A Basic Course in Real Analysis
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Lecture - 4
Continuum and Exercises (Contd.)

So, last time we were discussing about some example showing that a square root of a rational number positive rational number which is not a perfective square is a irrational number; similarly square root of a integer positive integer, it is also not a perfective square, it will be a irrational number. And in fact we have seen the first part is ok, second part we were just checking, and we come to that.

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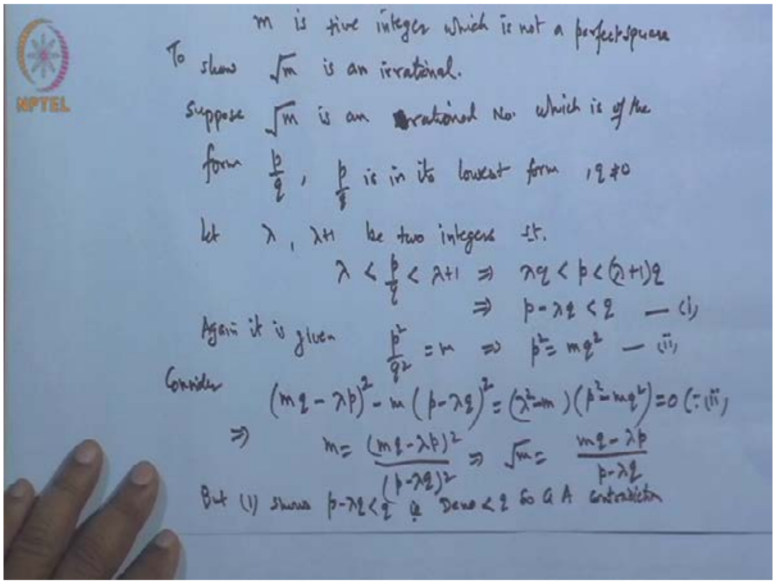

Suppose m is a positive integer which is not a perfective square, is it not? That is square root of m then we will show a square root of m is an irrational number, is it not, that we were discussing. So, we have proved this by contradiction. Suppose it is an irrational number suppose a square root of m is an irrational number is a rational number, sorry, is a rational number which is of the form, say p by q, where p and q where p by q is in its lowest form and q is not equal to 0. Then no common factor between p and q, and it is in the form, okay. So, what we get it and this number p by q is a rational number. So, we can identify the two integer in between this number will lie. So, let lambda, lambda plus 1 be the two integers such that lambda less than say p by q less than lambda plus 1. That is what we get is lambda q is less than p less than lambda plus 1 into q.

Now from here we can say that p minus lambda q is strictly less than q let it be 1, because this will be required; that is what we want, this will be required, okay. Again it is given that root m we are assuming it is a rational number which is the form p by q. So, p square by q square is equal to m; that is p square minus q square equal to mq square, p square equal to mq square or p square minus mq square will be see. So, this is also second. Now let us consider the expression mq minus lambda p whole square, consider this, minus m times p minus lambda q, oaky; let us consider this. So, basically what we are doing that this expression if I open it then it can be put it in the form of lambda square minus m into p square minus mq square; one can easily just verify it.

Now p square is equal to mq square because of the two so basically this will come out to the 0 because of two, okay. Therefore from here m can be written h, m will be written h; oh this is square of this sorry this is square of; otherwise this problem will be, mq minus lambda this is square will come. So, m will come out to be what? mq minus lambda p divided by p minus lambda q whole square or this square this square. So, root of m will be this; so, root of m will be of the form mq minus lambda p over p minus lambda q, just taking positive well roots okay. Now p minus lambda q is less than q from one. So, from one what it shows that this shows but from one shows that m square root of m that is m can be written h in the form of the, yes in another form is square root of m is another rest form where the numerator is this and denominator is this with denominator is lower than the denominator which we have assumed q.

But what we assume p by q is in its lowest form, so a contradiction exists, because m under root m is written in the form of p by q which in lowest form, but what we are getting under root m can also be expressed into this form where the denominator is lower than the q less than it. It means again the denominator is lower. So, it cannot be lowest form; that p by q cannot be lowest form. So, it is contradiction. So, it shows that p minus lambda q is less than q that is denominator is strictly less than q. So, a contradiction that p by q is in lowest form, and this contradiction is because our wrong assumption it is a rational one; therefore, root m is an irrational number, okay. So, that is what we get it, clear. So, this much we get.

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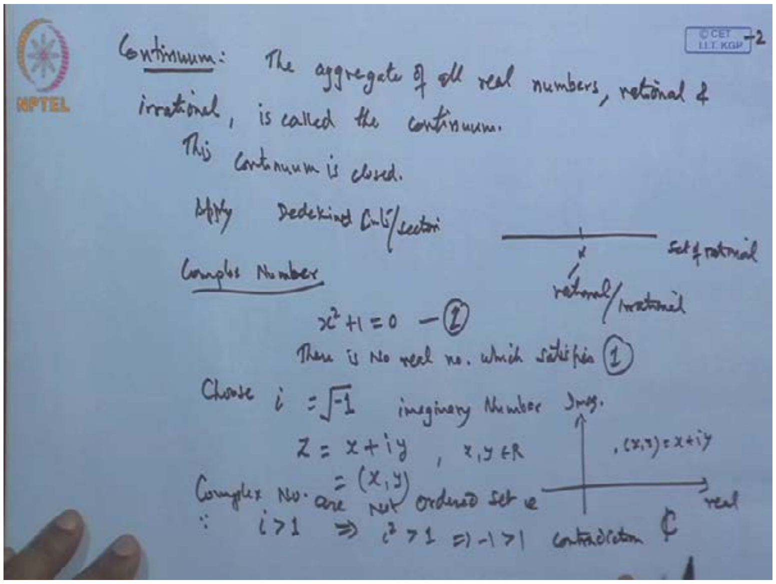

Now let us come to what is the continuum. The aggregate of the all real numbers that is rational and irrational is called a continuum. So, basically the set of all real numbers aggregate of the all real numbers rational irrational we call it a continuum. Now this continuum is complete; this is a complete set. This continuum is closed that is completeness is there; what do you mean by this? We have got this real discontinuum by applying the Dedekind cuts. So, when we apply the Dedekind cuts over the set of rational numbers rationales apply the Dedekind cuts or sections then this collection of the rational numbers in fact it is equivalent to the rational numbers when you apply the Dedekind cuts we are basically bringing this thing into form of the sections. So, each section will correspond to some number alpha.

Now this alpha may a rational number or may be an irrational number. When we apply the Dedekind cut or Dedekind theory over the set of rational number then it basically gives you the sections and collection of all these sections gives the real number. That is the continuum, because the sections which you are getting correspond to a number which may be real which may be rational or may be irrational. So, over the rational number we are applying the dedkinds cut but we are getting a class which is bigger than the rational numbers, because the sections which you are getting is larger than this elements itself that is the number of rational points, because the rational point will also be edit there. So, this gives you the total aggregate as a continuum.

Now the question is if suppose I apply again dedkind cut or dedkind sake method theory over the set of real numbers; that is we again find the sections occurs by using the dedkinds method over the set of real number, whether will you get a set of a numbers bigger than this set of real numbers. The question is, the answer is no. By this method we cannot enhance further the set of real number to a bigger class. We will get basically the set of real number itself. So that is why we call it this continuumis closed, but then somebody says why we are having the set of complex numbers also which is an extension of the real number. But when you talk about the complex number this basically is generated or obtained by some other trick, but the trick is that we are choosing the square root of negative quantity, because if you look the complex number, how this was introduced number.

The complex number is introduced basically when we look the solution of the equation x square plus 1 is 0; is it not. There is no real number which satisfies this equation, say, one which satisfies this equation, but this is also an equation. This may be some physical phenomenon may be governed by this equation. So, the question is whether can we further extend the system of real numbers not because of the cut because cut is not helping us; Dedekind cut is not helping us to extend it, but from other method and that method comes out to be searching the square root of minus one as an imaginary quantity i. We choose square root of minus one as an imaginary number, and then once it is imaginary number then we develop the set of points where it is of the form x plus iy where x and y both are real, and basically this will be represented in the form of the ordered pair x, y here x and y.

If x takes the position of real x axis that we call it as real and this is we called as an imaginary axis. So, the position of this point x, y basically is the same as x plus i y in a complex c in complexity. Now this leads to a extensive complex number system of the set of all complex numbers which is an extension of real, because once you but 0 then basically it gives you entire real line. So, it is an extension no doubt, but the way you have extended the real number is entirely different the Dedekind's method, clear. So, this is separate, and by this method when you are extending the real to complex this collection of the complex number does not behave as a smooth as a work set of real numbers, because between two real numbers one can easily ordered one can say this two given number either they are equal or one is greater than the other.

But in case of the complex no ordering can be defined; in fact if suppose the complex numbers are not ordered set. It means that is there no order can be derived. If z 1 z 2 are two complex numbers you cannot say z 1 is greater than z 2 or z 1 is less than z 2 except when equal to of course when you say z 1 is equal to z 2 we assume the real part is equal to real imaginary part is equal to imaginary. But otherwise greater than less than not be in fact we did a contradiction, because suppose there is an ordering defined. Suppose i is greater than 1 so i is complex number, one is also complex number, because one can written as one plus i 0, then what happens? One is positive so i is greater than one means you are assuming to be a positive quantity. So, let us multiply by i again. So, what you get? i square is greater than one again but i square is minus one is greater than one, so a contradiction.

Similarly if we assume less than we also lead to a contradiction, okay. So, this shows that ordering ration cannot defined over a set of complex numbers. So it is a different system differently steam where we discuss the complex number, but this is a very important area where all entire functions and all these things Cauchy various integrals real integrals can be completed with the help of complex integration. So, it is very important field where the people can use it in the application part. So, that is what?

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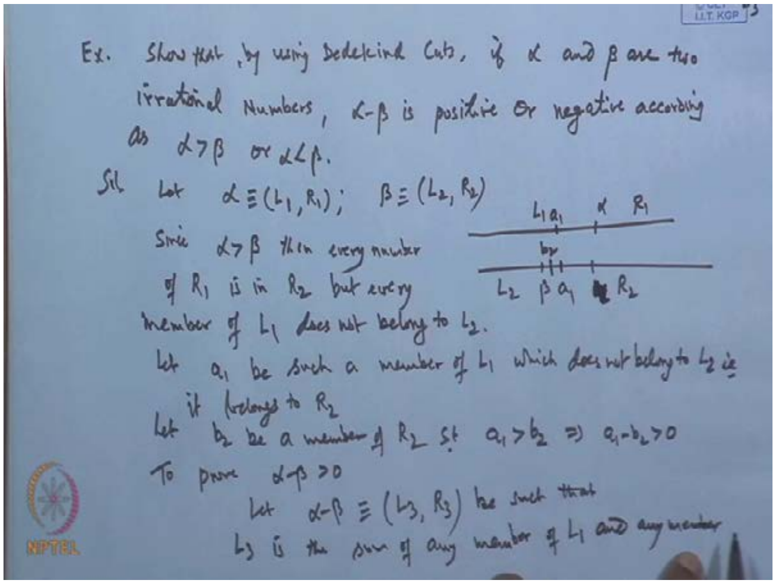

Now let us see a few exercises which I wanted to give you. Show that by using Dedekind cuts that if alpha and beta are two irrational numbers then alpha minus beta is positive or negative according age alpha is greater than beta or alpha is less than beta. So, using the Dedekind cuts we wanted to prove this part, okay. So, let us see suppose we have solution. Now alpha and beta are given as two irrational numbers; so obviously it can be represented by means of cuts. So, let alpha is represented by a cut L 1, R 1 while the beta is represented by a cut L 2, R 2, okay; now alpha is less than beta, so here we are having, so this is alpha is greater than beta. So, alpha is here; beta is somewhere here.

This is L 1, this is R 1, here this is L 2, here is R 2. Now when alpha is greater than beta it means every elements alpha is greater than every number or every element every number of R 1 or belongs to R 1 is in R 2. All the elements of the R 1 will be in R 2 but every member of L 1 does not belong to L 2, is it not; that is what is seen. So, every member of R 1 is in R 2, but every member of L 2 L 1 is not there. So, there may be some member which is in R 2. So, let a 1 be such a member of L 1 which does not belong to L 2. So, if it does not belongs to L 2 that is it belongs to R 2. So, here is some, say, a 1 this is number a 1 which is in L 1 but not in L 2, so it must be in R 2. Let us choose a number b 2.

Now let B 2 be a member of R 2 such that a 1 is greater than B 2. Suppose B 2 is I am choosing this number, okay. So, a 1 minus b 2 is positive b 2 be a number of R 2 such that a 1 is greater than B 2 no, a 1 oh sorry then yes. We should write like this b 2 here, is it not. So, let it a 1; this is b 2, okay. So, a 1 minus b 2 is greater than 0, is this clear. Now a 1 this alpha minus beta we wanted to prove this, what is to prove alpha minus beta is positive. So, when it is a positive number, what do you mean by that? A cut if I apply the definition of Dedekind cuts then a number is set to be a positive when its lower class contain some positive numbers, is it not. Then only it is considered to be positive, okay. So, lower class will be some L 3 alpha minus beta represented by a class L 3, R 3.

So, let alpha minus beta is represented by a section, say, L 3 R 3; if I prove L 3 contains some positive quantity positive number also then the alpha minus beta becomes greater than 0, okay. Now this L 3 what is this alpha minus L 3 is the sum of so it is alpha minus is given by the L 3. Now L 3 is the sum of what alpha minus beta; alpha is in where? Alpha this is alpha, is it not, and beta is here. Now, this a 1 minus b 2 is positive. So, L 3 we can say like this that L 3 is the sum of any member of L 1 and any member of minus L 2, and any member of let it not be such, let us write this thing be such that L 3 is the sum of any member of L 1.

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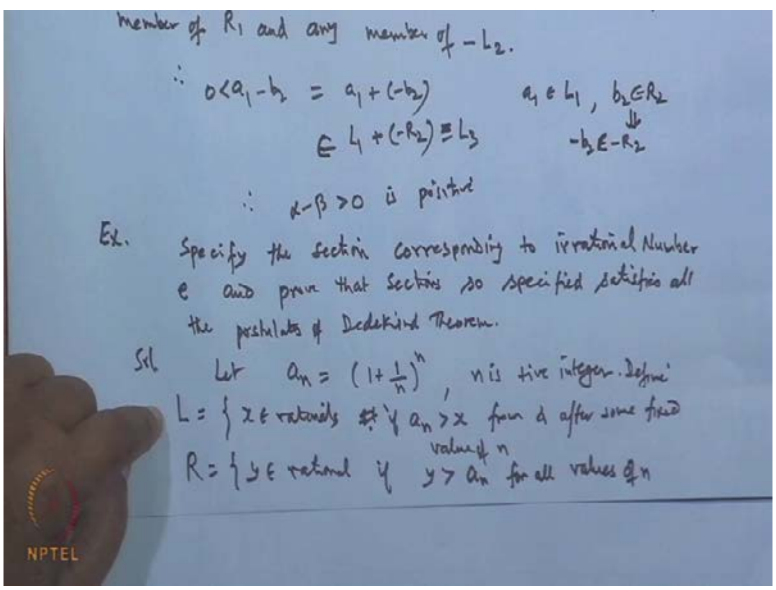

And any member of minus R 2 and any member of R 3 is the sum of any member of R 1 and any member of minus L 2. Let us see what is the meaning of this? We wanted to show the alpha minus beta to be positive. So, let us suppose alpha minus beta represents a cut by L 3 R 3, okay, and we are assuming that this cut is such that L 3 any member of L 3 is the sum of any member of L 1 and any member of minus R 2. So, if you take any element of L 3 it can be written as the sum of the elements of L 1 and minus R 2 member of minus R 2. Similarly if you take a R 3 then any member of R 3 we are assuming as a sum of the member of R 1 and any member of minus L 2, is it okay or not.

So therefore a 1 minus b 2, what is this; is it not the same as a 1 plus minus b 2. Now a 1 belongs to what? a 1 belongs to L 1; b 1 b 2 is in R 2. So, minus b 2, b 2 is in R 2. So, minus B 2 will be in minus R 2, is it not. Therefore this is the sum of L 1, this belongs to L 1 plus minus R 2, and that is nothing but L 3, and this a 1 minus b 2 is greater than 0 positive. So, a section L 3 R 3 which is assumed to represent the number alpha minus beta is such that it contains the positive numbers. So, alpha minus beta therefore this alpha minus beta is a positive cut is positive, clear. So, that is what is written, is it okay or not.

Now let us take another exercise, okay. Specify the sections corresponding to the irrational number e and prove that the sections so specified satisfies all the requirements satisfies Dedekind’s though or satisfy all the postulates of Dedekind’s theorem, because there are three postulates in Dedekind’s theorem that we can divide the whole system into real number into two classes lower and upper class, and each class is nonempty. Second one is that at least lower class every element in the lower class is the less than the elements of the upper class, and third is any irrational number will belong to any number either this class rational number or that class like this, is it not. So, three postulates are there, so we can show. So, what it says is that corresponding to e you first defines the sections, and then show that these sections satisfy the condition, okay.

Let us see this, what is e? If you remember e is the limiting value of one plus one by n to the power n. This is the e, is it not, but here we cannot take the limit because we wanted in terms of the sections. So, let us say let a n is 1 plus 1 by n to the power n here n is a positive integer. Now let us consider and define the class is as follows, define the lower class L as the set of those rational number x such that a n is greater than x from and after sometime from and after some fixed value of n; that is those rational number belongs to L for which a n is greater than x for after some number n naught for some value of n fixed value, say, after n equal to n naught, n is greater than x then we say x will be in L, and row upper class R, I am defining the this with upper class R, okay, the rational number by is an upper class R if y is or such that write if a n is greater than, okay, if y is greater than a n is strictly greater than a n for all values of n.

So, lower class I am defining the element x rational number belongs to the lower class if a n this number a n is greater than x after a certain stage, say, a n is n naught and R is the those rational number if y is greater than a n for each. We claim that this will be this section the way in which we have defined will satisfy all the conditions of Dedekind theorem. So, what this first condition is both the class L and R must be nonempty. So, there must be at least one element should be available in R from a n as well as some element of a an must be in R, okay. Now if we look what is n? n is a positive integer. Now if I take n any positive integer this sum will always be greater than 1; when n is greater than 1 this number will always be greater than 1, is it not.

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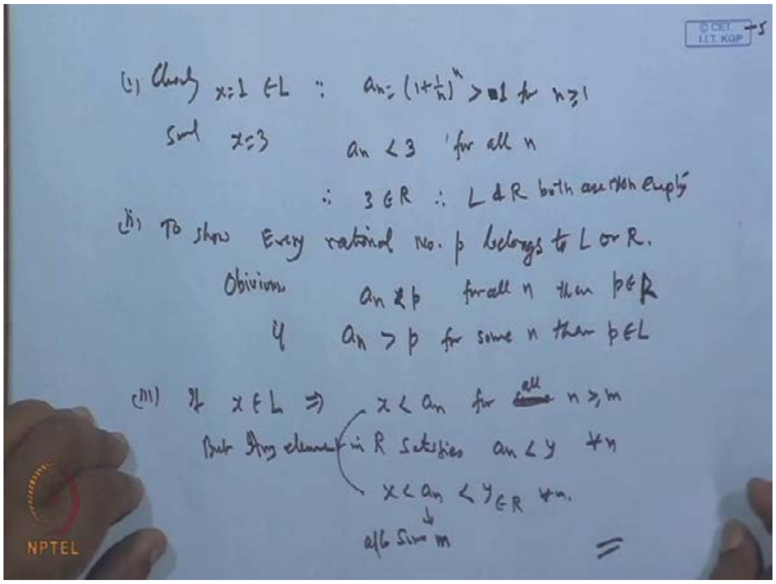

So, this one belongs to the lower class clearly number one belongs to lower class because a n which is 1 plus 1 by n power n will be greater than 1 for n greater than 1 or it may be equal to 1; n is equal to 1 also. You just check greater than sorry greater than 1; this is a n is greater than x no, so this is greater than 1, sorry. So, x is basically 1. This number will always be greater than 1 when n is greater than equal to 1. So, this therefore L is nonempty, okay. Similarly, x equal to 3 if I take then all the elements of a ns are less than 3 for all n 1 plus 1 by n to the power n whatever the n you choose it will remain less than 3. So, 3 is an element belongs to; therefore, 3 belongs to R. So, L and R both are nonempty. This is the first postulate results that be shown in the Dedekind. Second one condition which is in the Dedekind shows that every rational number p will be belongs to either L or R.

So, second one every rational number belongs to L and R. Second is to show every rational number p belongs to L or R; this we want to show, okay, which is obviously two which is obvious, why? Suppose a n is greater than p for all n, p is a number, p is a number I am choosing. So, there are two possibility either n will be greater than for all p or may be a n is less than p for some after certain stage, okay. So, an greater than p for all n then this p belongs to the upper class, is it not, upper class, and if a n is less than if a n is greater than, no, no, this is if a n is less than, sorry, what was the upper case. Yes, a n is less than y, y is greater than a n for all a n. So, if a n is less than p for all n then it is in upper class, and if a n a is greater than p for some n then this p belongs to the lower class.

So take any p. Suppose I take p any number say 2 then you can say 1 plus 1 by n to the power n then you can choose n such a way that this class after a certain stage will satisfy this condition a n is greater than because limiting value of this is 3, e is greater lying between this, is it not. Limiting value of this will be this; what is this? If I expand it 1 plus 1 by 2, this is the number actually, 1 plus n plus n minus 1 and so on. So, when you take the 2 plus something. So, if you take a number if all the numbers are less than 3, say, then this number, say, 4 then all the numbers are less than 4. If I take a number 2 then there are some number where it is greater than. So, we can identify all the numbers which is either in lower class or in the upper class. So, this is also true.

And third part is what? Every number a in the lower class is less than than the lower class. So, if x belongs to the lower class it means that x is less than a n for some n greater than equal to, say, m onward, is it not, for all n greater than it. After certain stage it is less than this, but if x will be in y but any element in the upper class R satisfy this condition; condition is that a n is less than y for each n. So, from here x is less than a n for some n but this is less than y if this is an element in R. This is true for every n; this is true for some n after some m after some n onward, okay. Then this will always be less than y. So, any element in lower class will always be less than the elements in the upper class, okay. So, this shows the property. So, all the conditions are satisfied Dedekind's, okay; all requirements of the Dedekind's theorem is satisfied, so this… Let us see the next problem.

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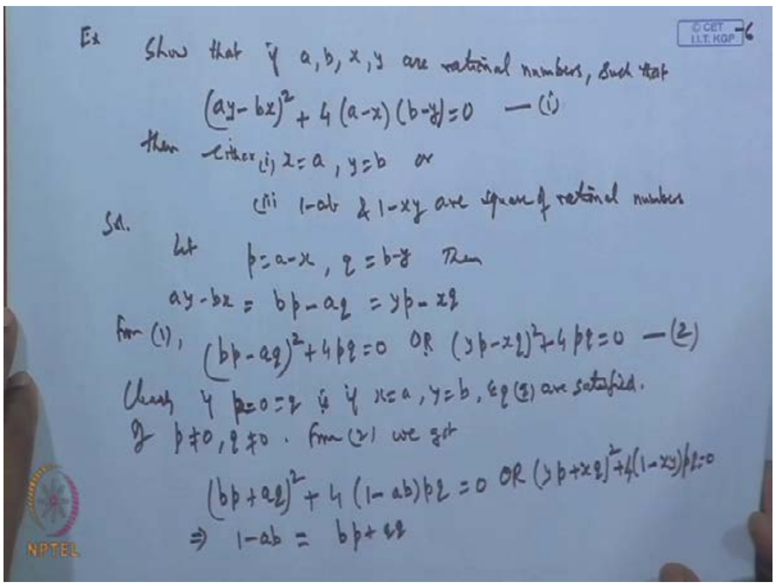

Suppose we have show that if a, b, x, y are rational numbers such that ay minus bx whole square plus 4 a minus x b minus y is 0, then such that either x equal to a, y is equal to b or then either this part or 1 minus ab and 1 minus xy are squares of rational numbers, okay. It is basically mathematics things, not much that Dedekind's cuts is required but somewhere is required the concepts, okay. So, let p is a minus x, q is b minus y, then if I solve this ay minus bx then we get. If I just substitute these values we will get the value to bp minus aq which is also the same as yp minus xq; this is what, okay. Just I take this p and q then. So, our given relation one so from given relation we can write ay minus a means bp minus aq whole square plus 4; pq is 0, or we can also write this thing as, because this is also equivalent to this. So, we can write yp minus xq whole square plus 4 pq is 0, okay.

Now clearly if p is equal to 0, q is also 0; that is if x is equal to a, y is equal to b, then p 0 q 0 the equations these two are satisfied, is it not. Obviously true, that is nothing. Now if they are not 0 if p is not 0 q is also not 0 then one can divide by 4 pq and one cam divide the equation two. So, from equation two we get the rational, okay; we get from here now we can write like this. This is now here, so we wanted to write bp plus aq whole square. If I just made the plus sign then two times of this will come in picture. So, here this is when you take the minus b and whole square so what you are getting is; you are getting this like equal to plus 4 1 minus ab, pq equal to 0, okay. Then only it is balanced.

Similarly if I take here yp plus xq whole square then you are getting 1 minus x by four times of this into pq is 0. This is from two. Now divide by pq. So, from here we get 1 minus ab is equal to bp plus aq whole square divide by minus 4 pq, but minus 4 pq from here is nothing but which is equivalent to bp plus aq divide by minus 4 p we can write as bp minus aq whole square. Similarly 1 minus xy we can also write in a similar way, and this will be equal to if I put it in this form we can yp plus xq divided by whole square divided by yp minus xq whole square. So, what we conclude that if first is satisfied then also this equation is satisfied; otherwise, second 1 minus ab can be literally square of the two rational numbers you see 1 minus ab is square of the rational number, 1 minus xy is also square of the rational number and that proves the result, okay. Then last exercise let us see.

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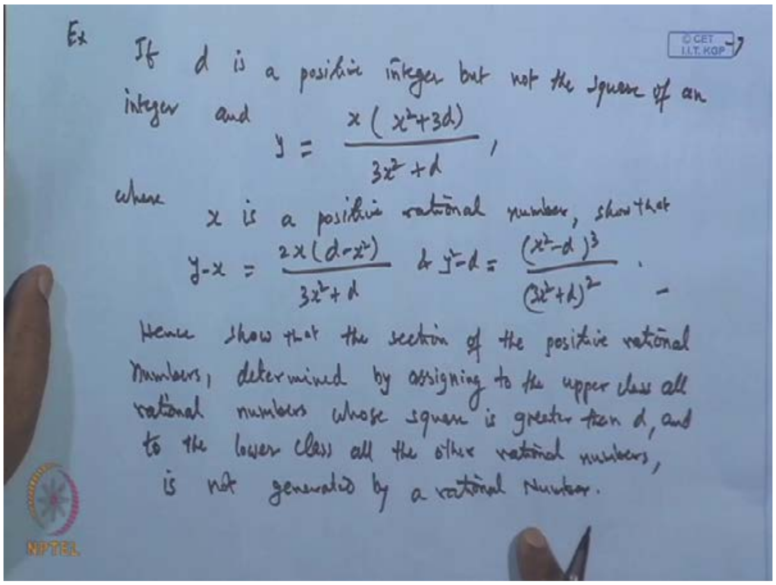

This is also interesting one. If d is a positive integer but not in the square of an, when d is a positive integer not the square of an integer, it means square root of d becomes that it is not a square of this integer. So, what we can say? It is not a perfect square, okay, square of integer, and y is written in the form of suppose y is x into x square plus 3d divide by 3 x square plus d where x is given x is a positive rational number, then show that y minus x equal to 2 x d minus x square divided by 3 x square plus d, and y square minus d is nothing but x square minus d whole cube divided by 3 x square plus d whole square.

Hence show that the section of the positive rational number determined by assigning to the upper class all rational numbers whose square is greater than d and to the lower class all the other rational numbers, so that the section of the positive rational number determined by assigning to the upper class all rational numbers whose square is greater than d and to the lower class all other rational number, is not generated by a rational number, okay.

Let us see, first let us see the first part of this I am not interested in this, because this is a routine thing. You are having the value y is given, x is already given, so y minus x you will find this expression, y square minus d you will find the expression just simple calculation. So, we are not interested in solving this, but this required by showing that this section is not a rational number, okay.

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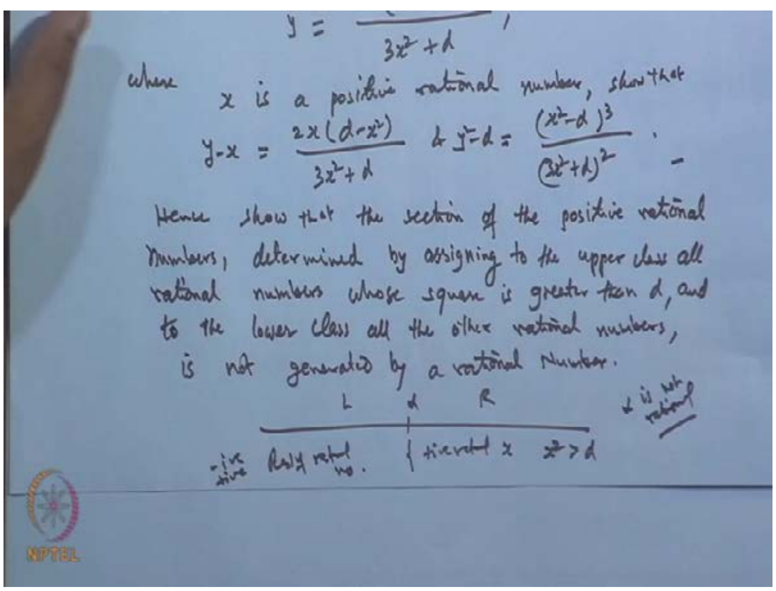

So what is the number of this? meaning is that this is our lower class, this is the upper class, we are taking those positive rational numbers which are whose rational number says x such that x square is greater than d and lower class rest of the rational numbers. So, this will include all negative and the remaining positive also some positive, but here exclusively positive is x 2 is greater than d. Now, this number say alpha we wanted to show alpha is not rational; this is all, okay.

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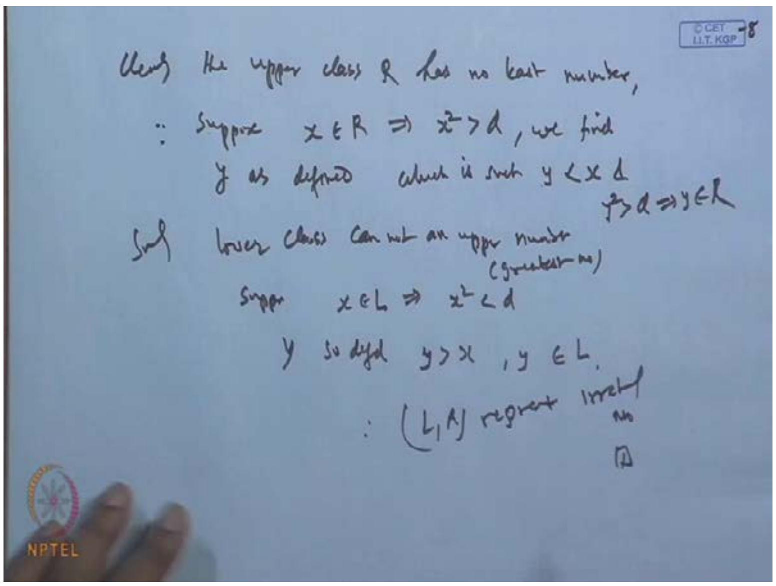

So, let see the proof is very simple, just go ahead, okay. Clearly the upper class R has no least number because if suppose x belongs to this class, suppose x belongs to the upper class R. So, by definition so this implies that x square must be greater than d; when x square is greater than d and the function is defined like this so x square is greater than d then y square is greater than d and here x square so y minus x is negative, okay. So, we get from here is this one defined so that then we find y as defined which is such that y is strictly less than x and y square is greater than d. So, if I take any number x in the upper class then what we are getting another number y which is lower than the x but still stays in the upper class. This implies that y is in the upper class R. So, upper class cannot have a least number.

Similarly lower class cannot have an upper number largest number we cannot have the lower class as greatest number, why? Because the reason is for if suppose x belongs to the class L then it x belongs to the class L then x square must be less than d; otherwise x will be in upper class, okay. So, x square less than d then the y so define will give what? If x square is less than d then this is positive. So, y minus x square is positive, so y is greater than x. So, again the y belongs to lower class. So, if x is in the lower class then we are also getting another number y which also in lower class but it is greater than L. So, L cannot have a largest number. If x square is less than d then this is positive. So, this is positive means y minus x is greater than 0. So, y must be greater than x. It means by another class another point in lower class which in this shows a contradiction. Therefore section L R represents irrational number; that is all.

Thank you. Clear?

Thanks all.