A Basic Course in Real Analysis
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Lecture - 37
L’ Hospital Rule and Taylor’s Theorem

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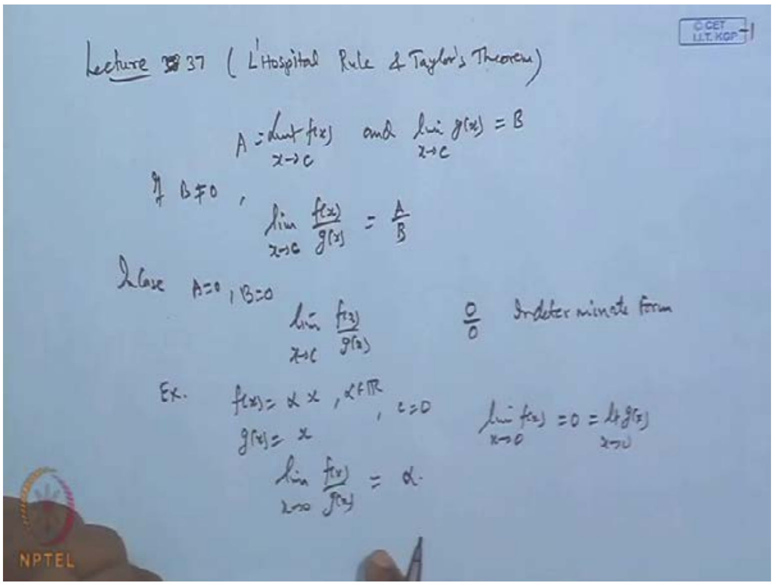

So, today we will discuss the L hospital rule and Taylor’s theorem. We have seen already if the limit of the function f x, then say x tends to say suppose A is our x tends to say c is say A and limit g x when x tends to c is say suppose B. Then if B is not equal to 0 then one can find the limit of the f x by g x when x tends to c, and basically this is equal to the A by B. But if suppose in case A is also 0, B is also 0, if B is 0 A may be anything then also we can discuss something, and it is shown that if the limit of the derivative exist then this must be say 0, and in case if this B 0 and A is less than 0 negative minus infinity and B is 0, A is positive then limit will go to the plus infinity.

Now, the case when A and B both are trending having the limit value with 0. So, when you consider the limit of the function f x over g x when x tends to c, it comes out to be 0 over 0 form which is known as the indeterminate form indeterminate form, why indeterminate? Because you cannot say the value of this is one or value is some finite number or value does not exist. In fact 0 over 0 the well it may exist, value the limit may exist, limit may not exist also and like this. For example, if we take the function f x say alpha times of x and g x say x, and choose the limit c as 0. Then obviously limit of f x as x tends to 0 is 0 which is the limit of g x when x tends to 0, both are coming to be 0.

So, when you take the limit of f x over g x as x tends to 0 then in fact, this is coming to be if you take the limit of f x over limit of g x then it is basically coming to be 0 over 0 form. But if I substitute the value f x is g x then the limit will come out to be alpha, because x will get cancel and x is independent of any x. So, limit will come out to be alpha a real number where alpha is a real number. So, it means the limit of this 0 over 0 f x by g x when it demands to be the 0 over 0 then it may, exist and may have a value a real number, or sometimes may not exist also that we will take few some example where the limit does not exist in the case.

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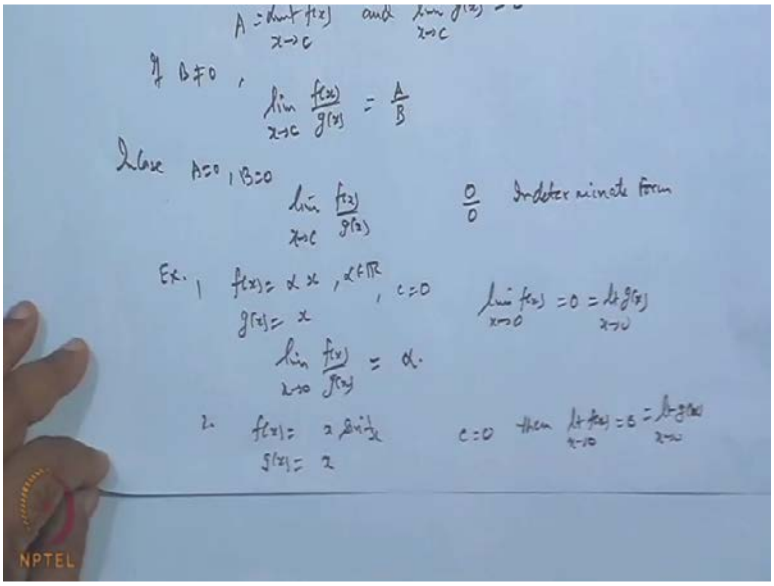

Say for example limit does not exist suppose I take, another one f x equal to x sin say 1 by x, and g x is equal to x. So, if I take c equal to 0 then both the limits f x when x tends to 0 is 0 limit g x when x tends to 0 is also 0.

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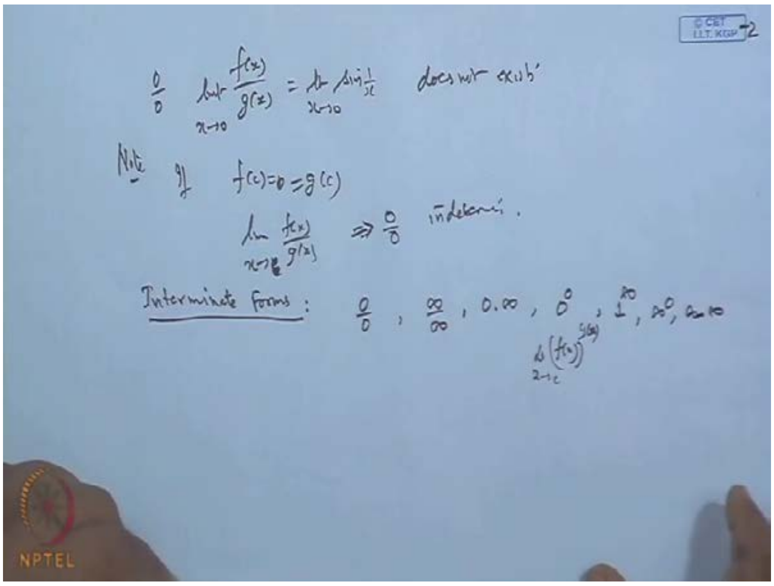

But when you take the limit of f x by g x, as x tends to 0 it is coming out to be the limit of sin 1 by x when x tends to 0 which does not exist. So, in the initial form it is 0 over 0 form, but basically if the limit does not exist. So, that’s why to evaluate such case you have to find the limit of f x by g x when it immerse to be 0 over 0 then we require some rules and that is given by the l hospital. Now, one more thing here we are choosing the limit even if f of c is 0 or g of c is also 0 then in that case limit of f x by g x when x tends to c is also in the same form 0 over 0 it takes this form, which is also in determinant form.

So, if the function is defined at the point c limiting point and if it is coming to be 0 over 0 then the limit of f x over g x to calculate the limit becomes a problem or even the limiting value is also coming 0 by 0 then it is also we cannot use any limit rules for the limits, that limit of the rational function is the numerator of the numerator over limit of the denominator which we cannot apply here. So, in such a case is we require the some concepts and that is to how to evaluate this limit of this ratio when it is coming to be 0 over 0 form and that is given by the l hospital.

So, before going to L hospital there is one we will say the in determinant forms, is the 0 over 0 is the only in determinant form, no there are many cases which can which are considered as a in determinant form. So, let me there indeterminate forms, this forms are first form is 0 over 0, infinity over infinity is also considered to be in determinant form 0 into infinity that is if f x tends to 0, g x tends to infinity then f x into g x this limit will go to 0 into infinity form or f x by g x if g x also, then 0 to the power 0 if f x tends to 0 g x tends to 0 the limit of f x to the power g x, this when x tends to c.

If it is coming to be 0 to the power 0 then as so considered to the in determinant form. 1 to the power infinity will also be taken as a in determinant form, and infinity to the power 0 is in determinant form infinity minus infinity is also in determinant form. So, they are various types situation where we can say the limit the forms which we are getting is an in determinant form, so evaluate this type we require certain tricks because just by substituting the value or taking the limit we cannot say the limit is correct or we cannot evaluate the limit also. So, before going for the l hospital,

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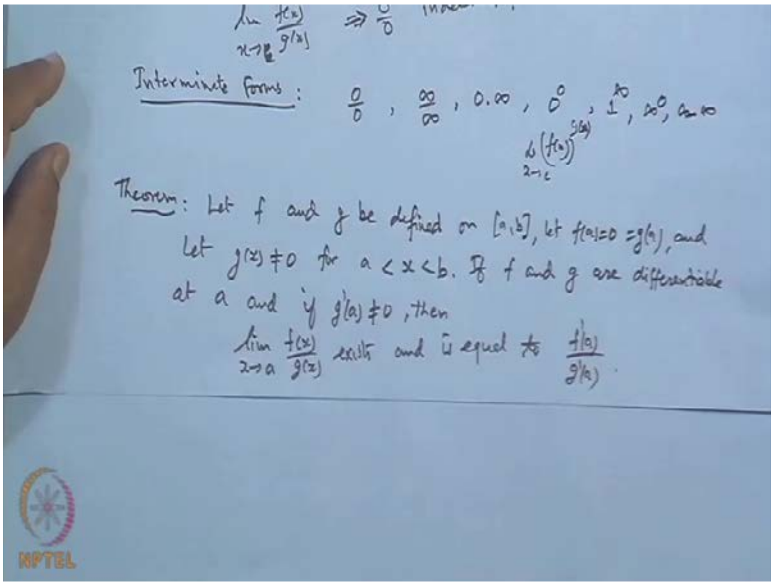

We have seen the one result which only requires the concept of the derivative and nothing more. What this result say, let f and g be defined, on a close and bounded interval a b, and let the value of the function at the point a is 0, value of the function g at the point a is also 0. And let g x is not equal to 0 for A less than x less than B. Now, if our f and g are differentiable function, at A and if, g prime a is not 0. Then the limit of f over g, then limit of f x over g x when x tends to A exist and is equal to equal to f prime A over g prime A, so this is the result. Now, here in this theorem we have assumed the function f and g both are attending the value 0, at the point a. So, when you take the limit of x over f x over g x when x tends to a basically it is of the form 0 by 0. But, what this theorem says if f and g are differentiable at the point a, and if the g dash is not 0 then basically this in determinant form has a value and equal to f prime A over g prime A the proof of this follows like.

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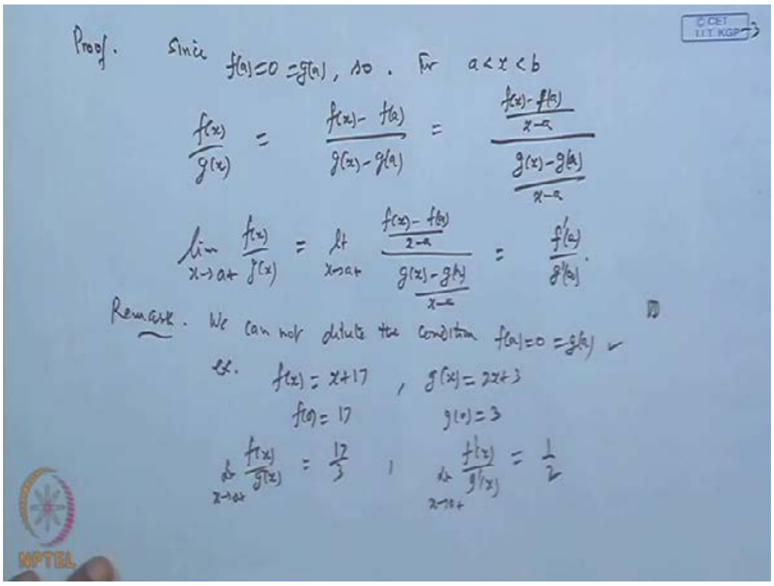

Now, since f A and g A both are 0. So, we can write it f x, by g x, for x so for x lying between A and B. We can write f x by g x f x minus f A over g x minus g A, because these two values are 0 and this can be written as f x, minus f A, divide by x minus A because A x is strictly greater than A I am choosing so the denominator is non 0, g x minus g A divided by x minus A. Now, if we take the limit as x tends to A. So, when you take the limit of this as x tends to A x tends to A f x, by g x, we are assuming here the limit a even we can take the limit a plus because we are not considering the function, simply we are taken function is differentiable at A and then g prime A 0 define and differential.

So, let us take the A plus here that is equal to the so we take this A plus, A plus in that case f x over g x is the limit of x tends to A plus f x minus f A over x minus A divide by g x, minus g A divide by x minus A. So, limit of the ratio is the ratio of the limits so we get this goes to f prime A this goes to g prime A, hence the result follows. So, but here they we cannot dilute, this condition the condition is must that the value at the point A both must be 0. So, this we cannot dilute so as remark we can say we cannot, we cannot dilute the condition, that f A equal to 0 equal to g A. Suppose, we have a function where the values are different and not equal to 0's also.

For example, if we take the function f x which is equal to say x plus 17, g x this is say 2 x plus 3 then obviously f of 0 is 17 g of 0 is 3. So, f x by g x when you take the limit of this x tends to 0 plus from the left hands right hand limit then it comes out to be 17 by 3 is nothing. But when you take that limit derivative of their limit, as x tends to 0 plus then derivative come is derivative here is one here is two so f A this comes out to be half, so basically both differs the reason is because we have diluted this condition, so this is not no longer this is are valid if both are not having the value 0 at the point A, that is very important and g prime is not equal to 0 that is very obvious from here because if g prime is 0 then we cannot divide it.

So, that's why the condition is taken g prime a is different from 0. Now, here we have assume the function attains the value 0 at the point A, and the derivative of the function, exist at this points, is it not then only we can find the value. If the derivative, or the functional value at the point A, is not defined but the limiting value is obtained, then also we can apply the result and that is given by the L hospitals, and that will be known as the L hospital rule, first.

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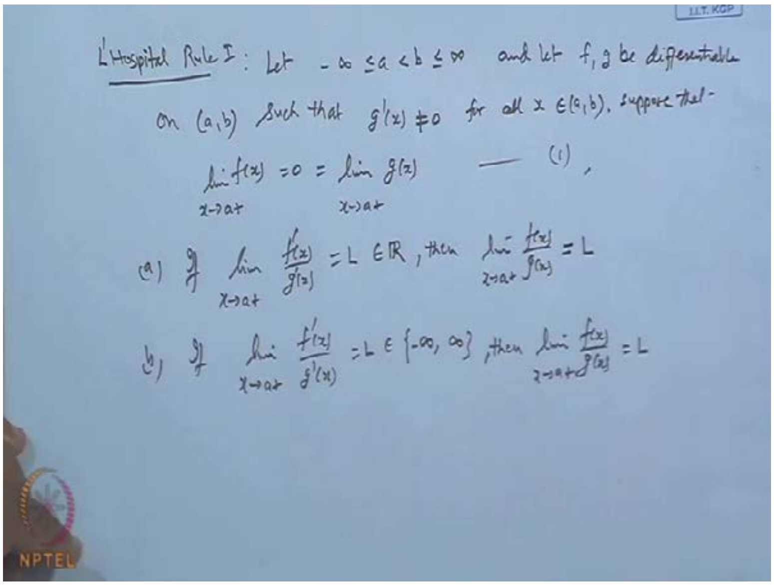

In fact, first is very important other rule is just extension case. So, that's let, A and B, are two real number, which may be real number which may be infinity or minus infinity also. And let, f and g, be differentiable, f and g be differentiable, on the open interval a b. Such that, such that the derivative of g, does not vanish any real inside this, interval a b that is non 0 for all x belongs to a. Now, suppose, that the limit of the function f x, when x tends to A plus, is 0 which is the same as limit, g x when x tends to A plus both are having the limiting right hand limit, exist and equal to value 0, let it be 1. Then the result says, if, this condition.

If limit of f x over g x f prime x over g prime x when x tends to A plus, if the limit of their ratio of their derivatives, exist and easily a number L, then the limit of the f x by g x when x tends to A plus from the right hand side will also exist and will be having the same value as the limit of f prime over g prime is their when x tends to a plus. The second is if suppose L is infinity or minus infinity so if limit of this derivative, f prime, x over g prime x when A tends to A plus, is suppose L which is, either in minus infinity or plus infinity only. Then the limit of this f x over g x when x tends to A plus will also be minus infinity or plus infinity depending on L.

So, this is what is known as the law. So, what this L hospitals rule says is that, if we assume the differentiability of the function, differentiability of the function but not necessarily at the point A. Then the behavior of the limit of f x by g x, and the behavior of the limit of f prime x over g prime x will be the same. That is the limiting behavior of f prime over g prime will be the same as the limiting behavior of f x by g x, whether L is finite or infinite or minus any value.

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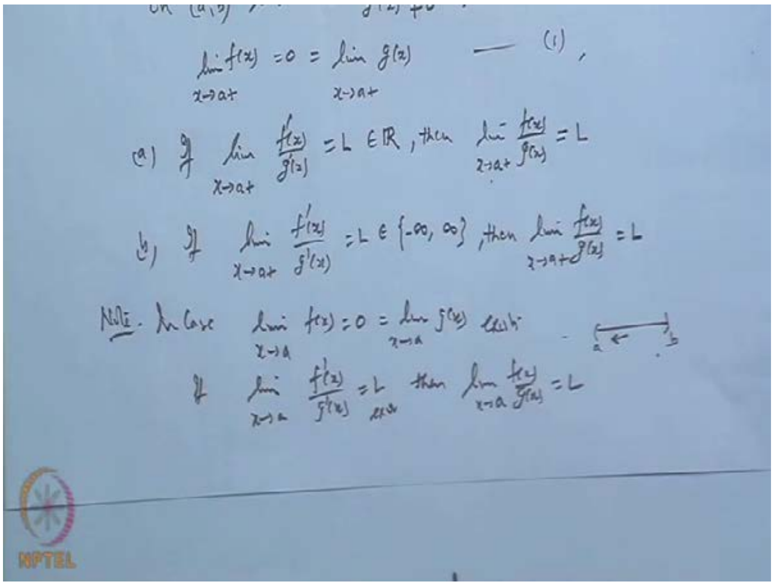

Now, here second, point which we also want to discuss, though I have taken the limit, a plus means the interval is this a b, and we are taking the limiting value from the right hand side. But there is same result continue to hold good when we take the limit, b minus or a minus suppose it is completely defined or limit exist, if the function has a limit at the point x equal to a then we can replace a plus by a, like this. So, if the limit of f prime over g prime at x tends to a is L, then the limit of f x over g x over x tends to a will also be L. So, in case, limit f x as x tends to a exist, and 0 and equal to the limit of g x when x tends to a when f and g both are having the limit I am not taking only the right hand limit let us limit exist. Then if limit of this ratio f prime over g prime x when x tends to a is equal to exist, then limit of their ratio, will also exist and will be the same as L.

It means the proof will remains the same whether we replace a plus by a or a plus by a minus provided the limits are they are. If the left hand limit exist we use this, if the limit exist we use this if the right left hand limit exist sorry right hand then this left hand limit exist then a minus and like this, but the result continue to hold good. So, we will establish this proof when the right hand limit exist and for others cases the proof is the same.

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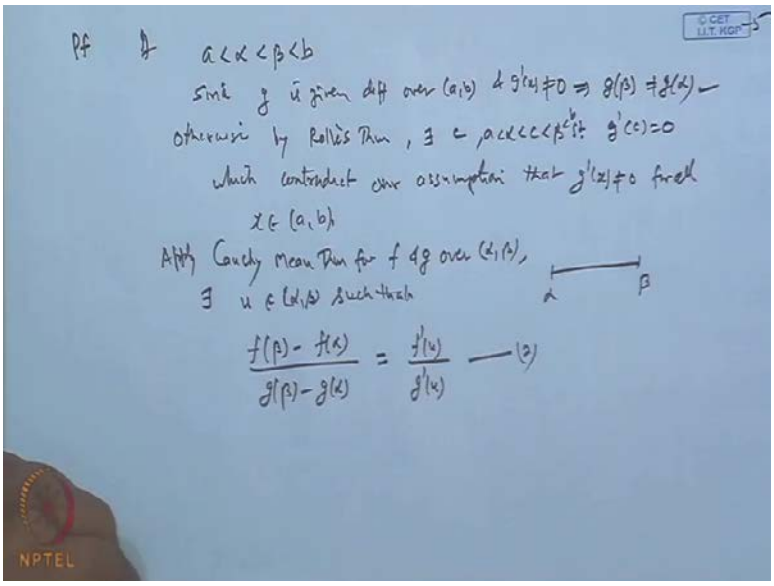

So, let see the proof. Now, let us take an alpha beta lying between a and b ok now, since the function g, since g, which is given to be differentiable over a b, since g, is given differentiable over a b, over a b and g prime x is not equal to 0. So, this automatically implies the value of g beta will be different from g alpha, why the reason is otherwise by Rolle's theorem, if a function g which is continuous over the close interval alpha beta, differentiable over the open interval alpha beta g is given differentiable over alpha beta, and if the end point alpha and beta the values are same then their will exist point c lying between alpha and beta such that is a and b such that, derivative of g at the point c must be 0.

Which contradicts, our assumption, that g prime x never vanishes for any x belong to the interval a b. So, always this result will be true that g alpha. Now, take the interval, alpha beta, apply cauchy mean value theorem, for the function f and g over the interval alpha beta, so what the mean value theorem says f and g both are continuous over the close interval alpha beta differentiable on the open interval alpha beta then f of b minus f alpha divide by g beta minus g alpha is the value of the f prime say c or f prime u divide by g prime u for some u lying between alpha and beta. So, by mean value theorem there exist an u, belongs to alpha beta, such that, f beta, minus f alpha, g beta minus g alpha is the derivative, this thing for some u let it b 2, clear this is value. Now, let us take the case one say…

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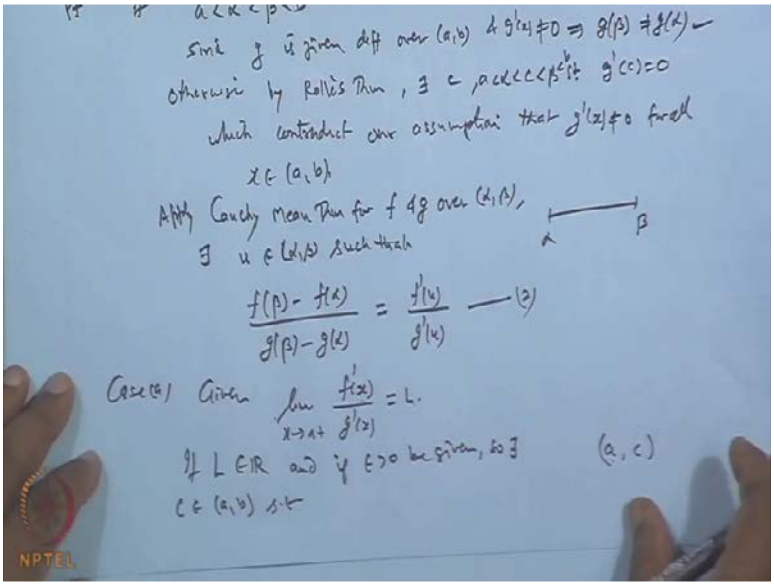

Now, let us take the case one, say, a what a it is given, limit of this is given, limit of f prime over g prime when x tends to a from the right hand side is L this is given, so given limit, x tends to a plus f prime over g prime derivatives is L so given. So, let us take so if L is given that is if L is known which is in R and let epsilon and if epsilon greater than 0 be given, a some positive number. Now, limiting value of this is L, it means when the point a close to some intervals say a c I take. So, whatever the point x in between this the limiting value of this prime is L means difference is very, very small, so this lies between L minus epsilon and L plus epsilon. So, we can say there exist, a c, belonging to the interval a b such that,

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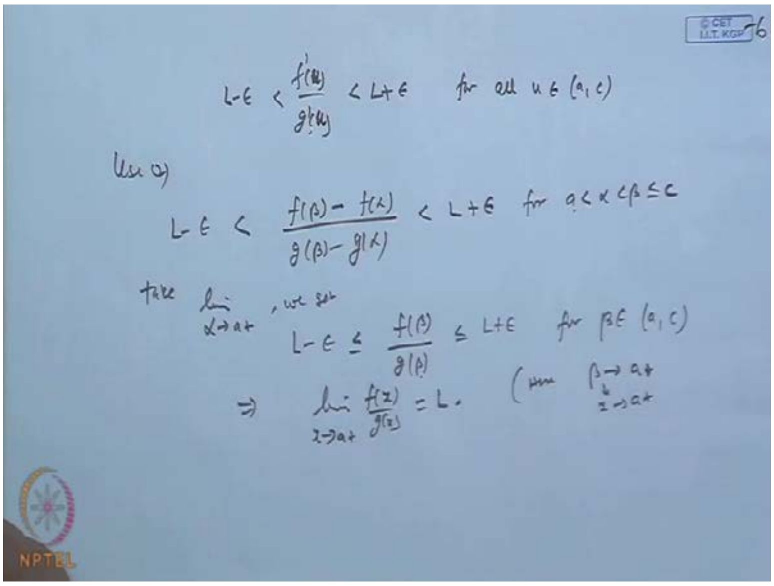

F prime x over g prime x lies between L minus epsilon and L plus epsilon, for all u let us be this u, so let us take the odd u, for all u belongs to the interval a c. We can find the neighborhood of right neighborhood of a, where if you take any point in that right neighborhood of a the this condition will be satisfied. But, f prime u over g prime u we have already, find out from there, is it not. So, from two we have discussed this from two f prime over g prime is this so substitute this value, so use two, hence from 2 we can say L minus epsilon is less than, f beta, minus f alpha over g beta minus g alpha, which is less than L plus epsilon, for a less than alpha, less than beta, less than equal to c.

Now, take the limit, take limit as alpha tends to a plus, because a plus this limit will be 0, is it not so this limit will be 0 this will be 0 so what we get is so we get, L minus epsilon is less than equal to when you take the limiting equal sign may also come, so this is less than equal to epsilon and this is true for all beta, belonging to the interval a c, therefore, this shows the limit of f be f x, by g x, When x tends to, because this limit is it not, when this one is there so limit beta tends to a. So, when you take x in place of beta I am taking x so x tends to a plus is also L. Because here, let beta approach to a from the positive side, that is equivalent to say that x, approach to a plus, where x belongs to this interval, so this shows…

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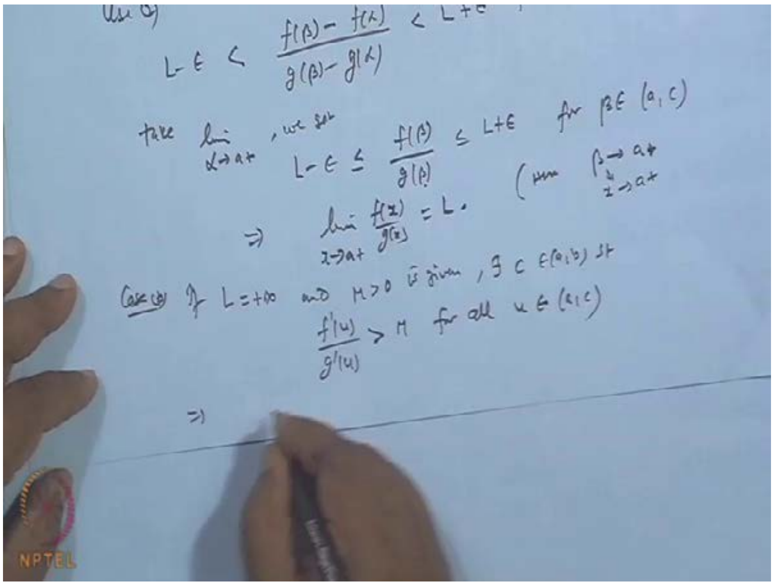

The second case, when l is plus infinity, l is plus infinity and, if m we choose greater than 0, is given because the limit is given to be L which is, infinity. So, what do you mean by this it means that they are exist, a c belongs to a b, because this is given, here this is given, this is suppose infinity. It means when point is closer to a in the right in the neighborhood right neighborhood of a then this limit can this value can exceed any positive number. So, I am taking m to be greater than 0, then a number, u can be obtained there exist c such that, for all u, this condition is satisfied, is greater than m for all u belongs to a c. And but since epsilon is arbitrary number this sorry then from here this implies f (b) this implies.

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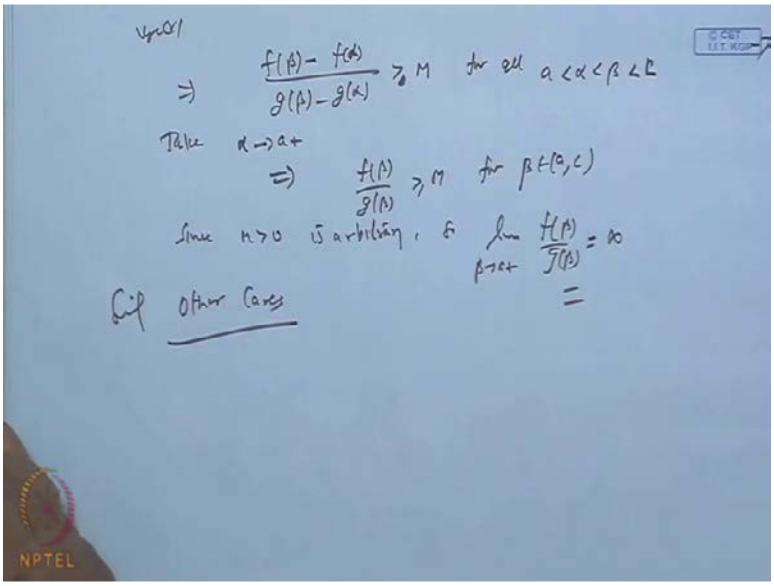

Again substitute use, to so f of beta, minus f of alpha, divide by g of beta, minus g of alpha, is greater than equal to what, m is greater than m, is strictly greater m for all alpha and beta lying between this bound, ok in that right neighborhood of a. So, take the limit as alpha tends to a plus and immediately we get the f beta over g beta is greater than equal to m for all betas, belonging to a c, and this shows since m is greater than 0 is arbitrary, large number. So, limit of this f beta, over g beta when beta tends to a plus will be infinity and that’s what. The other cases follows in the similar.

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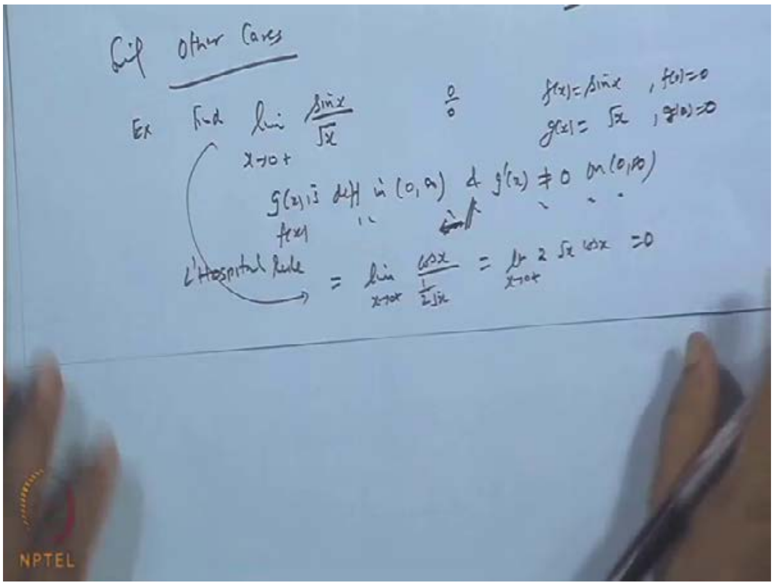

Similarly, we can show for others case, so that's proved the results. Let us see the use of this suppose I take the example, find the limit of limit of sin x, over under root x, when x tends to 0 plus. Now, if you look here, the function f x is sin x, g x is root x. When x tends to 0 the limit of this basically f (0) is 0 g (0) is 0. So, value at the point 0 is 0 so when you take the limit it comes out to be 0 by 0 form. But the is root g x is not differentiable at 0. So, the first theorem which we have shown it will not be applicable. However we can apply the L hospital rule. Because g x, is differentiable in the open interval 0 to say any number 0 to infinity, and g prime x is not 0 on 0 to infinity, and f x is also sin x is f x is also differentiable so there is sin function, is differentiable and like this.

So, we get and this is not required, this is not required up to here only. So, we get we can apply the L hospital rule. So, L hospital rule says that limit of this is will be the same as the limit of their differentiation derivatives, if we differentiate the numerator, and denominator separately what you get it this limit will be the same as the limit of this, that this one but this is equal to limit 2 root x cosine x as x tends to 0 plus and now, this is dominated by x so limit will come out to be 0, so answer is 0.

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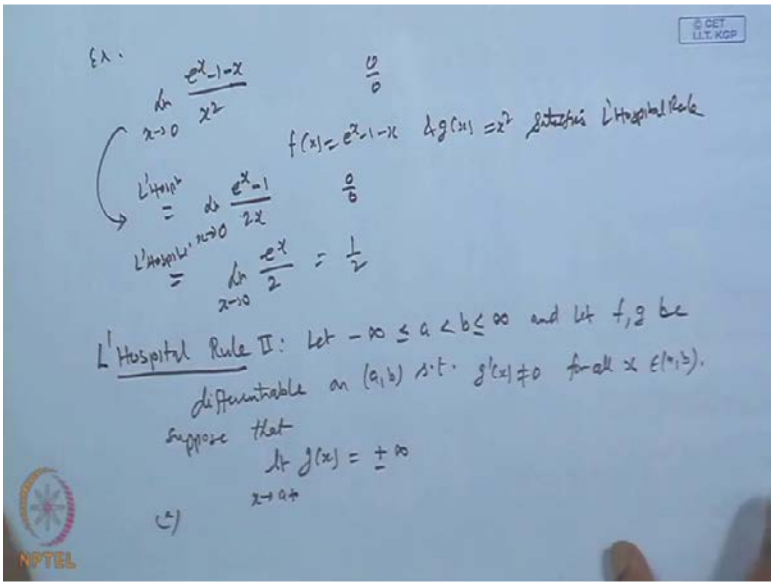

Another example we can say suppose I take the this example say, e to the power x, minus 1 and both side, take this example log e to the power x minus 1 over say x square limit x tends to 0, x tends to 0. So, it is x minus 1, let us take this also x minus 1. So, what happen when you take x tends to 0 it is 0 over 0 form, because e to the power 0 as 1 and this is 0. So, it is in determinant, both the function f and g satisfy, L hospital rule, L hospital rule so apply it. So, by L hospital rule differentiate the numerator, and denominator separately. So, if I differentiate e to the power n and denominator will give and then take the limit.

So, by L hospitals we get this thing these limit is the same as this limit. But again it is 0 over 0 form, because again 0. So, again apply the L hospital rule. We can repeatedly apply the L hospital so far the conditions of the L hospital rule are satisfied. So, if we apply again then we get e to the power x by 2 limit x tends to 0 now, this is half so answer will be half. So, this is one, then other forms let us take L hospital rule, II. Here a slightly different than the previous ones. Let minus infinity which is less than the case which is not covered in the previous one, minus a is less than equal to a less than, minus infinity less than equal to a less than b less than infinity, or may be at the most equal.

It may be infinity also and let, f and g f g be differentiable, f and g differentiable on the, interval a b. Such that such that, g prime x, is not equal to 0 for all x, belonging to the interval a b, and suppose, that limit of g x, when x tends to a plus is infinity or may be minus infinity, then if the following condition holds.

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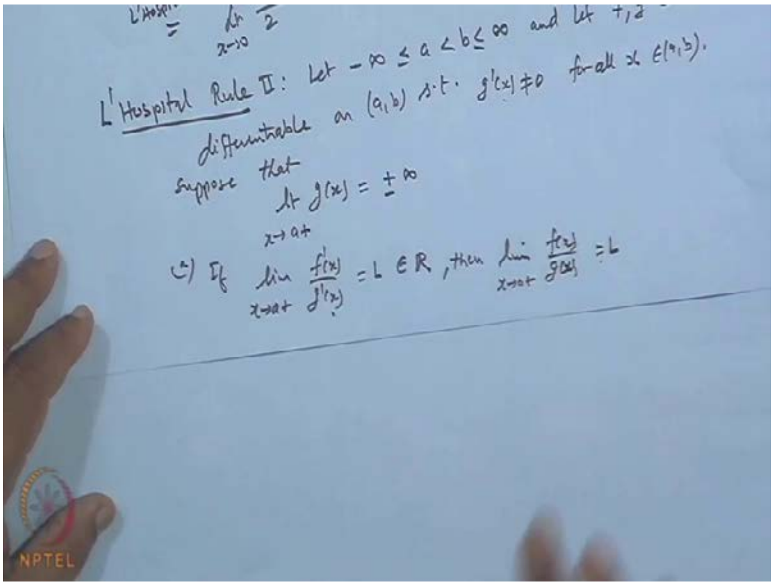

Then if, limit of f prime x, over g prime x, when x tends to a plus is suppose L belongs to R a finite number then the limit of f x, over g x, when x tends to a plus will also be L. So, here the case is this, if f and g are differentiable, and the limit of g x is infinity or minus infinity suppose I take plus infinity, and the derivative of g does not vanish anywhere in the interval. Then if the limit of their derivative exist, then the limit of their ratio will exist. So, here is silent about by f x f x whatever the f x value may be this will be L, is silent on the f x, but it is defined and different so it is finite value and this limit will be this one silent about this.

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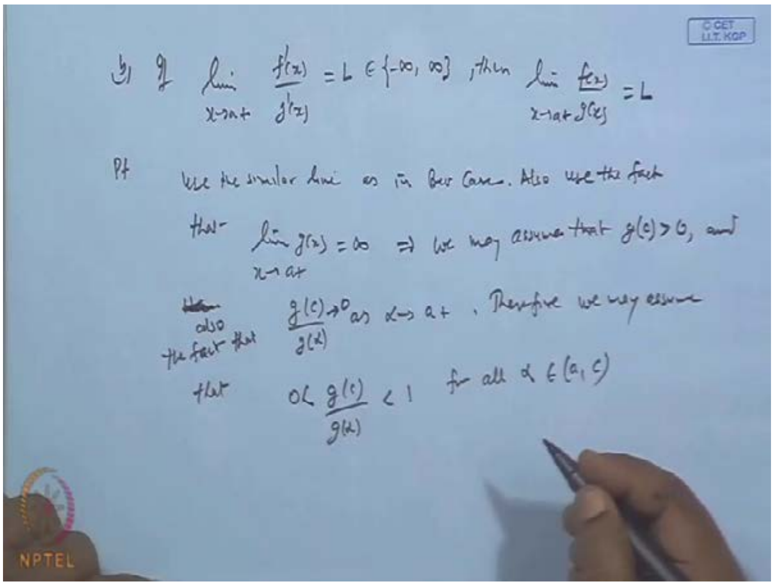

Second part says if, limit of x tends to a plus, f prime x over g prime x is say L, which is either infinity or minus infinity, then the limit of their ratio, f x over g x as x tends to a plus will also be L. The proof I am just dropping because the proof based on a similar lines, as a except that here just I will give use the similar lines use the similar lines, as in previous case, and then x and also, use this use the fact, that limit of g x, x tends to limit of g x as x tends to a plus is infinity. So, case one when it take the infinity, suppose I take the infinity minus infinity will deal. So, if this it implies that they'll exist we can assume, we can assume, we may assume that the value of g at some point c will be positive because it is going to infinity.

So, after certain state it will cover 0. So, that a point c will come then it is greater than 0. Enhance, and enhance the g of c by g of alpha. When alpha tends to this when as alpha tends to, tends as alpha tends to a plus, and also sorry and also, the fact, also the fact that, this thing when alpha tends to a plus so condition is given that a g x as a plus is infinity, so this will go to 0. So, said that g x over alpha as alpha intends is tending to 0, as x alpha tends to a plus because this tends to plus infinity and this is positive, so this will go to invent therefore, we may assume, that we may assume that 0 less than g c over g alpha which is less than 1, for all alpha which lies in the right hand neighborhood of a.

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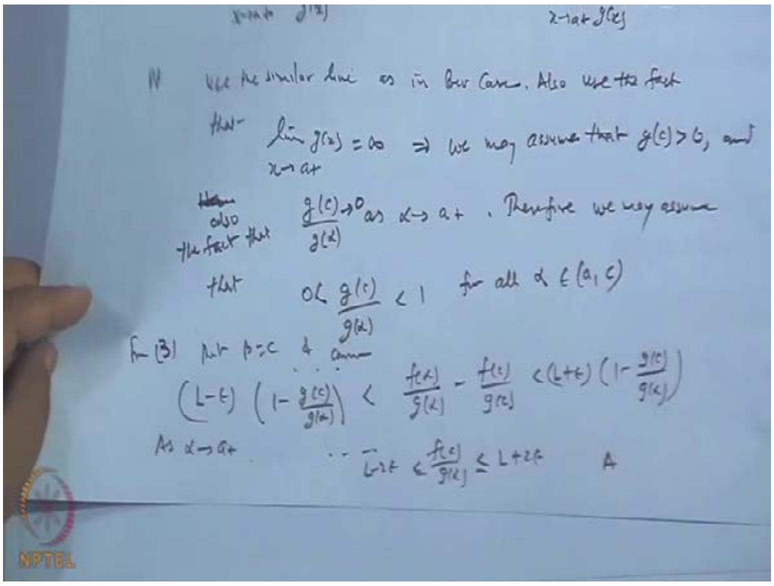

Now, using this fact and a similar lines we can show the previous just from the three third, proof of this, as we have seen we can go the proof is, in fact, from 3, if you remember this was third yeah, this was this was the case when f alpha (( )) so beta equal to c if I put it, then we get from here say third here, in third if I put beta equal to c, put beta equal to c and then consider this, we get that L minus epsilon multiply all side by g c over g alpha, because this is positive is less than f alpha, by g alpha, minus f c, by g c, is less than L plus epsilon into 1 minus g c by g alpha in fact, this so and from here as alpha tends to, a plus we will get this limit will exist and limit will come out by calculations you will get it limit to be less greater than equal to 2 epsilon less than equal to L plus 2 epsilon.

So, limit of this will be the same like this. I am not giving the detail proof, but you can work out and get the results. So, because this time is not permitting for that, we wanted to have some more results here.

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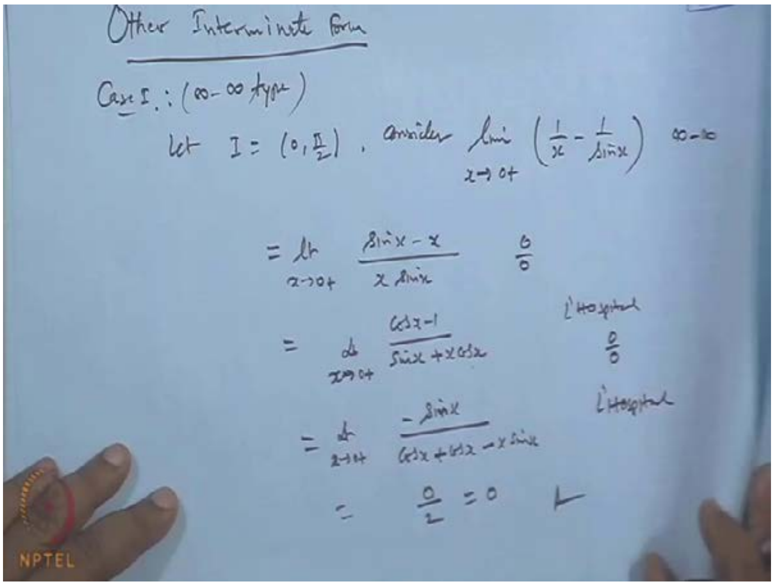

So, let us take the other determinant, forms other in determinant forms so case wise let us take say first case is, infinity minus infinity I will give by example, seen that in case of this where infinity minus infinity type. So, say for example, let I is the interval 0 pie by 2 and consider, the limit of this, as x tends to 0 plus 1 by x minus 1 by sin x. So, when you take x tends to 0 it is of the form infinity minus infinity. So, how to solve find out the value what we do is we take the l c m, so limit x tends to 0 plus this is the same as x sin x, minus sin x minus x. Now, it reduces this is infinity minus infinity now, it reduce 0 by 0 so apply the L hospital. So, when you apply the L hospital rule differentiate the numerator, and then differentiate the denominator, so we get sin x plus x cos x and take the limit as x tends to 0 plus. If this limit exist then limit of this will be the same.

Now, x tends to 0 it is 0 again 0 so it is again 0 over 0 so on. So, further apply the L hospital rule. So, when you take there is because all the f x and g x satisfies the conditions of L hospitals. So, we get x tends to 0 plus, cosine x is minus sin x divided by say cosine x, and then x, so again cosine x, then minus x sin x. So, when you take the limit of this what you get is x tends to 0, so that comes out to be now, 0 over 2 so this is 0, answer is this.

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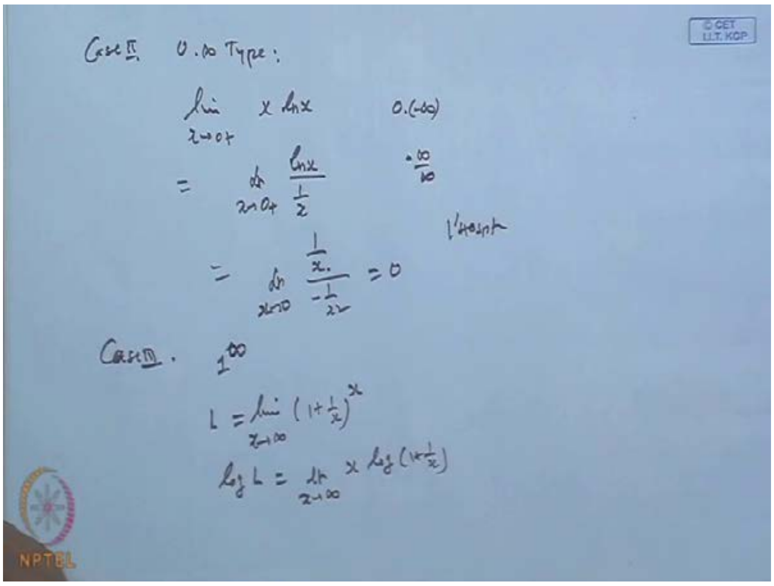

Second case is, say 0 into infinity type. Suppose I want to take the limit of this x tends to 0 plus, x l n x. So, when is x tends to 0 it is 0 into minus infinity that is 0 into infinity type. So, how to get it this one is what we do is we put it this in the form of 0 over 0 or infinity over infinity. So, if I take the l n x, divide by 1 by x as x tends to 0 plus, then it is infinity by infinity type, off course minus infinity makes no difference, then apply the L hospital rule now, so once we apply the L hospital rule one by x and here denominator we get the minus 1 by x square, as x tends to 0, and then when it comes up you get the value 0, so it is coming to be 0.

Third case, when, it is of the form say 1 to the power 1 to the power 0 1 to the power 0, for example, suppose I take the limit of this 1 plus 1 by x, as x tends to infinity power x 1 to the power infinity sorry, x tends to infinity, 1 to the power infinity. So, it is when x tends to infinity this is one this infinity so infinity to the power infinity. So, what we do is we consider this L limit L, take the log. So, when you take the log it is come to become limit is a continuous function log will go inside and we get log 1 plus 1 by x and x tends to infinity. So, basically when x tends to infinity is infinity into 0 type. So, this can be put it again either 0 over 0 or infinity over infinity type.

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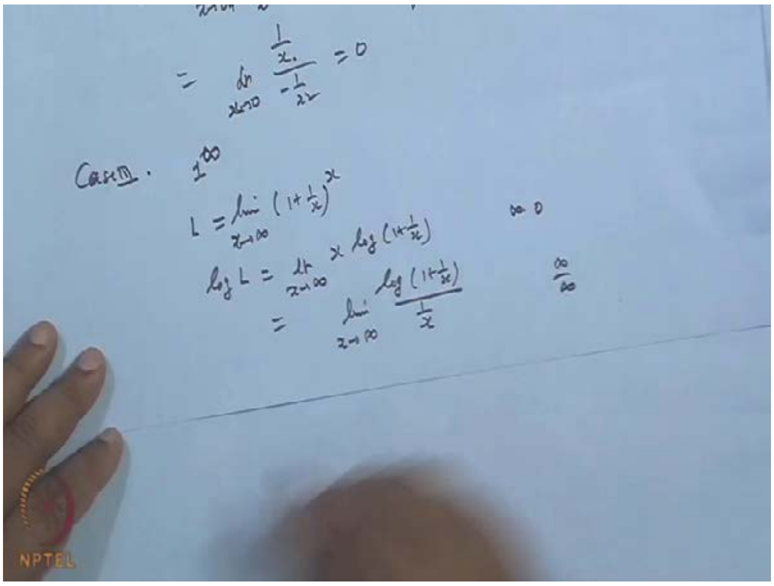

So, we can put it this as 1 plus 1 by x, divide by 1 by x, limit x tends to infinity that is infinity over infinity type, and then apply now, L hospitals rule.

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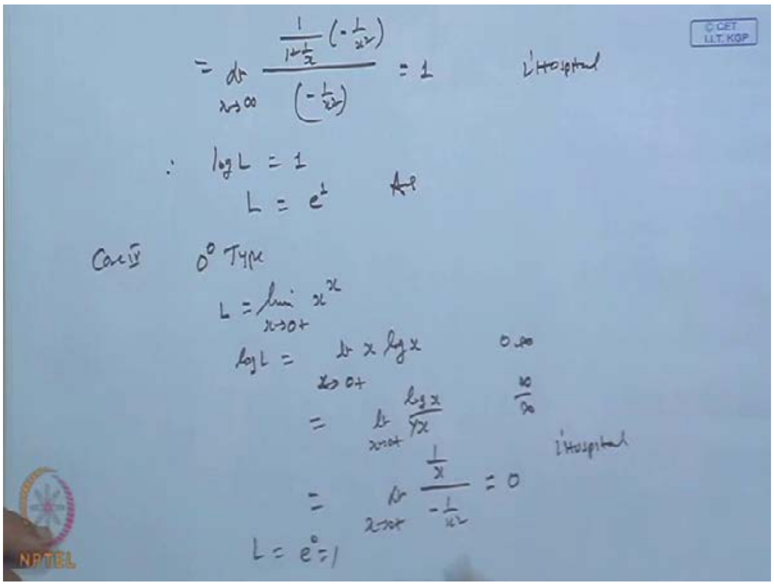

So, if we apply the L hospital rule, what we get differentiate the numerator numerator differentiation will give 1 over, 1 plus 1 by x, and then derivative of this will be minus 1 by x square, and then minus 1 by x square so and then take the limit as x tends to infinity. So, basically this limit comes out to be 1. So, since log of L is 1, is it not log of L is 1 therefore, so log of L is 1, therefore anti log of this will e to the power 1 that is e, so answer is basically L e. Then next case is four, when it is of the form, say 0 into infinity 0 into 0 type, 0 power 0 type. So, for example, if you take the limit x to the power x when x tends to 0 plus, so it is 0 to the power 0 again in this case take the L log, and when you take log this problem is reduced to, the case when it is 0 into infinity type. So, just like the previous one eth we can take it log x, by x, as x tends to 0 plus then apply it is infinity sorry 1 by x, so infinity over infinity. Now apply the L hospital rule, so when you apply the L hospital rule differentiate the numerator and denominator, separately, and take the limit when x tends to 0 plus and this limit will come out to be what, 0. So, the L will come out to b e to the power 0 that is 1 answer.

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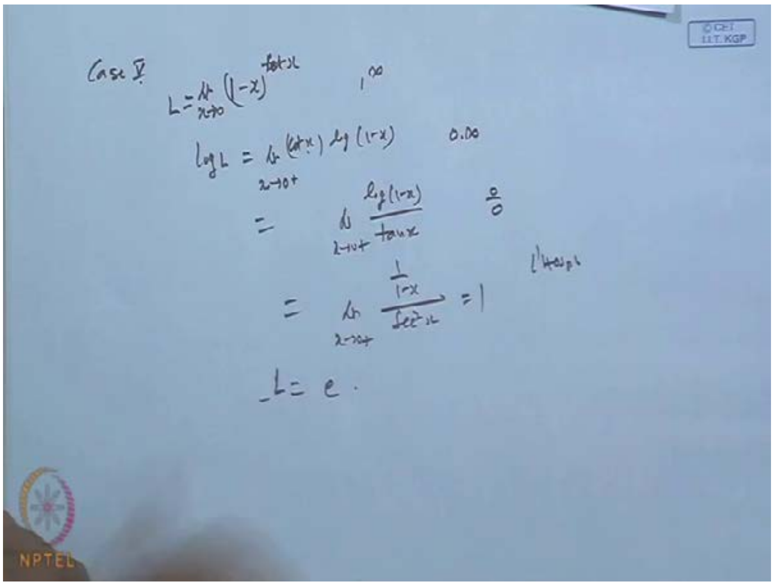

Then another case fifth, which is of the form say, 1 to the power say, again what form left now, infinity minus infinity we have taken and then let us take another case is limit of this x tends to 0 0 into minus infinity x log x that we have already considered. So, I think this is all forms we have discussed is it not. So, let us see so we can say another form infinity let us take one more example hence what we, suppose I take 0, say 1 minus x, x tends to 0 into say 1 to the power infinity so let us take that tan x.

Now, cot x, and x tends to 0. So, what this is the 1 to the power infinity, so what how will you do it take L, and then log L, so log L will be equal to what, cot x, into log 1 minus x, and x tends to 0 plus, so cot 0 is infinity this is 0 so 0 into infinity then again you can take log 1 minus x divide by tan x and x tends to 0 plus, so it is 0 over 0, apply L hospital rule, and we get from L is differentiate numerator, differentiate denominator, take the limit when this goes to 0. So, when you take the 0 upper limit comes out to be 1, and this is sec is also what the limit will be 1. So, L will be equal to e to the power 1 answer (( )), so that is what is shown. So it's almost we have completed this one.

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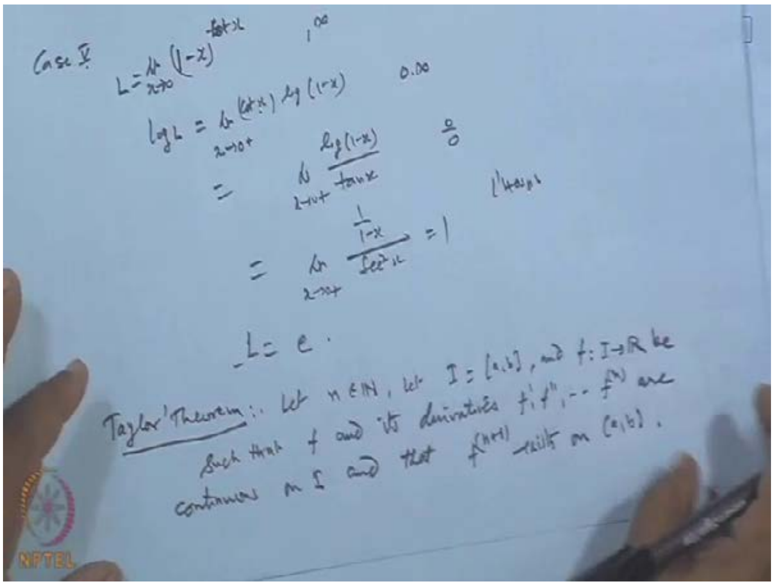

Now, let come to the Taylor's theorem, the Taylor's theorem it is an extension of the mean value theorem. Mean value theorem we get the relation between the functional value and its first derivative, whether it is a Cauchy Lavengen mean value theorem, f b minus f a or b minus a derivative of the function, or may be the Cauchy mean value theorem, where the two functions f and g are there, and then we get the relation between f, g and their derivatives. Taylor's expansion Taylor's theorem is an extension of the mean value theorem, where we have a relation between f and its higher order derivatives, and we basically used, to approximate the functions with the help of the first few terms of the Taylor series or Taylor theorem. What this theorem says is let n, belongs to capital N, and let, I is the close interval a b, and let, f is a mapping from I to R.

Be such that be such that f, and its higher order derivatives, say up to order n f prime f double prime up to f n. The nth order derivatives up to r continuous r continuous on the interval I, and that, the n plus 1 derivative exist, on the open interval a b.

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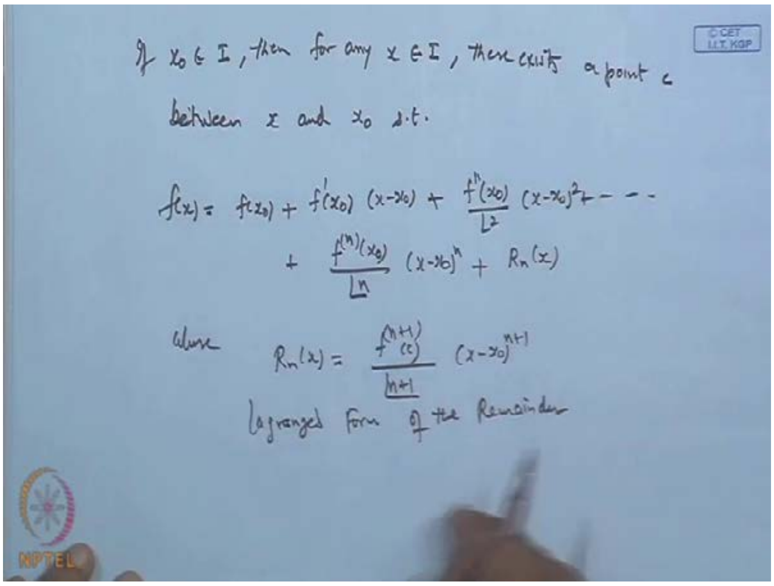

Now, if x naught is a point in the interval I, then for any x, belongs to I there exist, a point c. Between x and x naught such that, f x can be expressed as f x naught, plus f prime x naught, into x minus x naught, plus f double prime x naught over factorial 2, x minus x naught whole square and so on. And then plus f n, at the point x naught divide by factorial n, x minus x naught to the power n plus the remainder terms r n x r n x. Where r n x is given by f n plus 1 c, over f n plus 1 c over factorial n plus 1 into x minus x naught to the power n plus 1, and this is known as the Lagrange's, form, of the remainder, Lagrange's form of the remainder.

So, the function f is such, which is differentiable, and continuous, up to say n plus 1 the order time. And n plus one nth order derivative exist in the neighborhood of the point x naught in the neighborhood of the point x naught. Then there exist some exist in the open interval a b off course, and the neighborhood of the x naught. Then there exist a point c, where the expansion of the function f x can be written in the form of the series. Where the first n terms, of the series the polynomials, of degree n and that this term is called the remainder term and it is known as the Lagrange's form of the remainder. Now, in case if the remainder r n goes to 0, then this same expansion is known as the Taylor series expansion for the function f x. So, we will discuss it next time with the proof of this thing.

Thank you very much.