A Basic Course in Real Analysis
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Lecture - 39
Reimann/ Reimann Stieltjes Integral

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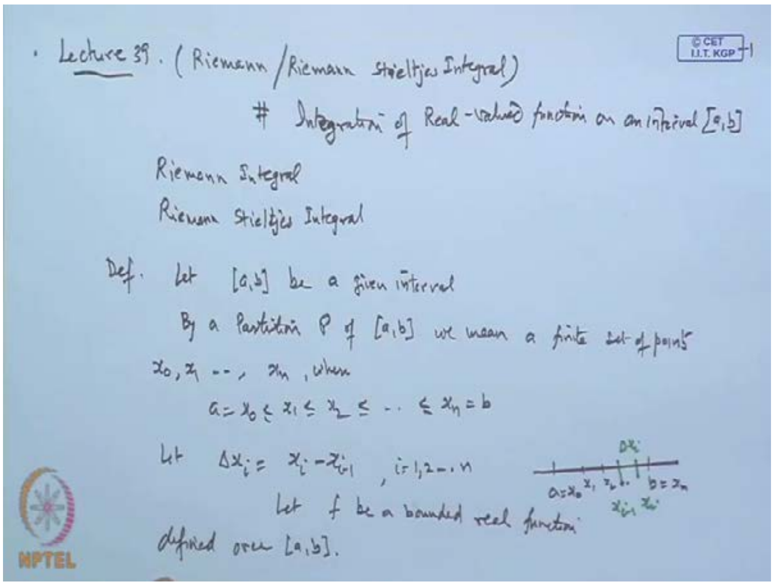

So, today we will discuss the integration of the real valued functions on a interval. Basically, we will do the integration integration of real valued functions on an interval say a, b. So, we will discuss today this part. Now, in this integral of the real valued function we have the two types of integral, we will discuss. One is the Reimann Integral another one is the Reimann Integral, and then Reimann Stieltjes Integral. In fact, the Reimann Stieltjes Integral is the generalisation of the Reimann Integral and as a particular case we can say this.

Now, this is all, is a definite integral you know. So, it is an extension part of the definite integral when we go for the Reimann Integration. So, let us see before going the Reimann Integral, let us see first the definition, how to define the Reimann Integral. Suppose, a, b be an interval, let a, b be a given interval interval. By a partition of a, b by a partition P of a, b we mean we mean a finite set of point, set of points say x naught, x 1, x 2, x n where a is say x naught which is less than or equal to x 1, less than equal to x 2, less than equal to x n and less than equal to x n which is say b. So, basically this is our interval a, b. What we are doing? We are partitioning this interval into a sub interval by choosing the point x naught, x 1, x 2, and x n in between a, b; where the x naught is the initial point coinciding with a, x n is the terminal point, last point coinciding with b. This x 1, x 2, x n are the distinct point and may be sometimes it may be overlapping, that is we can start with x naught x 1 then go for this x 2, start with x 1 like that way which also possible for that one.

So, let this set of collection, which final set of these points over the interval a b which satisfy this condition is called the partition of the interval a b. So, let delta x stands for delta x i stands for x i minus x i minus 1 where i vary 1 to n. Suppose, we have this point say here we have x i minus 1 and this is say x i. So, this interval we are denoting as delta x i. x i minus x i minus 1 is delta. Now, let us suppose f be a bounded function, let f be a bounded real function real function bounded real function defined over the interval a b over the interval a b. Now, since function f we are choosing to be a bounded function and we have divided the interval into a sub intervals like delta x 1 delta x 2 delta x n, it is the length of this subintervals and each subintervals x i minus each (( )). So, we can choose for each partition.

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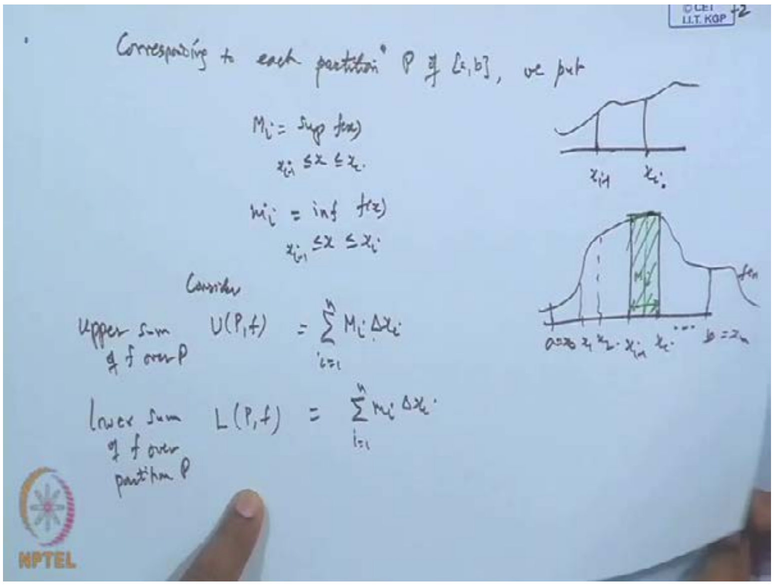

So, corresponding to each partition P of a b we take or we take or we assume or we put M i as the supremum value of the function f x when x varies from x i minus 1 to x i. If this is our interval x i minus 1, this is x i, function f is a bounded function, need not be continuous, but it is a bounded function. So, once it is a bounded function over the interval, closed interval x i minus 1 to x i then obviously, it will attains its supremum and minimum value and infimum value over this interval because it is bounded function. So, supremum will exist, infimum will also exist. So, let the infimum is denoted by small m m i, this is the infimum value of the function f x over this interval x i minus 1 to x i.

Now, let us take the sum, consider the sum M i delta x i and i is 1 to n, means over this interval, this is our interval a b. We have partitioned this thing as x naught x 1 x 2 x i minus 1 x i and so on, this is x n and here is something like this function f x. So, over this interval, over this interval function have attains a supremum value say at this point and infimum value is suppose this point. So, we multiply the supremum value of the function that is, this is our capital M i by the length of this interval. So, when you multiply this by the length of the interval you are taking basically this rectangle, area of this rectangle, is it not. So, this we are doing for each sub intervals, over each sub intervals we are calculating this and taking the sum.

This sum we denoted by as you know by U P f because this sum depends on P as well as the function f because supremum is taken of, for the, of the function f over this subinterval. So, over each interval M i may change depending on the function as well as the partition, delta x i depends on the length of the partition. So, if they change the partition delta x i will also change. So, this sum we call it as a upper sum, upper sum of the function f over this partition, corresponding to the partition P.

Similarly, then we write m i delta x i i is equal to 1 to n this sum we denote by L P f and is called the lower sum of the function f over the partition P over partition P. So, upper sum of f over P and lower sum of f over P. Now, if we change the partition the upper sum and lower sum will keep on changing. So, but what is this, but this upper sum and lower sum will always be a bounded function, bounded thing. Why? Since, function f is bounded over the interval a b. So, it means.

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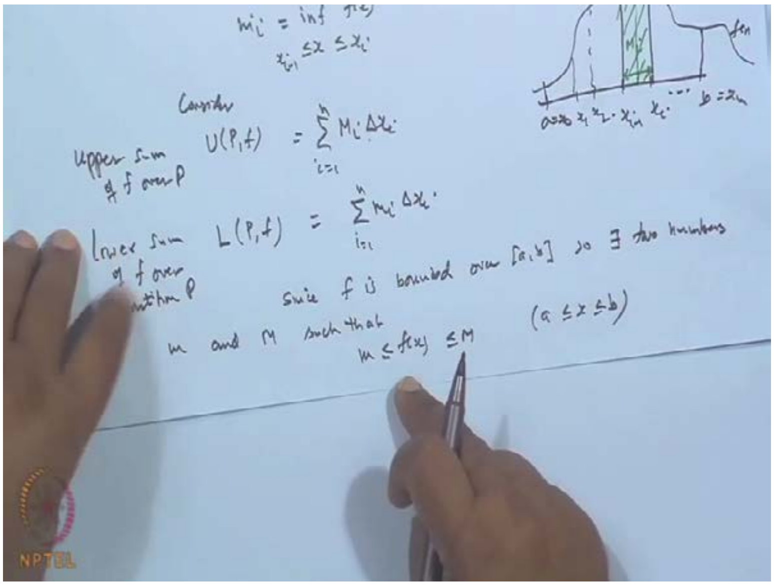

So, there exist the two numbers small m and capital M such that the value of the function f x will always fall between these two range, because f is bounded. So, bounded means it will have the least number and the largest number. So, m and capital M will lies for all x lying between x and b a and b. So, this is true, is it not?

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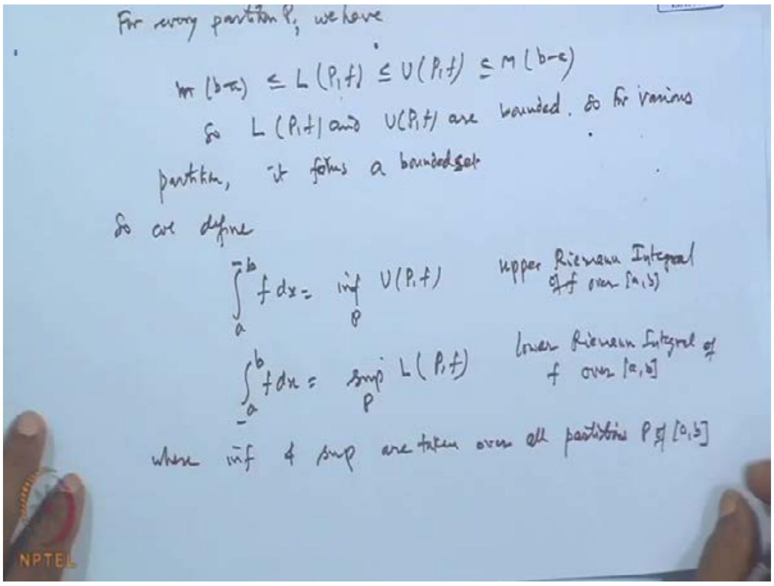

So, if we take any partition P. So, for any partition P or for every partition P we have m times b minus a will always be less than equal to lower sum of this which is less than equal to upper sum of function f over the partition P which is less than equal to m into b minus a. Why? Because this lower sum and upper sum over this interval x i minus 1 lower sum will always be less than equal to upper sum over each subinterval because m i is the infimum value capital M i is the supremum value.

So, because of this it will be less than equal to upper sum. Then take the summation over all the, such sub intervals. So, obviously, the lower sum will total lower sum will always be less than the upper sum. This is one thing. Second one is when we take the function f, f is bounded by M interval. So, this is the length of the interval. So, if we multiply this by the b minus a the total length of the interval then this will be the minimum area bounded by a curve whose lower value is m and the length over the length b minus a and this is the upper bound for the function f. So m into, so m into b minus a will lie between this two bound. Therefore, our lower sum and upper sum it is a bounded function. So, is a bounded say.

So, lower sum and upper sum are bounded are bounded or they form the bounded set are bounded. So, for the partition, for various partition, various for any various partition it forms it forms a bounded set. You keep on changing the partition, the lower and upper sum will change, but it will remain bounded between these two limits. So, it is bounded. So, once it is bounded it means we can take the infimum value and supremum value of this. So, this is upper bounded by this. So, the infimum value of this will also exist, supremum value of this will also exist because this is lower bounded by this infimum value will be at the most equal to this term and supremum value at the most equal to this.

So, what you see here that if we take from this, from this one. So, we define. So, so we define the if a to b bar upper f d x as the infimum value of the upper sum P f where infimum is taken over all the partition, infimum is taken over all such partition P. Similarly, a to b lower ward f d x is the supremum value of the lower sum P f where the supremum is taken over (( )). So, where infimum and supremum are taken over over all partitions all partitions P of a b, is it okay? And since these are bounded. So, supremeum infimum will exist. Hence our this integral will exist, this is called the upper Reimann Integral, this is known as the upper Reimann Integral and this one is called the lower Reimann Integral, this is called the lower Reimann Integral.

So, we get this lower Reimann Integral of f over the interval a b. So, lower Reimann Integral of f over a b. This upper Reimann Integral and lower Reimann Integral, thus what we say here f is Reimann. Now, if lower Reimann Integral and upper Reimann Integral coincides that is they have a same value and independent of course, it is a partition then we say the f is Reimann Intergable.

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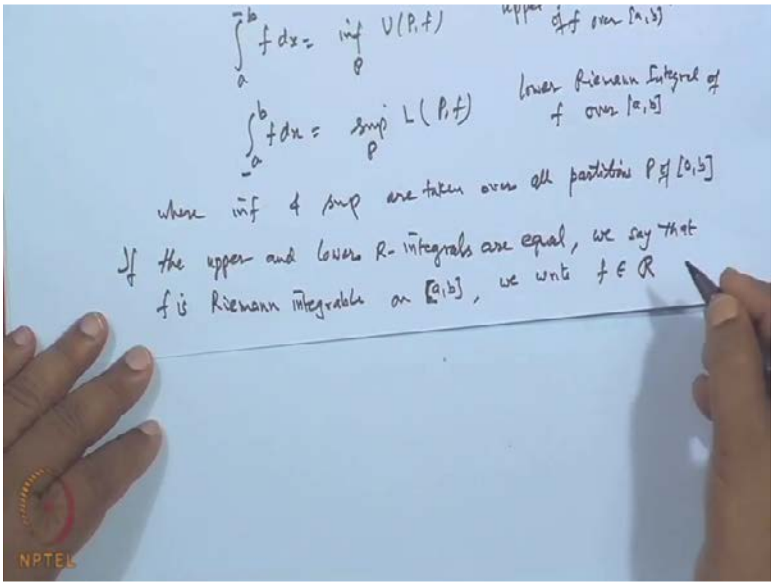

So, we say if the upper and lower, upper and lower Reimann Integral are integrals, Reimann Integrals are equal are equal then we say that f is Reimann Integrable, f is Reimann Integrable on the closed interval a b on the closed interval a b and we denote this and we write it as f belongs to R.

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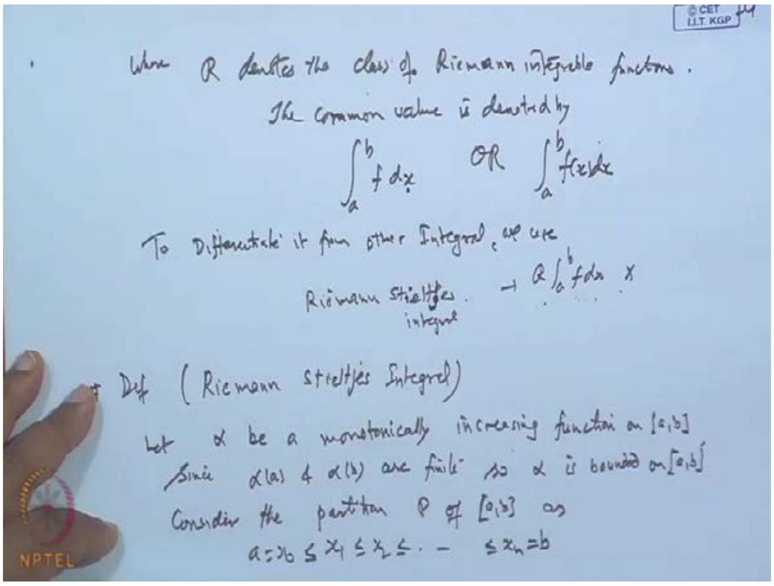

Where R denotes the class of, where R denotes the class of all or class of all Reimann Reimann Reimann Integrable functions Integrable functions Reimann Integrable functions. The common value we denote by, the common value is denoted by Integral a to b f d x or integral a to b f x d x or sometimes to differentiate between, to differentiate it from other integrals like Reimann Stieltjes Integrals we use this, say, we use the Reimann Integrable over R. So, here we will take the Reimann Integral a to b f d x as usual. Otherwise some author write like R a b also to show the Reimann Integral, but here this notation we will use to the Reimann Stieltjes Integral.

So, some author use, but here we would not write, we will take up only for the Reimann Stieltjes Integral Reimann Stieltjes Integral. This we will take up later on, what is this Reimann Stieltjes Integral, we denote this by R. So, that we will differentiate between these two. So, this is what. So, what they show that if f is a bounded function, f is a bounded function then upper sum and lower sum will definitely exist and upper integral and lower integral will exist. Now, the question of whether they are equal or not, if they are equal then we say the existence of the Reimann Integral is there, if they are not equal then we say the Reimann Integral does not exist. So, existence part we will take later on. First, let us see the other integral which is known as the Reimann Stieltjes Integral, a generalisation of our Reimann Integral and then we will study the Reimann Stieltjes Integral in detail.

So, as a particular case we can get all the results for Reimann Integral also. So, let us see the next definition for Reimann Stieltjes Integral. Now, what we do here in this is before going. So, we will take let alpha be a monotonic, be a monotonically monotonically increasing function on the interval a b on the interval a b. Now, alpha is monotonically increasing function. So, alpha a and alpha b are finite assuming that since alpha a and alpha b, these are real number. alpha a and alpha b are finite. So, we can say alpha is a bounded function. So, alpha is bounded on the interval a b because it is monotonically increasing.

So, alpha a and alpha b these are real numbers, if they are finite hence it has a finite all the values of alpha lying between alpha a and alpha b, because it is monotonically increasing function. Hence x will be a bounded function on a b. So, once it is bounded then let us consider the same partition, consider the partition P of a b as a is x naught less than x 1 less than equal to x 2 less than equal to x n which is say b. Then consider instead of this delta x i.

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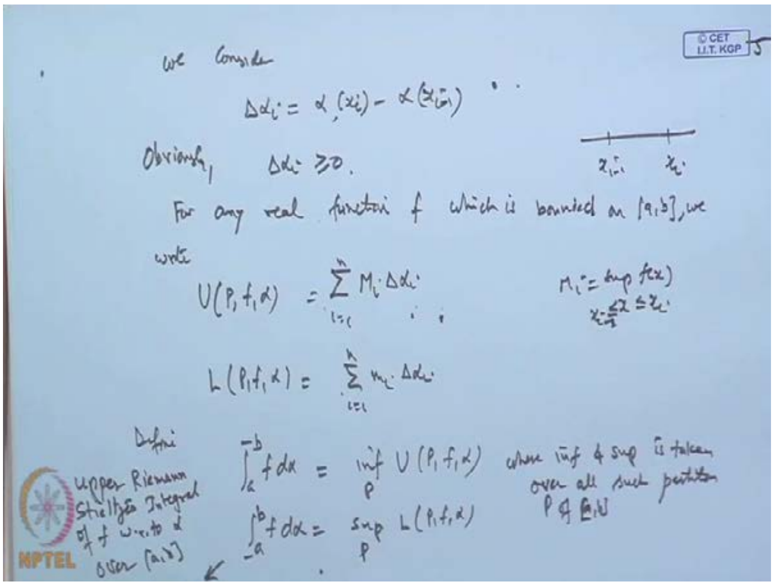

Now I consider the delta alpha i, we consider delta alpha i as the value of alpha at a point x i minus the value of the alpha at a point x i minus 1 because this is the interval x i minus 1, this is x i. So, earlier what we were doing we were taking the x i minus x i minus 1 as the delta x i. Now, alpha is taken as a monotonic function. So, I am considering the value of the alpha x i minus alpha x i and denoted by delta alpha i. So, obviously, this delta alpha i will be greater than or equal to 0 because alpha is a monotonic function, increasing function.

So, alpha x i will be greater than or equal to alpha x i minus 1. Therefore, this will be a non negative quantity. Now, for any real function for any real function which f which is bounded which is bounded on the close interval a b, on the close interval we write we write the sigma M i delta alpha i i is 1 to n as U of P f and alpha, where the M i means the supremum of f x over x lying between x i minus 1 to x i, same thing. And L P f alpha we are writing as i is 1 to n small m i delta alpha i.

Now, this is again we call it the upper sum and the lower sum of the function f with respect to the alpha, alpha it depends on alpha is it not. So, now, this upper sum and lower sum will be defined in terms of the function f that is M i any small m i and the alpha i, where alpha is this one, delta alpha i is this one. Now, in a similar way choose now. So, define the upper integral a to b f d alpha as infimum of U P f alpha where infimumis taken over all such partition p and lower sum is denoted by this supremum of L P f alpha where again supremum is taken over all partition where infimum and supremum is taken over all such partition all such partition all such partition P of a b or partition of a b supremum.

Now, if this, further because again this is bounded function. So, supremum infimum will exist. So, this will this two integral will exist, this is called the upper Reimann Stieltjes Integral of the function f of the function f with respect to alpha with respect to alpha over the interval a b and this we will called it as a lower and this we call it as a lower Reimann Integral, in a similar way we write lower Reimann Stieltjes Integral of f with respect to alpha over a b. Now, if both these values coincide then we say f is Reimann Stieltjes Integral. So, if let it be 1 on 2, is better we write it, let it be 1 2.

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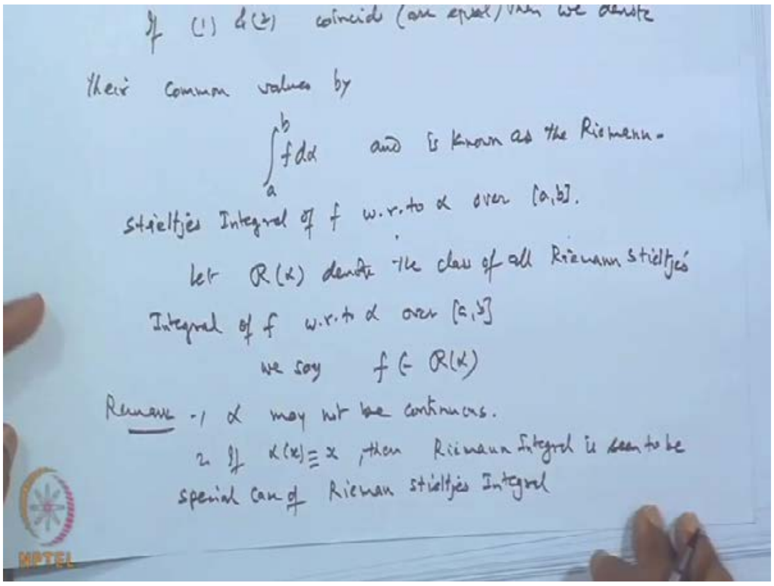

So, if 1 and 2 coincide, if 1 and 2 coincides have the same value that is are equal are equal then we denote we denote their common values their common values common value by integral a to b f d alpha and is known as is known as the Reimann Stieltjes Integral of f with respect to alpha over close interval a b, that is what is, so the class of all the class. Let R alpha denotes the class of all Reimann Stieltjes Integrals of f with respect to alpha over a b. So, in this case we say so we say that f belongs to R alpha.

There we are denoting simply by R, here we are denoting by R alpha Reimann Stieltjes Integral of the function f over. In both the case I am considering f to be a bounded function need not be a continuous. Here also note alpha need not be a continuous function remarks alpha may not be continuous function. It is simply an monotonic increasing function, still this clear. Second one is if we take alpha x equal to x then the Reimann Stieltjes Integrals converts then we say f is Integrable, then Reimann Integral is then Reimann Integral is the, is seen to be a special case of Reimann of Reimann Stieltjes Integrals. Thus simply alpha x equal to x will reduced the Reimann Stieltjes Integral will give the Reimann Stieltjes Integral and from there we can get the Reimann Integration. So, this is an extension Reimann Stieltjes Integral is an extension of our Reimann Integral.

Hence, whatever the property we will write for the Reimann Stieltjes Integral as a particular case when you choose a alpha x equal to x then you get the corresponding property of the Reimann Integral and remember we will always denote R by a Reimann Integral and Reimann Stieltjes Integral by R alpha. Second part is that earlier we have used the a to b f x d x, is it not. Where, x is the variable of integration. Now, here also one can write it f x d alpha x, but this is not a very common notation. So, normally the common notation is a to b f d alpha, but it means that a to b f x d alpha x. So, that is the meaning of this. Now, we come to now some properties of the, our partition P which we are choosing, partition P.

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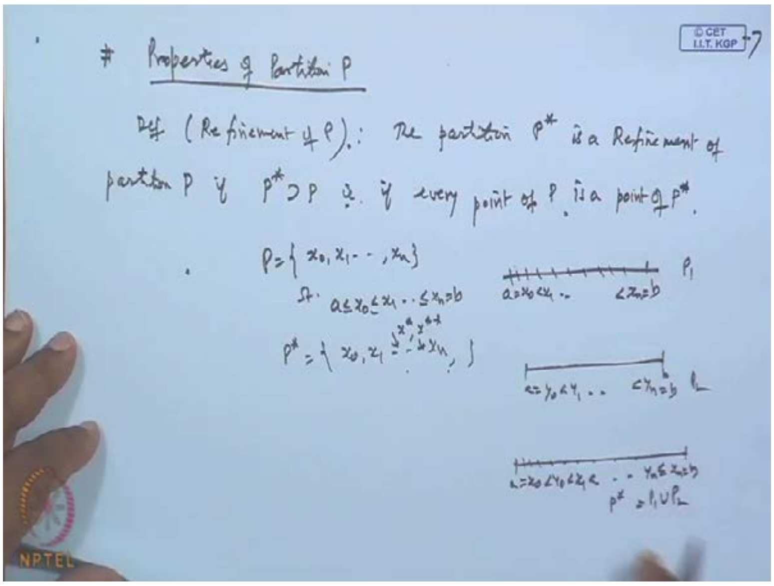

So, for over any partition P because this is our, taken as the P as the partition of this. Then what is this properties of the partition. Let us see, some properties which will be needed in establishing the existence of this integral because we have not so far defined or so far obtained any results or justify whether this integral will exist even if f is a bounded function. So, in fact, we will show the justification whether f is a bounded function both the Reimann integral and Reimann Stieltjes Integral will exist, f is continuous it will exist, f is monotonic we can also get the Reimann Stieltjes Integral provided under certain condition. So, existence result we will take up afterward. First, let us see what are the properties enjoyed by this partition.

So, let us see properties of partition P. So, we say the refinement of the partition, refinement of P, P is the partition. So, by the partition P star, the partition P star is a refinement, is a refinement of P of the partition P of partition P. If P stars covers P totally that is every point, if every point every point of P P is a point of P star then we say, because what is the partition? Because the partition P, this is our P partition means is a collection of the points x naught x 1, x 2, x n; such that satisfying this condition this condition that is all. So, if we take any other partition that will also contain the points of this.

Now, if this partition, if every point of P also a point of P star it means when you take the P star then these points will definitely there, apart from there may be some more point included here say x star x double star and so on. These points may also be included apart from this. So, that is why then every point of the P becomes the P star plus some extra points are available in this.

So, instead of partitioning these into the n sub intervals we are partitioning into n plus 1 sub intervals n plus 2 interval by introducing more point in between, but earlier partition you can as it is, then it is called partition or refinement of the partition because the partition may be different P 1 and P 2 are two different partition. Suppose this our P 1 say one of the partition x naught x 1 x 2 and say x n is equal to b, this is one partition. I take another partition a b say P 2 which is entirely differentiated y naught less than y 1 less than y 2 less than y n which is b. Now, this x naught x 1 x n and y naught y 1 and y n may not be the same point, but if we take the partition p 1 union p 2 this partition is P star, then what happen is a and b are there, then all points x naught then may be the y naught then x 1 and continues these like this x n equal to v n y n is this.

So, all the points are taken together which are partitioning the interval in a b into two n sub intervals basically. So, this becomes the refinement of the partition P 1 as well as you can say refinement P 2. So, in particular even if the partition P is there if I just include one more point in it then obviously, the number of sub intervals increases and in that way we are getting a refinement of the previous partition P. So, this is the concept of refinement, so obviously here.

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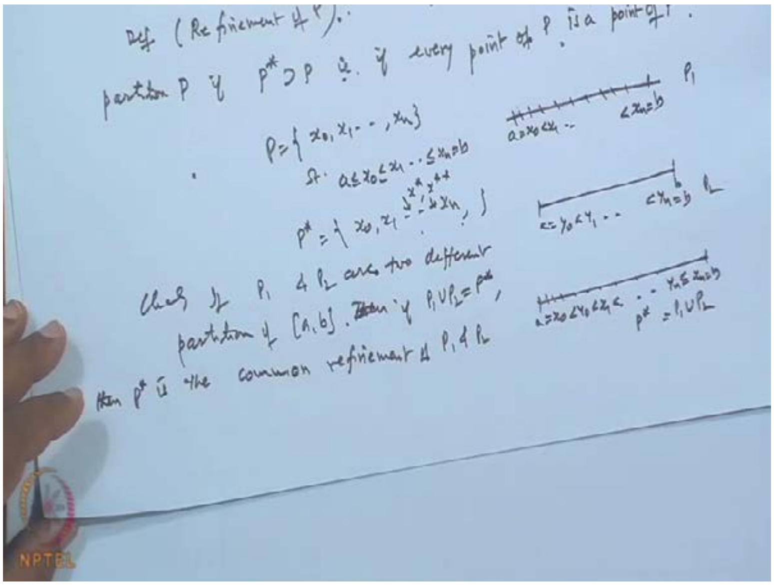

So, clearly if P 1 and P 2 are two different partition different partition of the interval a b then there union P 1 union P 2 is the say equal to P star is the common common refinement is the common refinement of these P 1 and P 2 provided of course, P star is this, union of this, if P star this then is the then P star is the common refinement, if P star is this, then a common refinement. So, this is what. Now, result which we are talking is.

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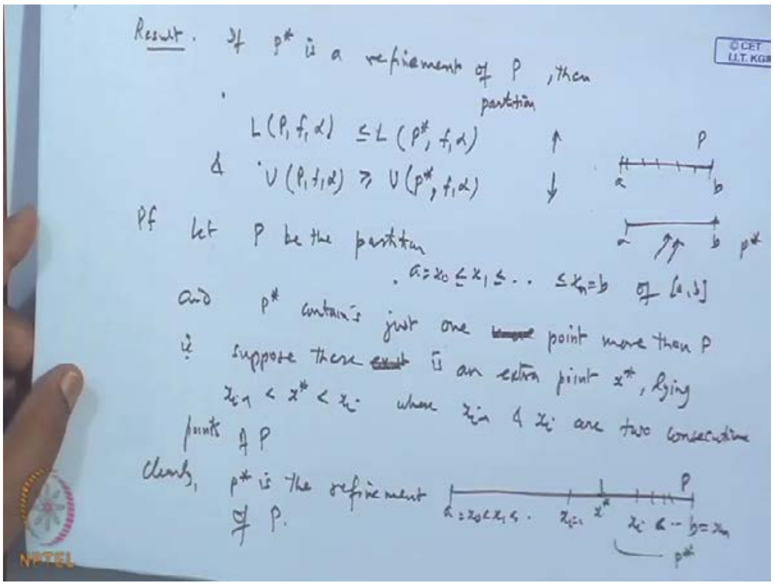

So, let the first result is if P star is a refinement refinement of the partition P, if P star is the refinement of the partition P then the lower sum increases, then lower sum L P f alpha will be less than equal to L P star f alpha while the upper sum P f alpha, this upper sum decreases that is upper sum of P f star P star f alpha. So, lower sum increases and upper sum decreases.

So, when you divide the interval a b into a partition P, into a partition P and if I include few more point, introduce few more point and getting the partition P star which is the refinement of P, then in that case the lower sum will be a will form an increasing function, will be a increase lower sum value increase while the upper sum will decrease. So, that is. So, proof is this. I assume let P be a partition, P be the partition. Say a is x naught less than equal to x 1 less than equal to x 2 x n equal to say b is the partition of a and P star is suppose contains just one more point. Say one more point, just one point more than, just one point more than the partition P that is suppose, that is suppose there exist there exist or there is an extra point extra point x star lying between lying in this subinterval x i minus 1 to x i lying this where x i minus 1 and x i are two consecutive consecutive points of P.

So, what we are doing is that this is our interval a b, here we are having the partition x 1 x 2 x n, here is x i minus 1, this is say x i and then x n is this, like this. Now, what we are doing is this is our partition P. Now, P we are just increasing one more point say here x star x star. So, this new partition becomes P star and it is the refinement of P. So, clearly P star is the refinement of P. Others, I am not changing, only in one subinterval I am taking one extra point that is all. So, it becomes a refinement of this. Now, with this point x star what we want to claim that lower sum will increase and upper sum decrease. So, let us prove first for the lower sum.

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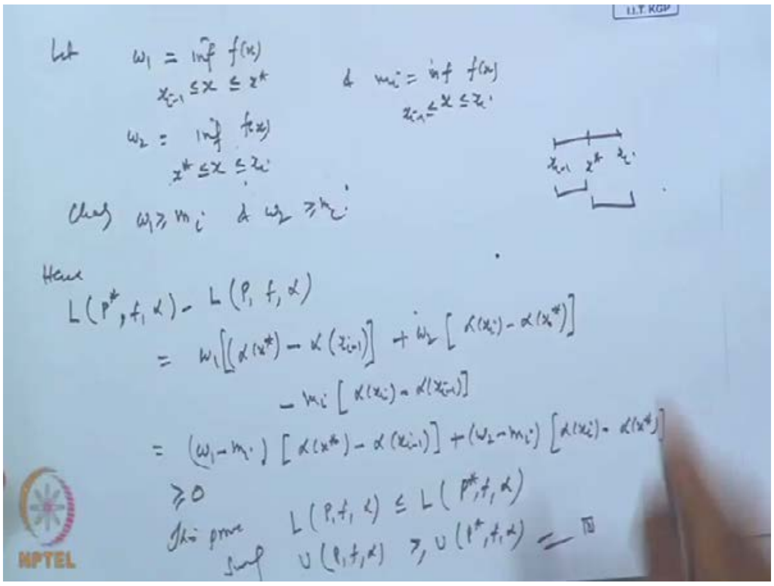

So, let w 1 is the infimum of the function f x over the interval x i minus 1 to x star while the w 2 is the infimum value of the function f x when x lying between x star and to x i and m i is the infimum value of the function f x when x is varying over x i minus 1 to x i. So, clearly this m w 1 will be greater than equal to m i and w 2 will also be greater than equal to m i because this m i is taken over the whole interval x i minus 1 to x i as the infimum value is taken and then these are the infimum value over the partition of the subinterval.

So, obviously, this infimum value may be more. So, if w 1 is greater than or equal to m i w 2 is greater than or equal to m 2 i and hence consider the lower sum with respect to the partition P star of the function f with respect to alpha minus the lower sum of with respect to the partition P and f and alpha. So, what we get is, over this interval P star is the sum of the two interval this is our x star. So, first you take over this and then this. So, here the lower the infimum value is denoted by w 1. So, it is the w 1 and then value at a point alpha x star minus the value of the function alpha at the point x i minus 1.

So, this is the value of this lower sum over this (( )) minus and then for. So, this will be multiplied by alpha, w 1 multiplied by this, then over the second interval w 2 w 2 this is alpha x i minus alpha x star. So, this is the lower sum of P over the partition P star. This plus this minus the lower sum of the partition function f with respect to the partition P. So, that is equal to m i into alpha x i minus alpha x i minus 1 with respect to the alpha. Now, let us combine just. So, w 1 minus m i and within bracket you get alpha x star minus alpha x i minus 1, is it not? Then plus w 2 minus m i alpha x i minus alpha x star. Now, w 1 is greater than equal to m i w 2 is greater than equal to m i.

So, these two things are positive. alpha is a monotonic increasing function, x star is greater than or equal to x i minus 1. So, value of x star at alpha at x star will be more than value of this one. Similarly, x i is here greater than equal to x star. So, this will be positive. So, it is greater than equal to 0. So, what it shows that when the, we increase the partition when we get the refinement of the partition by introducing the points, more point then lower sum increases and that proves the first part of this.

Similarly, so this proves that lower sum of the function f with respect to the partition P is less than equal to lower sum of the function f with respect to the partition P star which is the refinement of this increases. Similarly, upper sum we can show that P f alpha is greater than equal to upper sum of P star f alpha. So, this is what we proved. So, this is what. Now, we come to property which is related to using this partitioning interval and property of the lower sum integral and upper integral.

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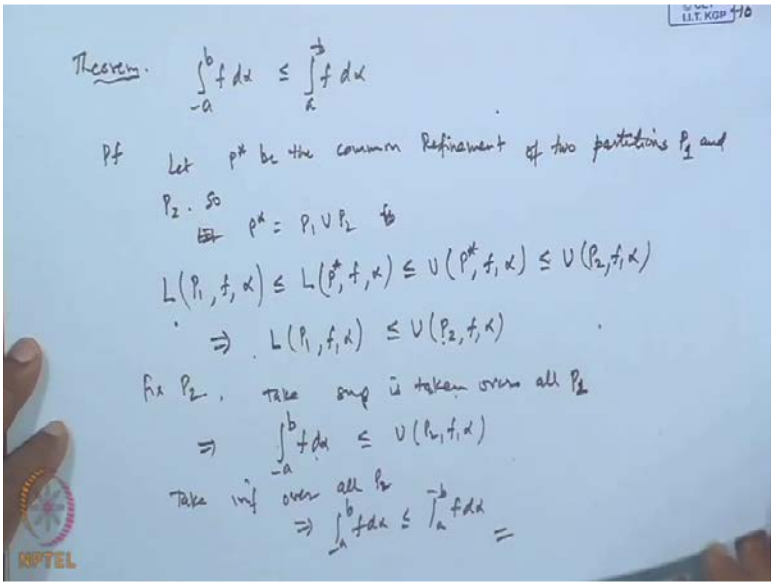

So, this we put it as a form of theorem. What the theorem says is the lower integral of the function f lower Reimann Stieltjes Integral of a function f over the interval a b will always be less than equal to the upper Reimann Stieltjes Integral of the function f with respect to alpha over a b. So, obviously, this result is also valid for a Reimann Integral because when alpha becomes x then lower Reimann Integral is always be less than equal to upper Reimann Integral, the proof of this. Let us see the proof, let P star be the common refinement refinement of two partition, two partitions P 1 and P 2 and P 2. So, let us take the partition P. So, let P star is the union of P 1 and P 2. So, is the, so obviously, so P star is equal to P 1 union because it is a common refinement. So, we can take P star is a P 1 union P 2. So, it is the refinement for P 1 as well refinement for P 2.

Now, since we have already proved that when you have a refinement of a partition then lower sum increases and upper sum decreases. So, using this one we get, so we get lower sum of the function L with respect to the partition P 1 over the partition P 1 L with respect to alpha is less than equal to the lower sum of the function f over the refinement P star with respect to alpha because P star is the refinement. So, lower sum increases, but this is always be less than equal to upper sum because lower sum is always be less than equal to upper sum with respect to the same partition.

However, we will prove it is also true for a general, in general lower sum will always be less than equal to lower sum whatever the partition we choose, that we will show it next.

And f and then alpha and then now upper sum decreases, upper sum decreases means it is less than equal to P 2 f alpha because P star is the refinement. So, obviously, the upper sum respect to P star is less than or equal to the upper sum. So, from here what we can conclude, it implies that the lower sum with respect to the partition P 1 f alpha is always be less than equal to the upper sum with respect to the partition P 2 f alpha.

It means if we take any two arbitrary partition P 1 and P 2 then always the and function f is fixed, alpha is fixed then lower sum of that function f with respect to alpha will always be less than equal to the upper sum of the function f with respect to alpha, whatever the partition we choose. For a same partition this is true, but for in arbitrary partition also we have shown that this lower sum is always be less than equal to upper sum. Now, what we do is here. Let us fix up the P 2, fix partition P 2 and let take the supremum value, supremum is taken over, supremum is taken over all P 2, all let us see first that (( )) or P 1 or P 1. This I am fixing and here I am taking the supremum over P 1. So, when you take the supremum of all P 1 then this will give the what? It will give the lower sum.

So, we get from here is a bar lower b f d alpha, this is the lower Reimann Integral Stieltjes Integral will give and then less than equal to U P 2 f alpha. Now, again you take the infimum value then in the right hand side take infimum over supremum is taken over all P 2. So, we get from when you take the infimum then it will give the upper sum, we get from here is f d alpha is less than equal to upper sum of this P 2 f d alpha, this proves the result. So, this proves. So, it is in results.

Now, the question arises as we have seen in the first way that what is the guarantee if f is a bounded function, we have defined the Reimann Integral, we have defined the Riemann Stieltjes Integral, but then what is the guarantee whether they are equal or not because if they are not equal then no point of as going further. So, the existence of their integral is important when the both the integral coincide both will have a same value. So, under what condition both these integral, lower integral and upper integral exist and have a equal value.

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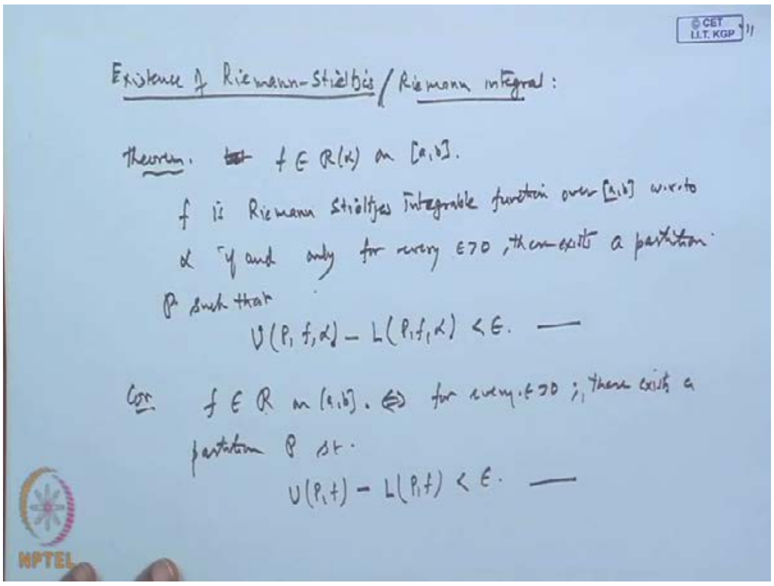

So, that we can say f is Riemann Integral Integrable function or f is Riemann Stieltjes Integral. So, this theorem gives a little bit about the existing part of this existence of Reimann Stieltjes or Reimann Integral because both I am dealing at a time or Reimann integral, that is all. The theorem is let f belongs to the Reimann Stieltjes Integral with respect to alpha on a b, then we say f is a Reimann Stieltjes Integral or removed it or f is Reimann Stieltjes Integral, f is Reimann Stieltjes Integral, Stieltjes Integral function, Integrable function over the interval a b with respect to alpha if and only if, if and only if for every epsilon for every epsilon greater than 0 there exist there exist a partition there exist a partition P such that such that the upper sum P f alpha upper Reimann sum minus upper Reimann Stieltjes sum lower sum L P f alpha is less than epsilon. So, this condition is necessary as well as sufficient for the existence of a function to be Reimann Stieltjes Integral, so in particular when we take alpha x equal to x.

So, we can say as a particular or as a corollary we can say f belongs to the Reimann Integral on a b if and only if if and only if for every epsilon greater than 0, there exist a partition P such that the upper sum of the function f with respect to partition P minus lower sum of f with respect to partition P is less than epsilon. So, so this is for the Reimann, necessary and sufficient condition for Reimann Integral, this is for the necessary and sufficient condition for Reimann Stieltjes. Now, we will see the proof next time. Thank you very much, that is all.