alpha x naught plus alpha x 2 minus alpha here. So, all gets cancel and finally, you are getting to be eta alpha b minus alpha a, but we are choosing the, for epsilon we have taken the eta in such a way that this is less than epsilon. So, this shows, this implies f belongs to the Riemann Stieltjes Integral with respect to alpha and that is proved, the result. So, every continuous function is a Riemann Stieltjes Integral and in particular every continuous function is a Riemann Integrable function.

Next, shows that is also a if f is a monotonic function, f is monotone, if f is monotonic on the closed interval a b and if alpha is a continuous function, is continuous. Now, apart from this alpha is also monotonic. Remember, this I need not to write because we are assuming this is already monotone function, but every monotone function need not be a continuous throughout. So, here we are assuming exclusively alpha to be a continuous, apart from its monotonicity on the closed interval say a b. Then f is Riemann Stieltjes

Integral or f belongs to the class (( )). Let us see the proof of this again. So, it means the monotonic functions, if f is monotonic and alpha is continuous then this will be a Riemann Stieltjes Integral.

Now, in case of the Riemann Integrable function alpha x we are choosing only be x. So, over the closed interval a b, if you take alpha x equal to s, it is automatically continuous function and it is a monotonic function. So, basically it is satisfy this condition. So, need not to take this condition when you are dealing with Riemann Integral. What simply say every monotonic functions on a closed interval a b is Riemann Integrable function , but in case of this we have to take a condition on alpha. See the proof of it.

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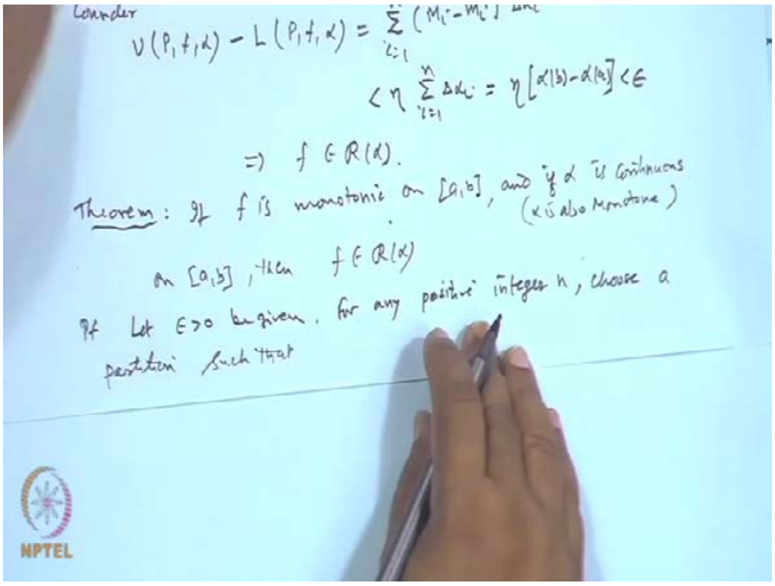

Let epsilon greater than 0 be given be given, then again because alpha is continuous. So, it will assume all the values from alpha a to alpha b, it will assume all values. Therefore, we can partition it delta alpha a b, we can choose the equal partition alpha b minus alpha a by n as a n partition for this sub partition of the interval a b like this. So, we can choose that. So, for any positive integer, for any positive integer n for any positive integer n choose a partition choose a partition P.

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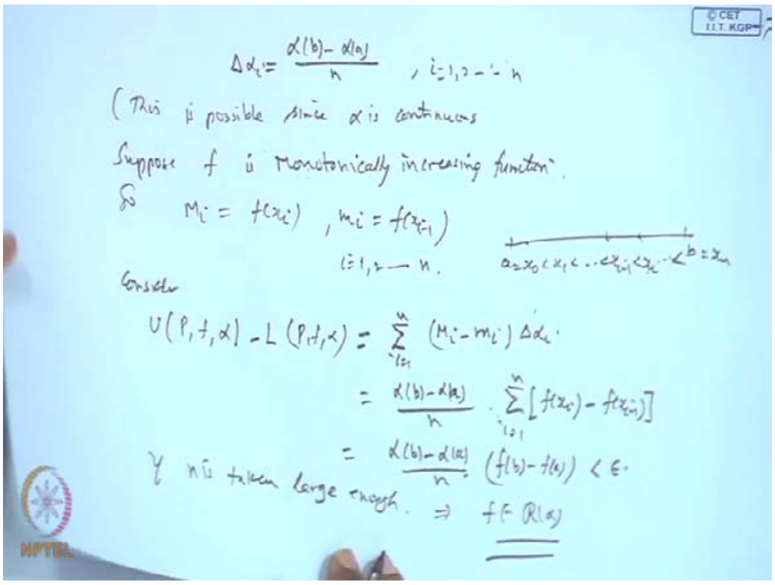

In such a way such that such that the alpha b minus alpha a divide by n is our delta alpha i, when i is 1 to n means equal partition delta alpha 1 is the same as delta alpha 2 is the same as this and the value is alpha b minus and this is possible since alpha is continuous. If alpha is not continuous, it means that the some point in between a and b the function alpha may not be defined at that point. So, we cannot talk about these things.

So, since it is continuous therefore, all the values in between alpha a and alpha b is possible. Therefore, you can divide it and get the equal values of delta alpha i. Now, suppose f is monotone which is given f as monotone. So, let f is monotonically increasing function monotonically increasing function. The similar case when monotonically decreasing function will can be proved in a similar way. So, there. So, once it is monotonically increasing function, it means when you choose a to b partition as x naught x 1 x 2 x i minus 1 x i and x n is b, then value of the function at a point x i is greater than the value of the function at a point x i.

So, over the interval x i to x i minus 1 the M i the maximum value of the function will attain at the point x i where the minimum value will attain at the point x i minus 1 because it is increasing function, monotonically increasing function and this is true for every i i 1 to n. Now, consider the upper sum of the function f over this partition minus lower sum of the function f with respect to alpha over this partition. Now, this will be equal to what? If we write this thing is sigma i is equal to 1 to n M i minus small m i into delta alpha i, but M i minus small m i is this. So, we are taking delta alpha i we are choosing same it is independent of i.

So, we can take it outside by n and this sum i is equal to 1 to n what you are getting is f of x i minus f of x i minus 1 this is the value. Now, this when you substitute i equal to 1 to n the terms get cancelled and only you get the f b minus f a. So, finally, you are getting alpha b minus alpha a by n f b multiply by f b minus f a. This you are getting. Now, this one is less than epsilon, this follows from, f b condition is, this is a small quantity alpha b minus alpha a is less than this. Now, this part when you are taking is less than epsilon (( )) why? So, if you are taking this alpha b minus alpha by n f b minus f a and this divide by n. So, n is sufficiently large.

So, we can take this is less than epsilon as n is taken sufficiently large. If n is taken sufficiently large. So, when n is sufficiently large the total thing can be made less than epsilon therefore, it satisfy the condition 1 which is necessary and sufficient condition for a function f to be in the class R alpha. So, this shows f belongs to R alpha. So, that is what. Another results which we.

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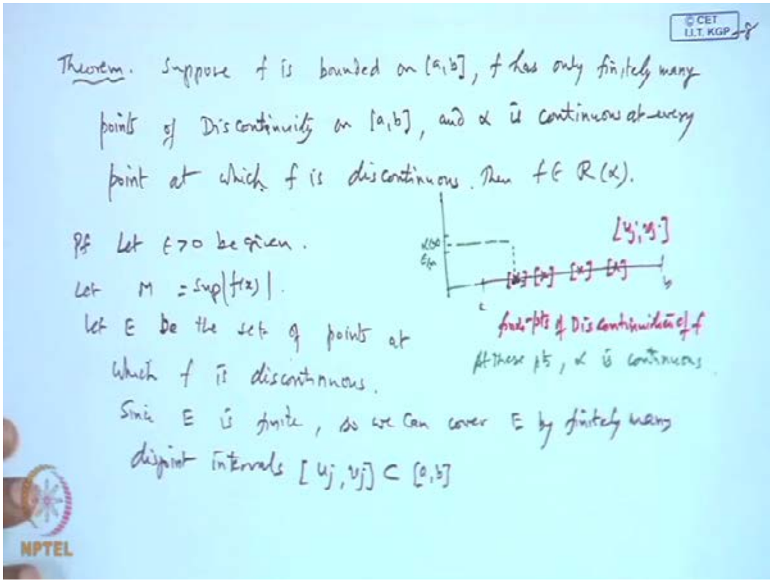

So, every monotonic functions now we come to the f is bounded function. Suppose, f is bounded on a closed interval a b and f has only finitely many finitely many points of discontinuities discontinuities on the interval on the interval a b and alpha is continuous at every point continuous at every point at every point at which f is f is discontinuous f is discontinuous then f is a will be an element of R alpha, that is f will be a Riemann Stieltjes. What it shows is a b interval is given, the function is given to be bounded on this, but as a point of say these are the points. This is the point of discontinuities, these are the points of discontinuities of f, but infinite number, these are finite points of discontinuities, not infinite number, it is a finite number of points are there which are point of discontinuities. And what is given is at this point alpha is a (( )) at these points alpha is continuous, that is very important part. If alpha is also discontinuous at this point then this result will not hold. That we will see that proving you can easily easily see the reason if both alpha and f has the same point of discontinuity over the interval a b, then the f cannot be in R alpha. This result will not work.

Only this result hold when the function is discontinuous at the point alpha must be a continuous at that point. So, this is the very very important point here to prove. Let us see the proof of this. Let epsilon greater than 0 be given given. Now, since f is bounded. So, let us see the supremum value of f x is suppose n, this is the supremum value. Now, let E be the set of, E be the set of points at which at which f is discontinuous f is discontinuous. Now, since E is finite because it is given that function f is has a finitely many point of discontinuity. So, since set E is finite. So, we can cover E by So, we can cover E by means of a finitely many disjoint intervals by finitely many disjoint intervals intervals say u j v j finitely many disjoint intervals which is a subset of a b. Because these are what happen these are the point of discontinuities.

So, only scattered point is a isolated points, we can cover it by means of these intervals like this. Say, u 1 u j v j, this is the interval covering. Now, since function f is given to be continuous at this point. So, it means by definition of the continuity, if I look the definition of the continuity, what happens? Say, this is the point. So, at this point the function is continuous. It means for a given epsilon greater than 0, if I take say alpha is this point then at this point alpha and for any number say epsilon by n. Suppose, these are n points are there. So, I can choose the epsilon by n say interval length. So, that all the points, image of any point inside this interval will fall in this.

So, correspondingly this we can identify a length delta neighborhood such that the image will fall. So, some of these values alpha x minus this alpha values at these interval can be made less than epsilon because of the continuity. So, that is the advantage of alpha taking to be as a continuous at the point where the function is a discontinuity. So, that is what.

So, let us take the difference of this. Now, such that the sum of the corresponding difference is less.

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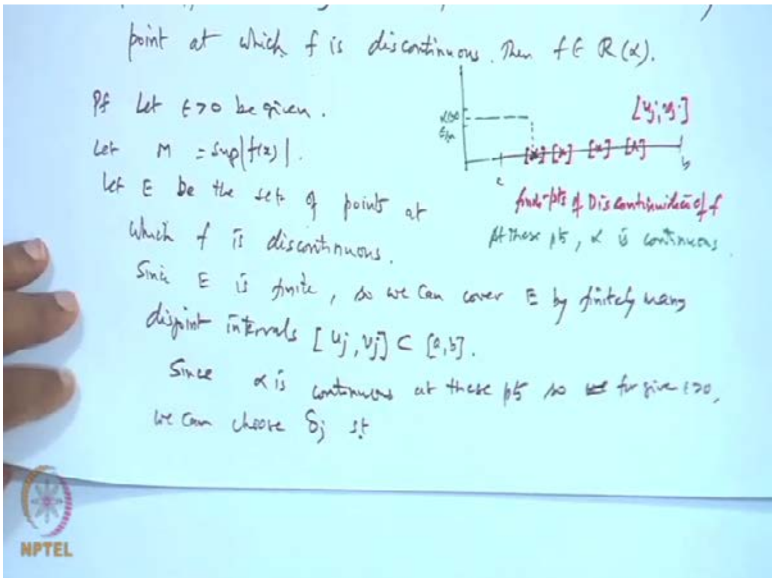

Now, since alpha is continuous at these points, at these point. So, we can. So, for a given epsilon for given epsilon greater than 0 we can choose delta j such that such that the sum of this, such that the corresponding difference is less than epsilon.

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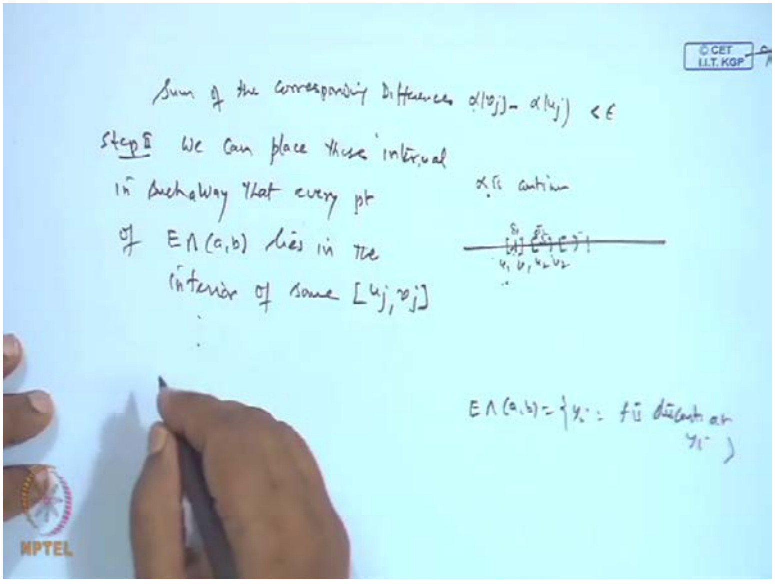

Such that the sum of the difference corresponding difference corresponding difference alpha v j minus alpha u j is less than epsilon. See why? I will again repeat suppose this is the, these are the points. So, let this point is covered by this interval u 1 v 1, here we are taking say u 2 v 2, here this is the point, then u 3 v 3 and like this. Alpha is continuous alpha is continuous at these point.

So, we can for a given epsilon greater than 0 we can identify here delta 1 delta 2 and so on. Such that image of this, for this epsilon, image of any point will fall within the epsilon neighborhood of this. So, what I am choosing is the total sum of this difference is less than epsilon. It means if the number is n then each one we can take epsilon by n. So, the total multiply by n will give this. So, that is what is getting. Now, we can replace these intervals in such a way.

Now, second step what we do is this is the first step, second step what we do we can replace, we can place these intervals in such a way in such a way that every point every point in such a way that every point of E intersection a b lies in the interior of interior of some interval u j v j. What is the meaning of this is E intersection a b, this is the set of those points say y i such that f is discontinuous at y i.

So, now, you are taking this, these points are enclosed by u 1 v 1 in such a way that each point of this lies in the middle of this, in the interior of this, you can take up u 1 v 1 in such a way that the first point lies in y 1 in this, u 2 v 2 so that the second lies in it and like this. So, this is the way we will construct and then proof will (( )). So, next time we will continue this proof.

Thank you.