A Basic Course in Real Analysis
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Lecture - 36
Application of MVT, Darboux Theorem, L Hospital Rule

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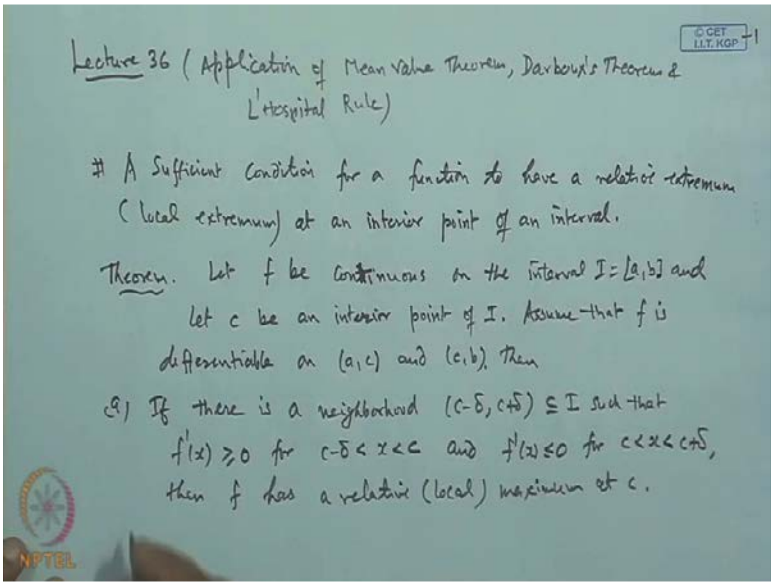

So, we have already discussed the mean value theorem various type of mean value theorem, Lagrangians mean value theorem, Rolle's theorem, then generalized mean value theorem, that is cauchy mean value theorem and also the few applications of the derivatives. And the most interesting application which we will discuss is the monotonic character of the function just by looking the sign of the derivatives. And we have seen if over the interval derivative is greater than 0 positive, greater than equal to 0, then the function will be greater than equal to 0, then function is monotonic increasing function. If it is less than equal to 0, the function is a monotonic decreasing function.

So, in this lecture we will give few more applications of the derivatives as well as our mean value theorems. With the help of the sign of the derivative near a point x in some neighborhood of the x, one can identify whether the point correspond to a maximum point, relative maximum point, relative minimum point or may be none of them. So, this is the criteria a sufficient. In fact, the a sufficient condition for a function to have relative extremum, that is local extremum at an interior point at an interior point of an interval.

These are the sufficient condition. Of course, the result says which is also known as the first derivative test for extremum.

Let f be continuous continuous continuous, let f be continuous on an interval on the interval say I which is a closed and bounded interval, and let c be an interior point interior point of the interval I. Assume that f is differentiable on a c open interval a c and open interval c b. Then the result says if there is if there is a neighborhood say c minus delta to c plus delta contained in I such that the derivative of the function f, that is f prime x is greater than equal to 0 non negative for c minus delta less than x less than c; means towards the left of this is positive non negative, and f prime x is less than equal to 0 non positive for the c in right hand side of the interval, that is c less than x less than c plus delta, then the function f has a relative or we also use the word local maximum at c.

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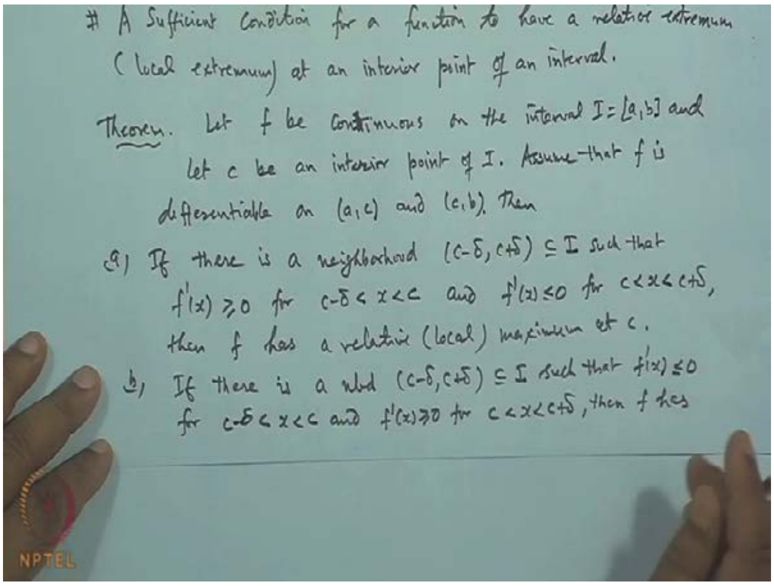

The second part is also says if there is a neighborhood there is a neighborhood c minus delta c plus delta in I, such that, the derivative of the function f is non positive for the left hand side interval c minus delta less than c, and non negative in the right hand side of interval of c, that is c minus less than x less than c plus delta.

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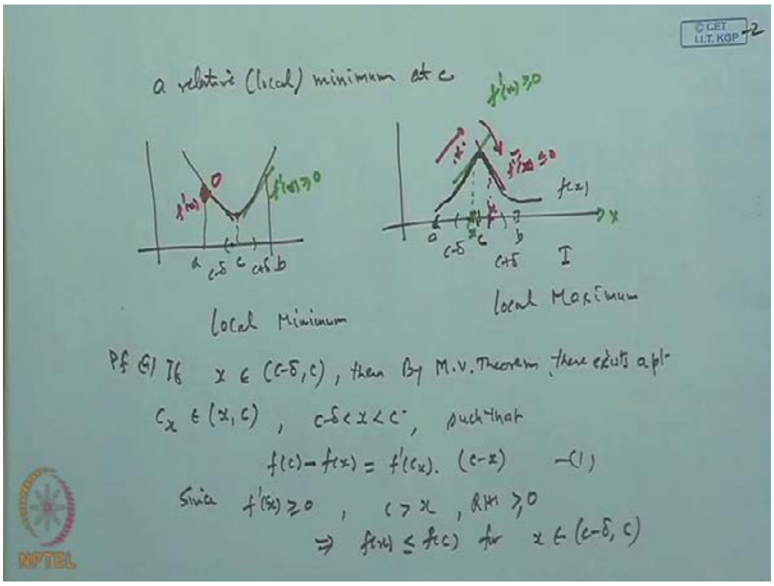

Then f has a relative or local we can say local minima minimum at c. So, before going to the proof let us see the geometrically what (( )). Suppose we have, two cases one is this say c is this interval. Now, a function is suppose is this, suppose this is the function f x. Now, what the property says? f be a continuous function on the interval a b, suppose this is the interval a b, where the function is continuous, on the interval and let c be an interior point in of the interval a b and assume that function f is also differentiable on this c, means this interval a a c and c b the function is differentiable, about c we are not talking right now.

So, in this the function is differentiable in this function is differentiable, the function is throughout over the closed interval is continuous, and then what he says is if there is a neighborhood if there is a neighborhood around the point around the point say c c minus delta, c plus delta this is the c minus delta it is c plus delta. So, in this neighborhood if, the function behaves like this if, the derivative of the function at this derivative of the function at this point is non-negative, while the derivative of this function at this point is, non positive, then the point c will correspond to the maximum point.

Let us see why so, if the curve is like this, this is our curve now, if we look this suppose I take a point here x point this is our x now, if we look this here the derivative means slope a prime x, this is the denote the slope of the function at the point x on this. So, if f prime x is greater than equal to 0 so, slope will be positive it means it makes an acute angle with the axis of x. Now, when in this interval if I choose in this interval if I take a point here, then correspondingly the tangent if I draw at this point the tangent will be something like this, the slope becomes obtuse, that is the derivative f prime x is less than equal to 0.

So, it means if near by x c if near by c the tangents changes, its behavior from positive to negative, that is it is keep on coming here and then as soon as it reaches to a point where it has a local maximum it has a change and the direction is changed. Is it not? And this change point the point where it changes its direction will correspond to the point of next one. The same case happens in second part. If we have say suppose, we have this curve and here is this a b and we have this point c so, let us take a neighborhood c minus delta c plus delta.

When you picked up the point below in the left hand side of this and draw the tangent that f prime x here is negative, here this was positive and here it was negative f prime x is negative is less than equal to 0 means non positive here this is non positive and then when it goes to here it is positive so, here the f prime x is greater than 0. It means the tangent again changes time from negative instead of changes the direction as soon as it crosses the point on the curve corresponding to c. And this point will correspond to the minimum point so it is a local minima for this. So, geometrically we can explain these things let us see the proof analytical also we can prove it, that in case of one it is a local maxima while in case of the two it is a local minima.

So, let us see the proof of it. if first we proof a part. Let if x belongs to the interval say c minus delta c in the left hand side so this is the figure 1 in this, then by mean value theorem by mean value theorem there exist by mean value theorem there exist a point exist a point say c x, in the interval x c, where x lies between the where x belongs to this means x lies between c minus delta less than this less than. So, I am choosing the interval c here x c and applying the mean value theorem.

So, there exist a point c x in this interval such that such that, the f of c minus f of x is equal to the derivative of the function at a point c x into c minus x. This one now since our let it be 1 since it is given over this side f prime x is positive, f prime x as c x this is c x is greater than or equal to 0 its given. Now, c is already greater c is greater than x because of this interval. So the right hand side right hand side is positive greater than or equal to 0 therefore, left hand side has to be positive. So these imply this is only possible when f (x) is less than equal to f c. For all x belongs to the interval c minus delta to c. Clear? It means in this neighborhood the function f attains the maximum value at c.

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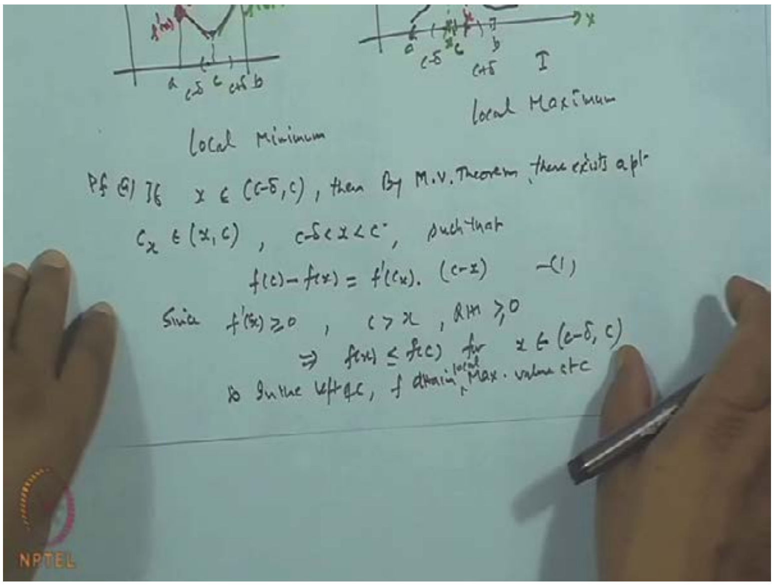

So in the left side in the left of the c in the left of c the function f attains maximum value at c local maximum value, you can say local because we are choosing only the neighborhood around this.

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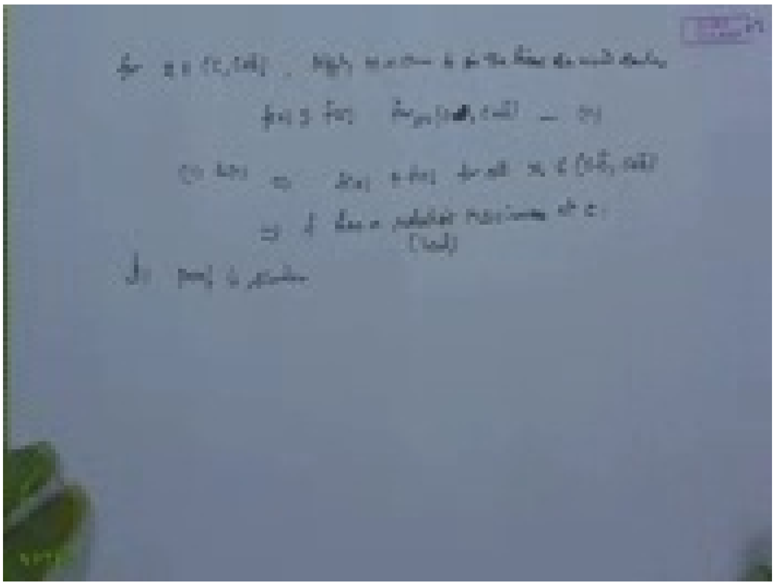

Now, on the other hand if I take this say x for x belonging to c to c plus delta and again apply the mean value theorem, then what we get? we get the x there exist some x mean value theorem over this same, so by mean value theorem we can say, f (x) is less than equal to f c for by mean value theorem and in a similar way, and on the same lines of as used earlier, proof means there exist a c x line between the say, c minus comma x where this will lie, so for all we can say this less than equal to this for x belonging to c minus delta to c plus delta. For all for all this is true this is c to c plus delta, for all x c to c plus delta therefore, 1 and 2 will gives that implies the f x is less than equal to f c for all x belonging to c minus delta, to c plus delta.

And that is sufficient to proof f has a relative maxima or local we can say maximum at the point c, at c this prove. The proof for the part follow similar proof is similar so, we just so that is gives the now, the converse of this is not true remark, because what we have seen is, we are assuming the derivative is non-negative in the left hand side and non positive in the right hand side, then there is a maxima.

Here in this case non negative here and positive non positive here, it is a non positive and non negative then minima, but if I say the converse part of it darboux theorem suppose, f has a maximum at the point c, then this or minimum at the point c, then we can can we derive so, that the function will changes its behavior from positive to negative so, that converse is not in general true. The converse is not true in general. For example, say suppose I take the function f x which is equal to say, let f is a function f is a function from R to R we define by f x is equal to 2 x to the power 4 plus x to the power 4 sin 1 by x for x not equal to 0 and f of 0 equal to 0.

Then clearly this function, then this function clearly what when x is different from 0 the derivative of the function will be eight x cube plus 4 x cube sin 1 by x and then minus x square cos 1 by sin cos plus and, then minus 1 by so it is minus sign so it will be the minus (x), because sin is cos 1 by x and 1 by x is minus 1 by x square so that is why minus sign is there. And when x is equal to 0, the derivative of this function f prime 0 is limit f x minus f 0 over x minus 0 when x tends to 0. And this comes out to be what? when you divide by this you are getting limit x tends to 0 2 x cube plus x cube sin 1 by x. And this is dominated by x cube and this is also total limit comes out to be 0.

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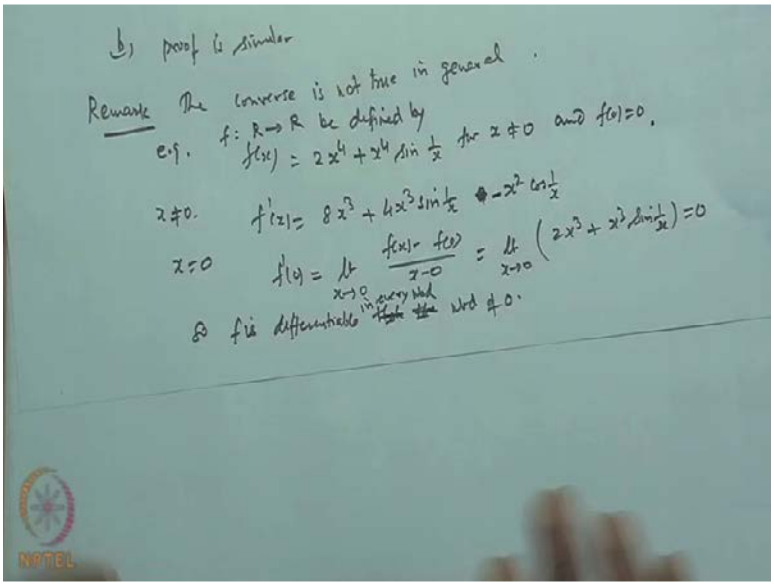

It means the function f, so f is differentiable f is differentiable through out the neighborhood of 0, in some neighborhood of 0 or any neighborhood in every neighborhood of differentiable in every neighborhood of 0 in every neighborhood of 0.

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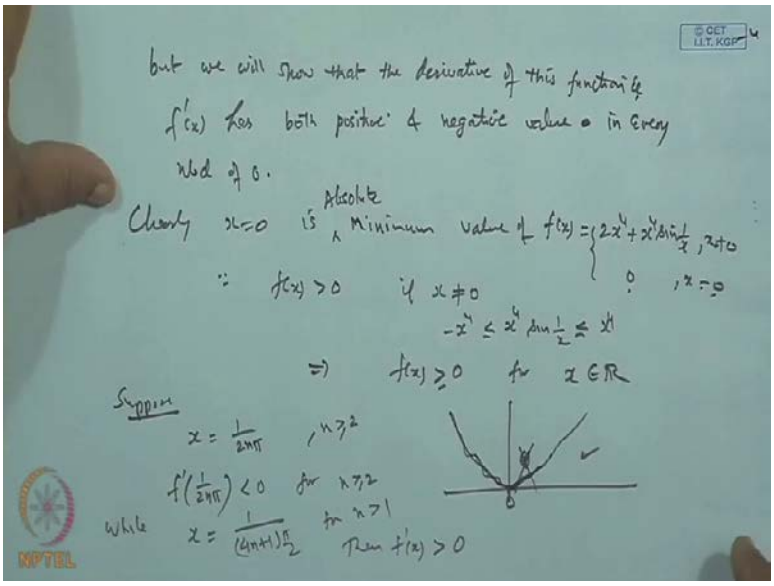

But, we will show that, but this function, but we will show that the derivative of this function that the derivative of this function that is f prime x has both positive and negative values and negative values negative values of in every neighborhood of 0. One more thing let me just clear it, the function is defined by this. The function clearly x equal to 0 is the minimum value of the function f x which is 2 x 4 plus x 4 sin 1 by x for x is not equal to 0 and 0 for x equal to 0. Whatever the x you choose this will be always a positive quantity why? Because so it is x equal to 0 is basically an absolute minimum value absolute minimum the reason is because f (x) will always be greater than 0 if x is different from 0.

The reason is the sin x to the power 4 sin 1 by x x to the power 4 sin 1 by x this is bounded by minus x to the power 4 and plus x to the power 4, this sign will be there. Now, 2 x 4 when we added both side, then x 4 is a positive quantity therefore, the function f x will always be greater than or equal to 0. 0 at the point 0, so it is 0 for all x belongs to R. So, 0 is the point which is the minimum point for this function and so, what we wanted to show this function is something at the point 0 it is 0, and rest of this I do not know what type of this function is, it will go something like this may be x 4 dominated something so, a curve will be there a curve may be like this something.

Now, this point 0 is the point where it happens the absolute minimum value. But what he says is, that this curve this is not correct, what we wanted to show, that this derivatives sorry derivative will have both positive and negative value in this it means, though it has a minimum value at the point 0. But according to the previous result the result was which we have shown is this. That if the derivative is non-negative, then their left hand side interval it is non-negative and right hand side interval it is non positive, if it has a maximum. And when it is a minimum point, then left hand side derivative it is all the values will be greater than equal to 0, and for the left hand side right hand side it will be less than say less than 0 and right hand side it will be greater than equal to 0. But, here in this case we are unable to get any interval whether in which the left hand side or the functional values are less than equal to 0 and right hand side the derivative of this is greater than equal to 0, the reason is like this.

So this follows from here. If we take the say point suppose I the suppose I take x which is equal to 1 over 2 n pi n is greater than 2. Then what happen this? the function f derivative of this function the derivative is defined like this this derivative, f prime x at this point at this point so, this is positive now this will be since it is 2 n pi this will be 0 this will be 0, and here it will be something negative, is it not? So, when we take n greater than 2 so it becomes what? 4 4 into pi becomes more than 8 in fact so, we get this value negative. So, here we are getting to be negative, and in fact this is we can show this thing is that this is negative for n greater than equal to 2 and positive. So, it is negative for n greater than equal to 2.

While, if I take x equal to 1 upon 4 n plus 1 pi by 2 for n greater than 1, then what happens? the derivative f prime x derivative f prime x this is the derivative f prime x so, when you take this term this of course, it will not effect this value will be 0 odd multiply so it is positive always so, it is always be greater than 0. So, in the same this sequence belongs to the same intervals 2 n pi and even the right hand side of this so we are getting the right hand interval where the derivative is having both positive and negative values therefore, it contradicts the that converse is not true, in fact it so this shows so, that's interesting example this we discuss it. Now let us come to the other applications of mean value theorem.

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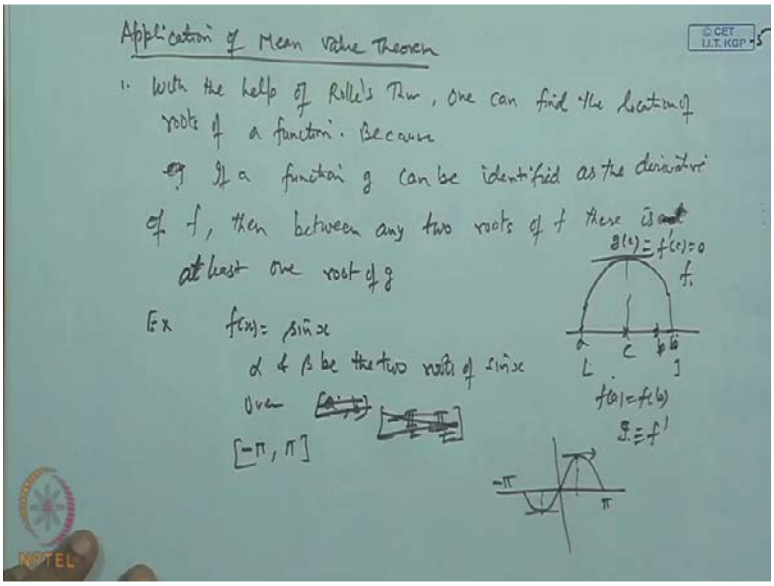

So, first is the Rolle's theorem with the help of Rolle's theorem, one can one can find one can find the location of location of roots of a function of a function for example, for if a function the reason is because sorry because if a function g can be identified can be identified as the derivative of as the derivative of a another function f, then between, then between any two roots any two roots of f there is a there is at least there is at least there is at least there is at least one root of g. Because what is the Rolle's theorem says, if a function f g between any 2 roots of f j if a function f such that f of this is interval a b f (a) equal to say f (b) this is b, this is b so, this is a function f if, the function f we will say continuous over the closed interval a b and differentiable over the open interval a b and at the point a both f a and f b is 0, then there exist a point c in between where the derivative of the function f prime c is 0. It means, if I say g is a function which is equivalent to its derivative so, between any two roots of a b function f there is a root of g because this is equivalent to g c. So, that is what for example, if we lo the function f x which is say sin x, then let alpha and beta be the two roots be the two roots of this function sin x.

The sin over the interval say any interval I take say between the interval say minus say a to b just a to b. Or let us take the exact root because this sin become 0 is 0 sin pi is 0 so, between minus say over the interval minus pi by 2 to pi by 2 let us take this interval. pi by 2 minus pi by 2 it will not help because 1 will be 0 other so, do not take it let us take the interval say minus pi to pi, let us take this interval. So, this is the function sin 0 is 0 sin pi, then minus pi so pi. So, the function f is such which is continuous and differentiable over this interval. Now, there exist at least one root obviously this is one of the point where the derivative vanishes and here this another point where the derivative vanishes.

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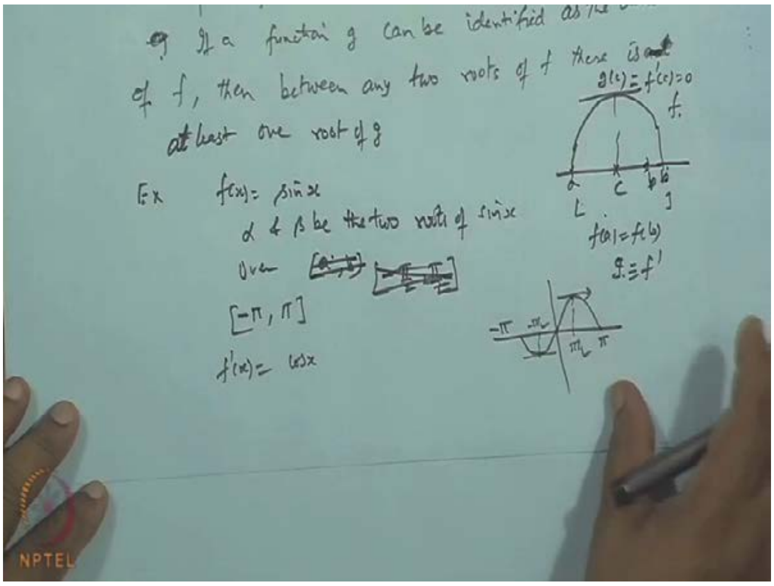

So what is the derivative f prime x? the f prime x is cos x so cos x vanishes at the pi by 2 and then minus pi by 2. So, it shows that between any 2 roots of the sin x there is a root of cos x, and vice versa also if I start with cos x, then its derivative is sin x minus sin x of course, then again between any 2 roots of cos x there is a root of sin x. So, one can locate the roots of the function with the help of Rolle's theorem.

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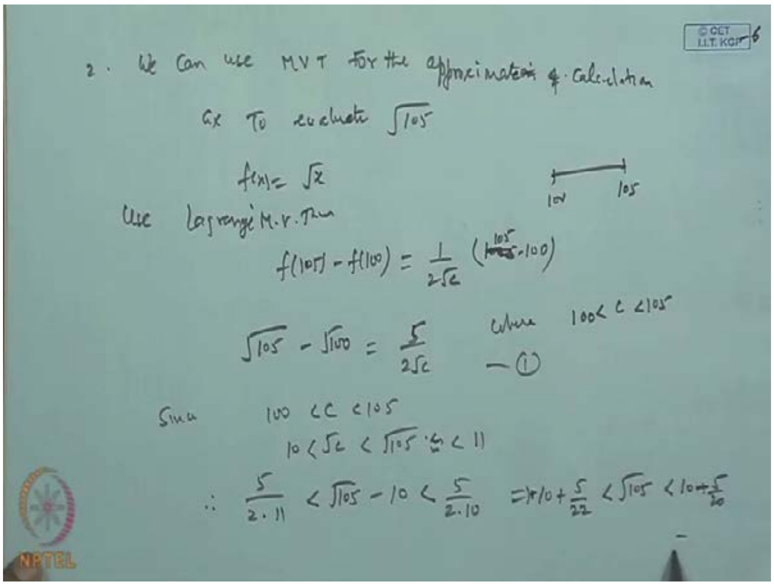

The second we can also use the mean value theorem, to approximate or for the approximation for the approximation of the roots for the approximation of some number say approximate calculation you can say for the approximate calculation let us see how. Suppose, we wanted to find to evaluate this, evaluate under root say 105. Now, if I lo this take the interval suppose, 100 to 105. And the function f x if I take to root x, then apply the lagranges mean value theorem mean value theorem. What we get is f (105) minus f (100) is equal to the derivative of this function at some point c into the length of the interval 105 minus 100. 105 minus 100 where c lies between 100 and 105 where c is this. So, what is this is? This is equal to 105 minus 100 this is equal to 1 by 2 root c into 5. So, 5 by root 2, we want can approximate root 2 now so, since our c lies between 100 and 105, so root c will lies between 10 and under root 105 this is almost approximately or less than say eleven less than 11.

So it means this value so what we get? Therefore, under root 5 105 minus 10 will lie between this 2 bound the bound will be 5 over 2 into 11 and 5 over 2 into 10. Therefore, the root 1 0 5 will lie between this 10 plus 5 by 22, and less than less than 10 this is 10 so minus 10 minus 10 here, because this will go from here and this will come so, 105 is this and this minus say 10 plus both sides so, this is 10 only. And then it will be 5 over 10 plus 5 over 20. Now, this is an approximate value for this. So one can get the value (( )).

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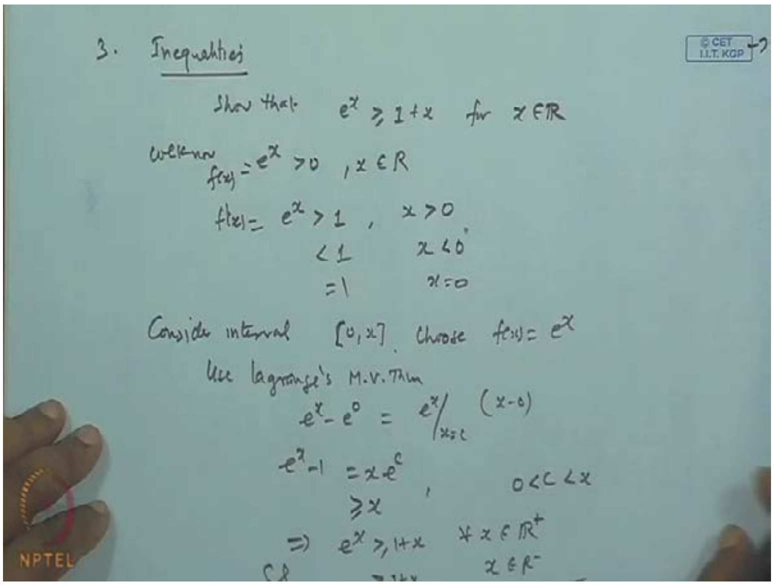

So this another application of this then inequalities can also be established. With the help of this the inequality suppose, I want to establish show that e to the power say x is always be greater than equal to 1 plus x for x belongs to R. Now, we know that e to the power x is always be greater than is positive quantity, is always be positive quantity when x belongs to R. Is it not? 1 plus x or if x is negative 1 by x and its derivatives and its derivative, f prime x is e to the power x itself. So, f prime x will always be greater than 1 when, x is greater than 0 and is less than 1 if x is less than 0. Because it is 1 upon e to the power a. And for the x equal to 0 it is 1. So, 1 occurs if clear? So, this will be the solution. Now, to establish this result consider the interval consider the interval say we take the interval 0 and x.

This closed interval and choose the f x as e to the power x. Now, this function is continuous differentiable in the interval inside the interval a 0 x so, by the mean value theorem lagranges mean value theorem. The value of the function at the end point is equal to the derivative of the function x at the point x equal to c, into the length of the interval x c so, we get what? e to the power x minus 1 equal to e to the power c into x where, c lies between what? c lies between 0 and x. c is greater than 0 so, e to the power c will be greater than 1 so, it is greater than or equal to x. At the most 0 when c is equal to 0 it is 1 so it is z. So, this shows e to the power x is greater than equal to 1 plus x for every x belongs to R positive. Now similarly, we can show for x belongs to R negative it is true. greater than equal to this hence, for all R it is true. So this can be shown. So, this also inequality we can use it mean value theorem. Now there are certain darboux theorem we will require in the darboux theorem.

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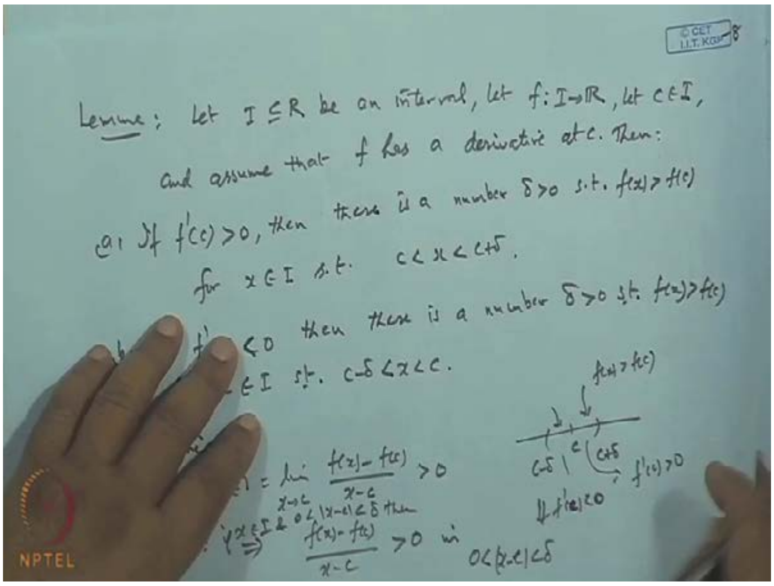

So, in order to prove the darboux theorem, we first need this lemma proof we can skip because let R I which is subset of R, be an interval and let, f is a mapping from I to R let c is a point in I and assume, that f has derivative has a derivative at c, then this lemma says if, derivative of a function at the point c is strictly greater than 0, then there is a number delta greater than 0 such that, such that f (x) is greater than f (c), for x belonging to I such that, it lies in the right hand side of the interval of the interval neighborhood of c. And if f prime x c is negative, then there is a there is number delta greater than 0 such that, f of x f of x will be greater than f c for x x belongs to I such that c minus delta less than x less than c.

So, what this says is suppose, f is a function and c I is a interval c is a point belonging to this and function has a derivative at this point. Now, if the derivative is strictly greater than 0, then what he says is we can always find a right hand side neighborhood of this c where, the function will have a positive value, where the function f x will be always be greater than the value at the point c. So, here in this interval the function f x will always exceed the value at the point c. It means, that f (c) will behave as the lowest point in this interval If derivative is positive. And if this f (x) is and if derivative is negative, then in the left hand side of this neighborhood, the value of the function will always be positive if here. So, here if the derivative is negative at the point c, if the derivative at a point c is negative, then we can get this interval neighborhood left hand, where the function is exceeding the value f c if, derivative comes out to be positive at point c, then we get the right hand side interval, where the functional value exceed the value at the point c.

So, using this lemma we can prove the darboux result. In fact, the proof of this is very simple since, the derivative exist since this derivative f prime c exist which is the limit of this f x minus f c over x minus c, this exist. And it is given to be say greater than 0. Now, when the limit greater than 0 x extremum c, then obviously this entire thing must be positive. So, this is only possible when, limit is strictly greater than 0 its only possible when the this quotient should be greater than 0 otherwise, if it is negative we cannot get the limit to be strictly equal to 0.

But, x is if I choose x in the interval for so, there exist this is greater than 0 in some neighborhood 0 less than x less than mod x minus c less than delta. So, means there exist if it is greater than 0, then there exist a delta greater than 0 such that, if x belongs to I interval and 0 less than mod x minus c less than delta, then this holds. So, if x satisfy the condition if x belongs to I and satisfy the condition x is greater than c, then from here we get implies so, f x must be greater than f (c) otherwise, this will be negative. So, that proves the result for the first part similarly, for the other.

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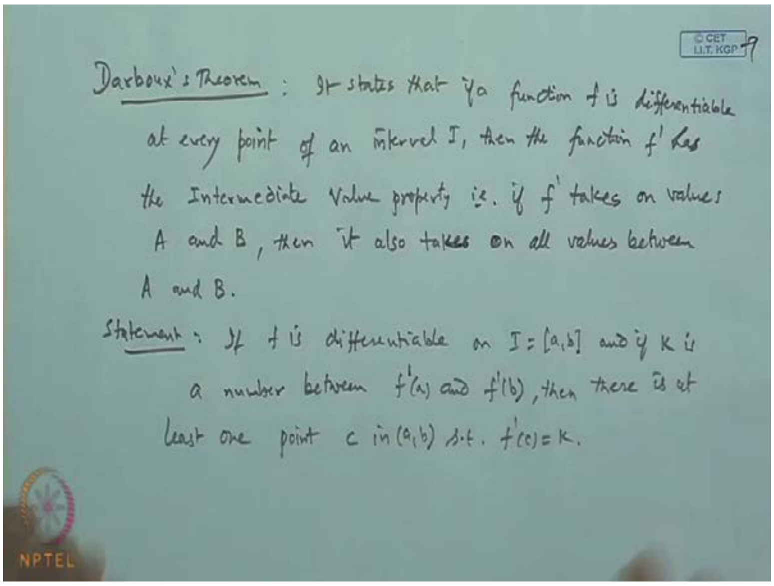

So, this we can see now, darboux theorem which is darboux theorem. Basically, what is the darboux theorem is? it states is states that if a function if a function f is differentiable function f is differentiable at every at every point of an interval say I, then the function f prime that is the derivative of this function f prime has the intermediate value a has the intermediate mediate value property. It means, that is the meaning of this is that is if f dash takes if f dash takes on values a and b

a and b, then if it then it also takes it also takes on takes on all values between a and b a and b. So, this is our so, exact statement is, we can see exact statement if f is differentiable on the interval I, as close and bound interval I and if k is a number k is a number between f prime a and f prime b, then there is at least one point c in the interval a b such that, the derivative of the function at the c is k. It means, if f is differentiable through out this means, derivative of the function exist including the derivative at the end points, then if I picked up any arbitrary number any number which lies between f dash a and f dash b, then there will be a some point where the derivative of the function at that point will coincide the number k. So, it can attend all the values in between a and b which is taken with this. So, which is maximum and minimum value.

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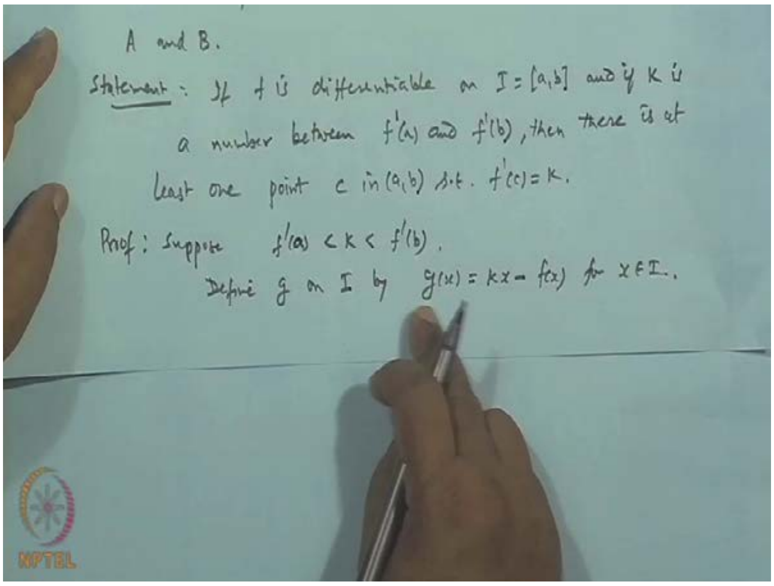

So, proof is like this suppose that, k lies between suppose, k lies between f prime a and f prime b. Now, we define the function g, on I by g x equal to k x minus f x for x belonging to I. Let us define this. Now, since function f is continuous and differentiable so, g is also continuous and differentiable.

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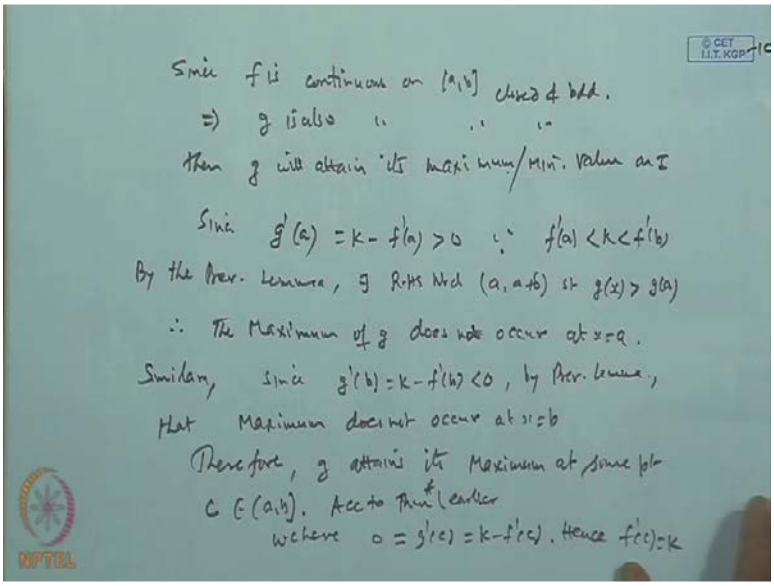

So, since f is continuous on the closed and bounded interval, this is closed and bounded interval so, this shows g is also continuous on the closed and bounded interval. So, if it is continuous on a closed and bounded interval, then it will attain its maximum value on the interval, then g will attain its maximum value similarly, minimum value also maximum or minimum value maximum and minimum value of course, on I so here, we require the maximum and minimum when this is less than 1. So, we this I. Now since, our derivative g prime a, that is by definition is coming to be what? k minus f prime a this is why just differentiate, and this g prime a is positive why? Because our chosen k lies between f prime a and f prime b so, this is positive this is positive.

Now, it follows from the previous lemma so by the previous lemma, what he says? if the derivative is positive, then there will be a 1 neighborhood. Is it not? So, if the derivative is positive, then there is a number delta such that, in the right hand side of this neighborhood the function f x will be greater than this. So, we get from by the from the previous lemma we can say, that the maximum of g by the there exist a right hand side neighborhood say, a to a we can say a to a plus delta. Such that, the derivative the functional value g (x) g (x) is greater than g (a).

Therefore, the function g cannot attain the maximum value at the point a. So therefore, the maximum value maximum of g does not occur does not occur at the point x equal to a. Because it already violate, if it is maximum it must be g x must be less than equal to g a, which is not true not occur similarly, since g dash (b) that is equal to k minus f prime b is negative. So, again by previous lemma again there is a left hand side, where the value will be greater than the less than this. So, again this shows that g by the previous lemma that maximum does not occur that maximum does not occur maximum does not occur at x equal to b. So, neither it occurs at x equal to a nor at x equal to b. Therefore, but maximum should be attained because it is continuous over the closed and bounded interval therefore, g will attain g attains its maximum value maximum at some interval, at some point at some point c which lies in the interval a b.

Now, again a function g is such which has a derivative and maximum is attained at some point. So, according to the theorem according to we have discussed is according to the theorems earlier, we have seen that there exist a point we have 0 which is less than equal to g dash c, which is k minus f prime c, and hence f prime c is k. According to the theorem, this theorem I am putting star and this star is we can say this theorem is there.

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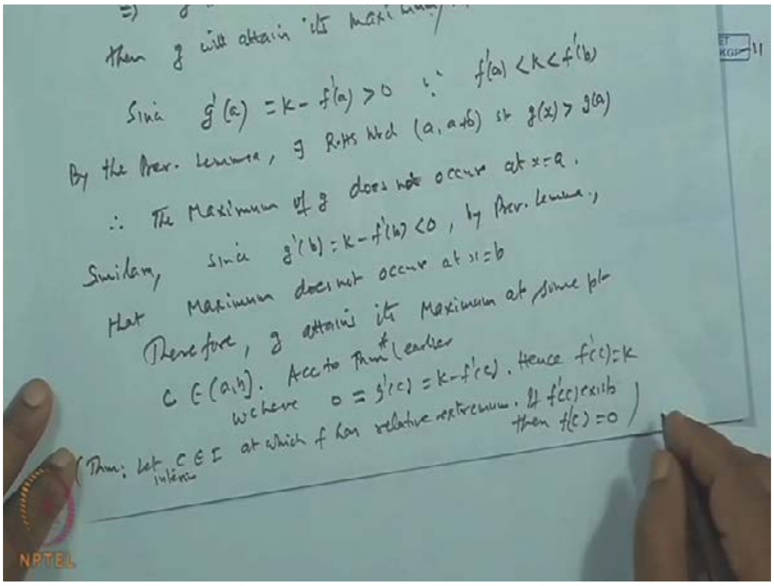

The theorem says which is interior theorem is that let c be an interior point of the interval I interior point and, then at which the function has a relative extremum. At which f has relative extremum, then if the derivative if the derivative of the function at this point c exist, then the value of this must be 0. This was the result which we have already proved so, using this result we get this therefore, this completes.

So, thank you very much we have not covered the hospitals next lecture we will do it.

Thank you very much thanks