A Basic Course in Real Analysis
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Lecture - 35
Mean Value Theorem (Contd.)

So, we will discuss the Mean Value Theorems, Rolle's Theorem, Lagrange's Mean Value Theorem, and Cauchy's Mean Value Theorem, Generalized Mean Value Theorems.

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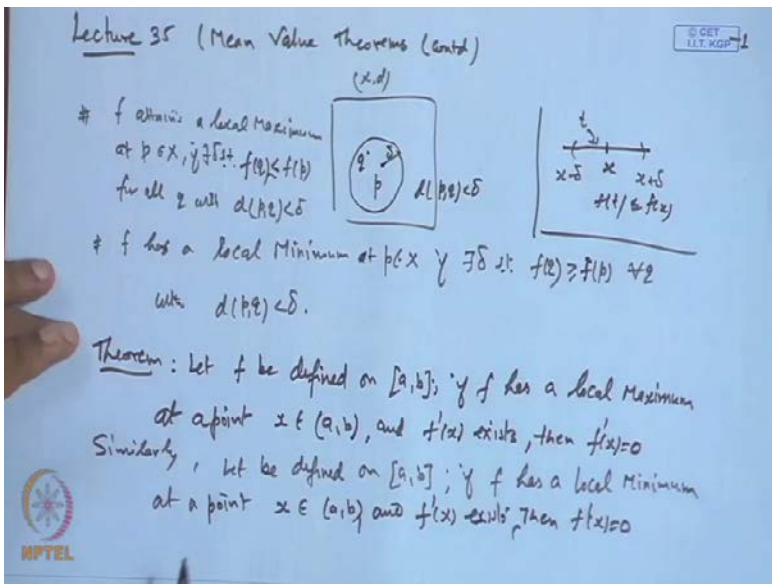

Yesterday, we have discussed about the local maxima, and local minima, and we have seen a function f defined over an interval or metric space x d, f is a function, this is real valued and defined on a metric space c, then we say f has local maxima at a point p, if there exist a delta. Such that the value of the function at the point p is greater than the value of the any number q, which lies in the delta neighborhood of p.

If it lies in the delta neighborhood of p, and if the image of any point q is less than or equal to the value of the function f p, then we say function f attains or has a local maximum at p, if there exist p belongs to x, if there exist a delta. Such that this inequality holds for all q belonging to this, that a d of p q is less than delta. Similarly, when you say it has a f has a local minima at p belongs to capital X, if there exist a delta, such that f of q is greater than equal to f of p, for all q with d of p q is less than delta.

So, this is the case of local maxima or local minima and intervals in a similar way. When we say a function has a local maxima or local at the point a, it means there is a neighborhood x minus delta x plus delta, such that whenever any point t lies in l then f of t is less than equal to f x for all t belonging to this range, if it has a maximum local maxima, and similarly, for the local so with this we have a result which is useful in establishing the mean value theorems. The result is: let f be defined on a closed in and bounded interval a b, and if f has a local maximum at a point x belongs to the open interval a, b, and if the derivative f prime x exist at this point, then the derivative must be 0.

So, what it says is, if the function attains a local maxima at a point x in the interval a, b, where, the function is totally defined and also the derivative of the function at the point exist, then derivative must be 0. The same case is the similar statements similarly, if let f be defined on the closed interval say a, b, and if f has a local minima, at a point say x belonging to a b, and if the derivative of the function exist at this point, where, it has a local minima, then the derivative of the function is 0. So, this is the condition, at the point of local maxima or local minima, if it attains the derivative exist, then the derivative must be 0.

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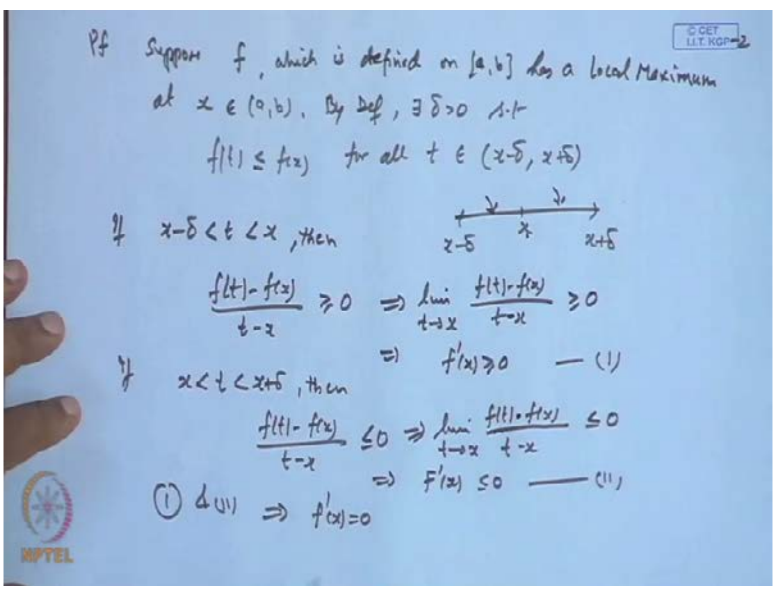

Let us see the proof of this, the proof you say, suppose the function f which is defined on the closed and bounded interval a, b has a local maximum at the point x, belonging to the open interval a, b. Suppose, let us see, so what is the definition of this by definition of the local maxima; there exist a delta is greater than 0, such that the functional f of t will remain less than or equal to f x for all t belonging to the neighborhood of x, that belonging to this. That is, this is the point x, and here we are having the neighborhood x minus delta and x plus delta. So, if the function a attains a local maximum at this point x, corresponding to this, we can identify a delta and a neighborhood such that, whenever, the t lies here, then the value of the function is also less than equal to x or t lies here, the value of the function will also be less than equal to f x.

So, if suppose t lies between the left hand interval of x, then the ratio f of t minus f of x over t minus x. Now since, t lies between this left hand interval, and function attains the local maxima at the point x. So, f of t will be less than equal to f x. So, this part will be negative, and t is less than x. So, again this part is negative. So, total is greater than equal to 0. So, this implies that the limit of this, as t tends to x of this quantity that is f of t minus f of x divide, by t minus x which gives the derivative of the function at a point x is greater than or it cannot be negative. Therefore, the derivative f prime x is greater than or equal to 0. This is the first one. Now, if the t lies between the interval right hand to toward the right of x, then the ratio f of t minus f of x over t minus x, when t lies here, the f t is always less than equal to f x. because x is local maximum point and t is greater than x.

So, this is positive. So, this will be less than equal to 0. Therefore, taking the limit as t tends to x of the same f of t minus f of x divide by t minus x will be less than equal to 0, and that will implies the derivative of this is less than equal to 0. So, 1 and 2 combined will give you that derivative will be 0 at that point. So, if it is a local maxima, the derivative exist, then it has to be 0 there, or if it is a local minima and derivative exist. But if the derivative does not exist, then also the function will attain the maxima, minima also.

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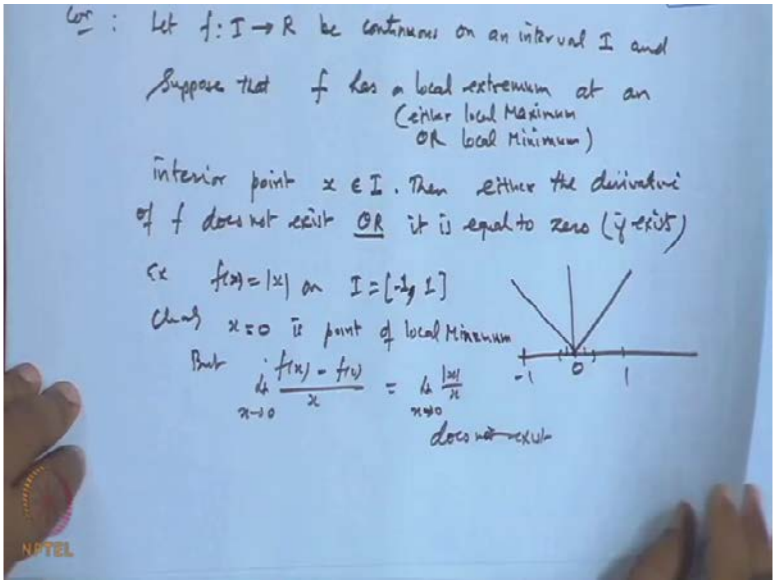

So, that is what we have the corollary here, as a corollary says, that if let f is a mapping from I to R, I is in interval be continuous on an interval I, and suppose and that f has a local extreme, when you use the word extremum is either local maximum or local minimum. So, a one word is used for that extremum. So, if it has a local extremum is either it has a local maxima or local minima. So, if it has a extreme at an interior point, say x belonging to I, then either the derivative of f does not exist or it is equal to 0, if exist.

So this part, we have already shown is it not, that if the function attains the local maxima or local minima, and if the derivative exist, at then at that point derivative has to be 0 now, what is says this result is that once it has a extreme minimum point, the point of the extreme, the derivative either will not exist or if it exist it must be 0. So, this part, we have for example for this, if we look the function f x which is equal to mod x on the interval say I which is closed and bounded interval minus 1 to 1, then if you look this graph of the function the graph of the function will be this, here is minus 1 here is plus1.

So, 0 is a point which as a local minimum point 0; therefore, 0 the clearly, the x equal to 0 is the point of local minimum point, in fact it is a minimum point of this, the reason is when you take any point closed in the neighborhood of the 0, if you picked up any point of this then the value at the point 0 will always be less than equal to value at any point arbitrary point in this neighborhood.

So, 0 becomes the minimum point or local minima we can say, but the function f x minus f 0 divide by x as x tends to 0, that is nothing but what mod x by x when x tends to 0, that does not exist that we have already discussed, because when x is positive the value will come out to be 1, when x is negative the value will come out to be minus 1. So, the limit does not exist. So, the derivative is not defined is does not exist, but the point correspond to a maximum, a minimum point of it. So, this is one of the ways, Now, this is the one way we have discussed it, that if the function attains the local minima, it has to be.

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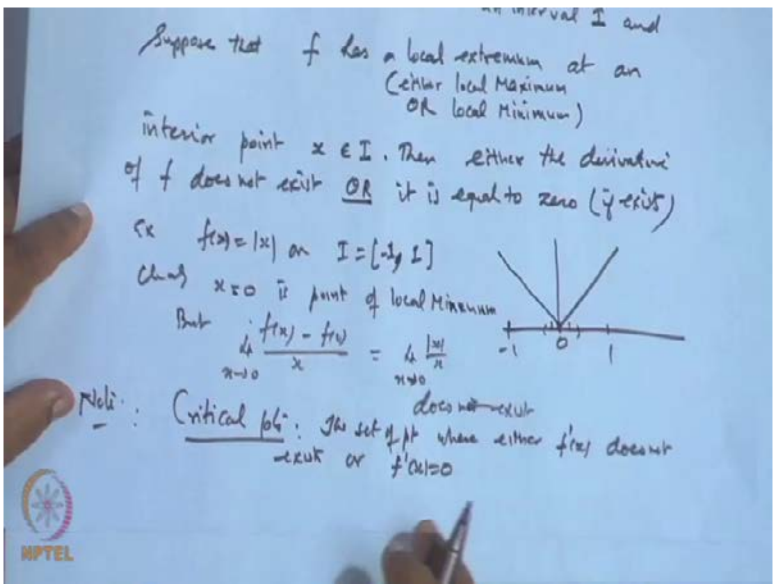

So, the criteria critical points we can say. So, as a note the critical point; critical points means, if the function f is such the set of all points set of points, where either derivative does not exist or equal to 0 are known as the critical points. So, obviously, the critical points are only the point where the maxima, minima may occur. So, this is the way of function is given first we will find the derivative, if it exist put it equal to 0. So, you will get the set of points which are known as the critical point, and then we have to test the local maxima, and local minima at this point it means, if the function, if the point is not a critical point, then there is no question of discussing the critical maxima, minima in fact, there will be no maxima, minima will be occur at the point other than the critical points. So, that is the important part here.

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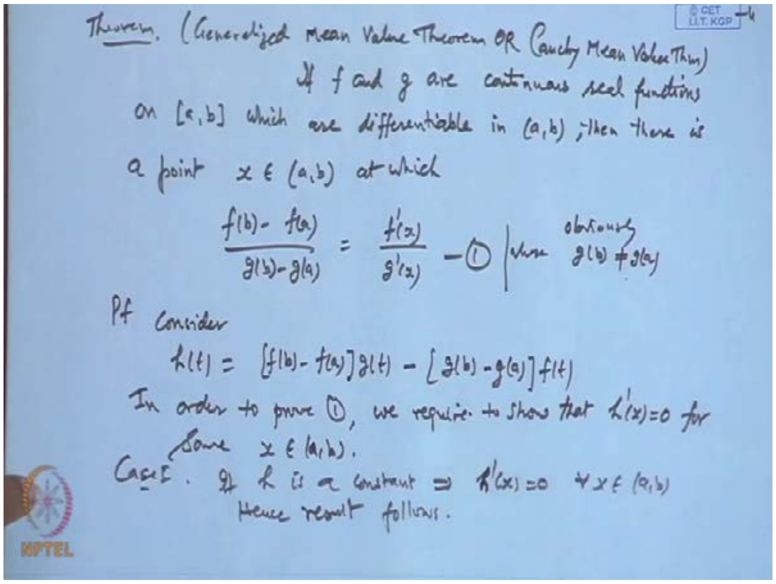

Now, we come to the result that is called generalized mean value theorem or also known as Cauchy mean value theorem, what this statement says, if f and g are continuous real functions on the closed, and bounded interval a b, which are differentiable in the open interval a b, then there is a point x belonging to the open interval a b at which f b minus f a over g b minus g a is equal to the derivative of the function f prime x over g prime x off course, here we are case, we will take up 2 cases, when g b is different from g a, and g b is not equal to g a, that we will take.

So, g b is not equal to g a, they are continuous function and attains, where the g b is different from g a derivative, because that we will said it is not required, but we will just put it why it is not required that we will come to, when we go for the Rolle’s theorem in fact, the Rolle’s theorem says that, if the g b equal to g a and g is a continuous and differentiable, then there must be some point where the derivative will banish.

So, derivative banishes means this denominator will be 0. So, it is not defined at all. So, obviously, this is true, we can say in fact, this will now, here one thing which we can judge, we don’t require the differentiability of the function f or g at the end point a and b, because we don’t need, we need only the derivative of the function in the interval a b in the point interior to a b.

Let us see the proof. Let us consider a function h t as f of b minus f of a multiply by g t minus g of b minus g of a into f of t, consider this function now, to prove our results, what we want that there exist a point in order to prove the result 1, we have, we require to show that that there exist that the derivative of the function h dash x 0, for some x belonging to the open interval, this we needed. So, case one, let us say, if our function h is a constant function, once it is a constant function then; obviously, the derivative of this function will be 0 for every x belonging to the open interval a b, hence the result follows, nothing to prove.

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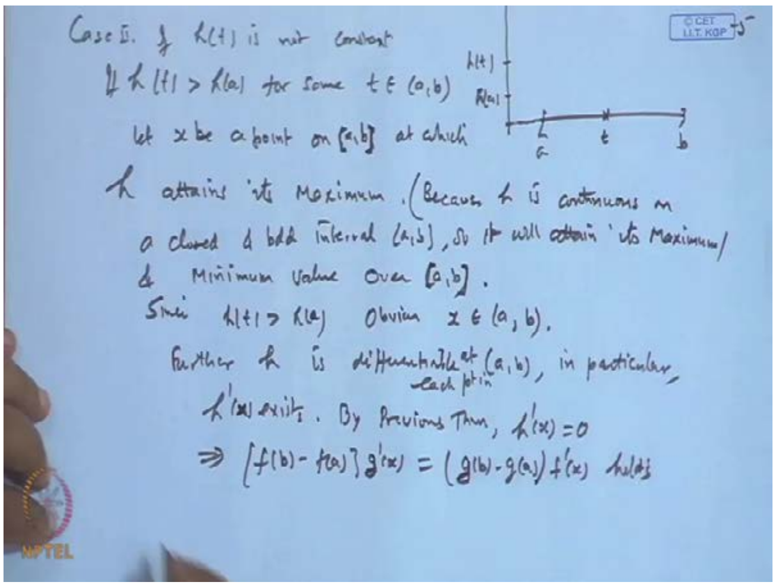

Now, case 2; if h is not constant, if h is or h t is not constant, let us take the 2 cases, suppose h of t is greater than h of a for some t belonging to the open interval a b, this is our closed interval, and t is somewhere here. So, the image of the t at this point is; say, here this is our h a, and here is somewhere, say suppose, h t is there, and let take, let x be a point on the closed interval a b at which h attains its maximum, it attains its maximum value Now, why it so the reason is because this will exist, because h is giving to be this function h, we have defined like this f t minus f a g t minus g b minus into f t, f and g both are giving to be continuous over the close interval. So, h will also be continuous over the close interval a b, h g and f is differentiable over the open interval h.

So, h will also be differentiable over the open interval a b, and since, h is continuous over the close interval a b, and close and bounded interval. So, it will attain its maximum value, and minimum value at least at some point in the interval a b, that result we have seen every continuous function on a bounded, on a closed and bounded set I will attain the maximum or minimum, and minimum value at some point. So, because h is continuous on a closed and bounded interval, say a b.

So, it will attain at some points, attain its maximum and minimum value over the interval a b, that is and so we can get a x belongs to this. Now, we have seen here is a b, the point x over a b, this is first, since we have taken h of t is greater than h of a so; obviously, the point x will not be a point, it will be in the interval a, and in a similar way, we can say it is lying between a b. So, x lies between this. Now, further the function h is differentiable inside the interval a b, differentiable at h point in the interval a b, at each point in the interval a b.

So, in particular the derivative h prime x exists. Now, use the previous theorem, says this result we have proved that, if the function, if f is defined over a closed interval a b, and f has a local maxima at some point, and the derivative exist, then the derivative must be 0. So, according to the previous theorem.. So, by the previous theorem, the derivative of the function at this point must be 0, and this will implies that f of b minus f of a into g prime x is equal to g of b minus g of a into f prime x holds.

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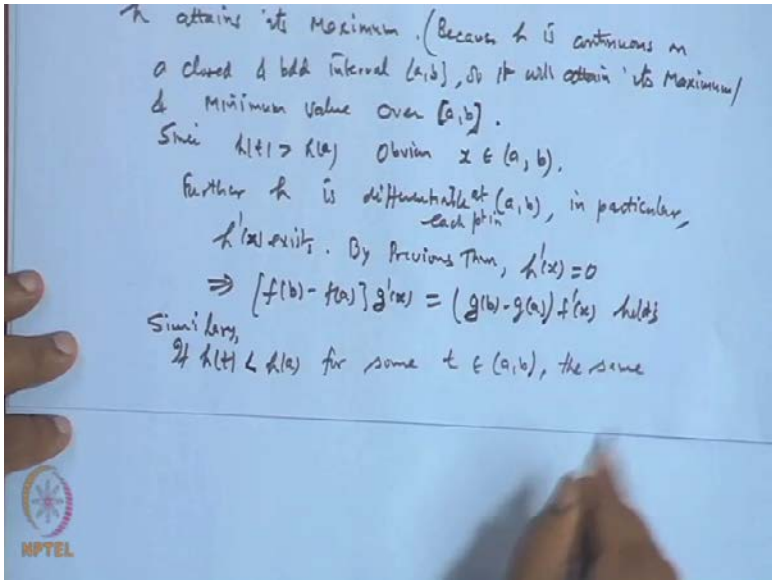

Similarly, if we assume if h t is less than h of a, for some t belonging to the interval a b then the same argument, by the same argument.

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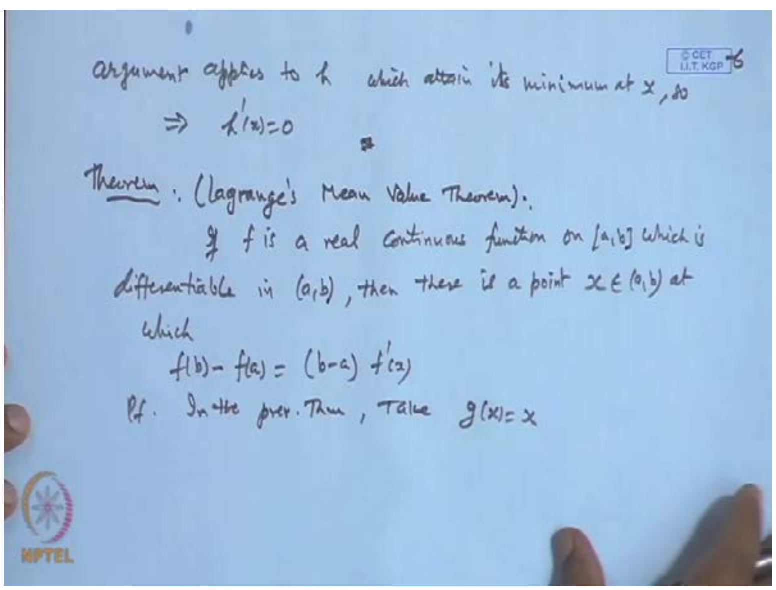

Then by the same arguments applies to the function h to the function h, which attains its maximum value, its minimum value, minimum at the point x, hence we get this implies, we get the derivative h dash x will be 0 in the similar, so derivative of this 0, and this proves the result.

So, this is very now, this theorem which we call it as a generalized mean value theorem in fact, it is as particular case, we can tried our result for this function, for Lagrange's mean value theorems, and Rolle's Theorem

So, next result is we get Lagrange's mean value theorem; what this says is, if f is a real continuous function on the close interval a b, which is differentiable in the open interval a b, then there is a point x belonging the open interval a b at which the f of b minus f of a equal to b minus a into the derivative of the function f prime x. The proof follows from the previous in the previous theorem, take g x is equal to x and the result follows. So, the result follows immediately, then come to the next; what is the meaning, if this lets geometrical meaning of the Lagrange's mean value theorems; what they says.

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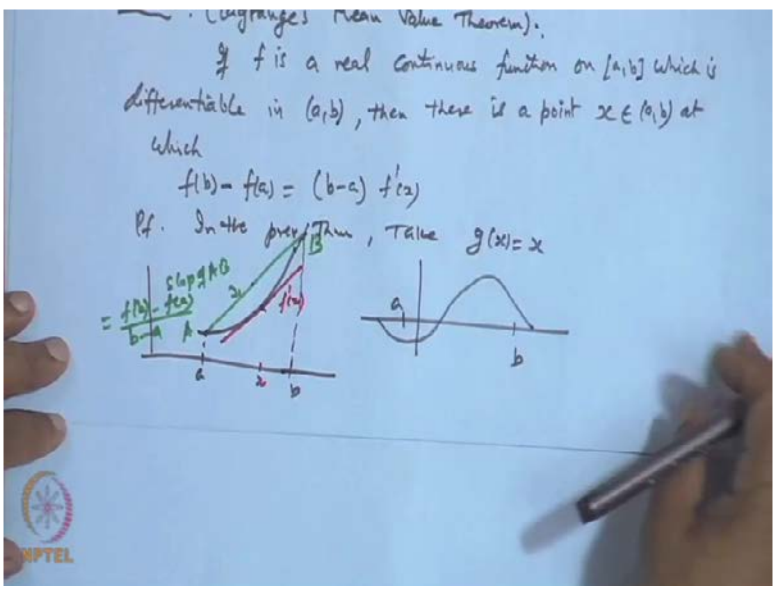

Then suppose, we have this function f x over the interval say a b say here is a here is suppose I say b. Now, let us take another graph, because that is very a, and b its, let's take this graph, then its more clear, suppose I take suppose I take this graph, here is say a, and this is say b. So, this point is corresponding to say like this. Now, if I draw the cord joining these points, this is the cord the slope of this cord a b is nothing but what, f of b minus f of a over b minus a this is the slope of this cord.

So, what this result says is that if a function f, which is continuous function over the closed interval a b and differentiable in the open interval a b, then there will exist a point x in the interval a b, such that the slope of the segment joining the end point of this curve will be parallel to this line on the curve that is this slope of this, and slope of this is the slope is equal to f base x, this is the slope of this, and here this is the slope. So, both will coincide. So, what this Lagrange's mean value theorem is gives that result regarding the slope. Now, this is the case when both f a, and f b are equal.

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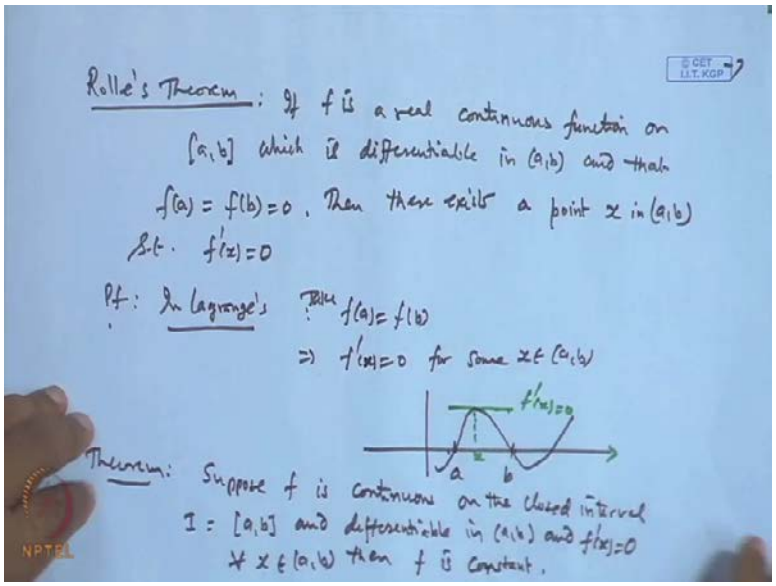

So, we can choose the Rolle's Theorem later on. Now, from here we can also drive the results which is known as the Rolle's Theorem, the statement of the Rolle's Theorem is suppose that f is continuous, if f is a real continuous function on the close and bounded interval a b, which is differentiable in the open interval a b, and that the value of the function at the end point coincides f a equal to f b, and say equal to 0, we can even take without 0 also it will work 0, then there exist a point x in the interval a b, such that the derivative of the function at this point is 0.

The proof follows again from the Lagrange's mean value theorem, in the Lagrange's case, if we take f a equal to f b then implies the derivative of the function must be 0 for some x belonging to the interval a b, that is the again the meaning is suppose, we have a curve, say suppose we have a curve of this type, say like this and here is the point a, this is the point say b, function is continuous, and differentiable over the open interval and at the end point both are attaining the same value, and equal to say 0, then the according to the there will exist a one point x, where the derivative banishes f dash x will be 0 at this point means, the line is parallel to x is of x slope will be 0.

So, this shows the; now, using this, we can come to know consequence of this result, we can get one more result, what this result says is suppose f is continuous on the closed interval, closed interval i and f is differentiable and differentiable in the open interval i at this point in the open interval i and the derivative f prime x equal to 0, for every x belonging to the open interval a b, then f is a constant function constant, and obviously, the proof is very simple, because if we apply the this theorem Lagrange's mean value theorem, then over the interval we get ah x.

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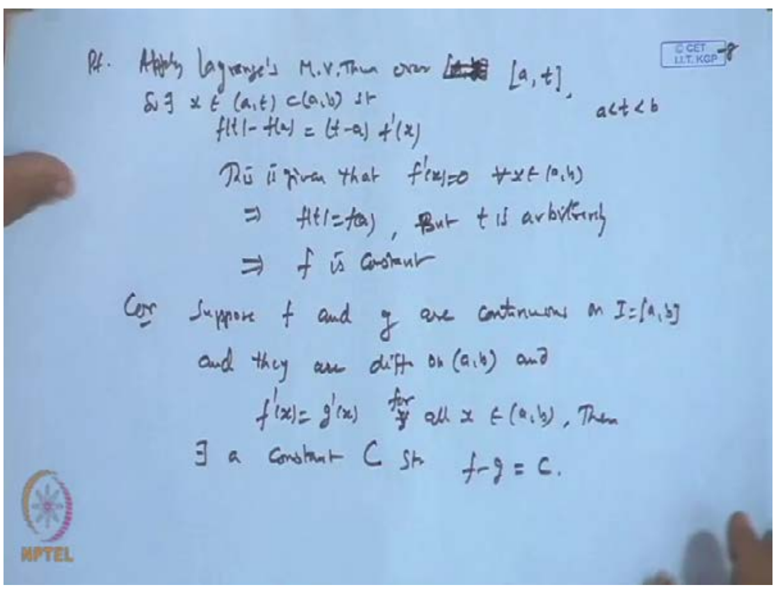

So, proof is apply Lagrange's mean value theorem over the interval say a x ah there exist some point say a x is only. So, let us take the point a y or a t, let us take the point a t where the t is lying between a less than b ok t is this over this interval. So apply the this, then what you get is f of t minus f of a equal to t minus a into the derivative, So, there exist a point x belongs to the interval a t which is off course, subset of a b, such that this result hold, now if at this point it is 0, but this is given that the derivative banishes for every x belongs to the interval a b. So, this implies that f of t equal to f of a, and for every t, but t is arbitrary.

So, this implies the function f is constant function, and as consequence of this result, we can say is a corollary that suppose f and g are continuous on the closed interval a b, and they are differentiable in the open interval a b, and the satisfy the condition that f prime x equal to g prime x for all x belonging to the interval a b, then there exist a consequence, then there exist are constant capital C such that the difference of this is equal to c means, they differ by a constant. So, proof follows immediately. So, I will not go for this proof further.

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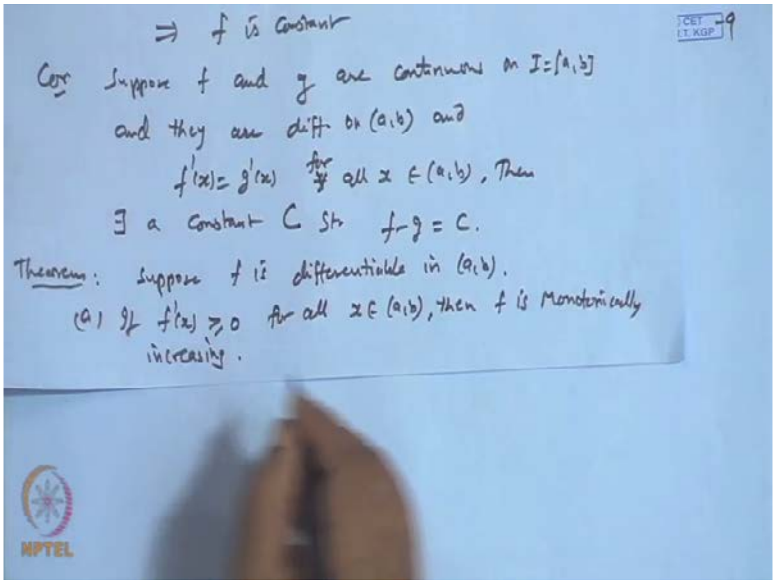

Now, using this thing, we can also drive the results for the function which are monotonically increasing, and decreasing. So, suppose f is differentiable in the open interval, say a b, then the following result holds, then number 1; f if f prime x is greater than equal to 0, for all x belonging to the interval a b, then f is monotonically increasing monotonically increasing.

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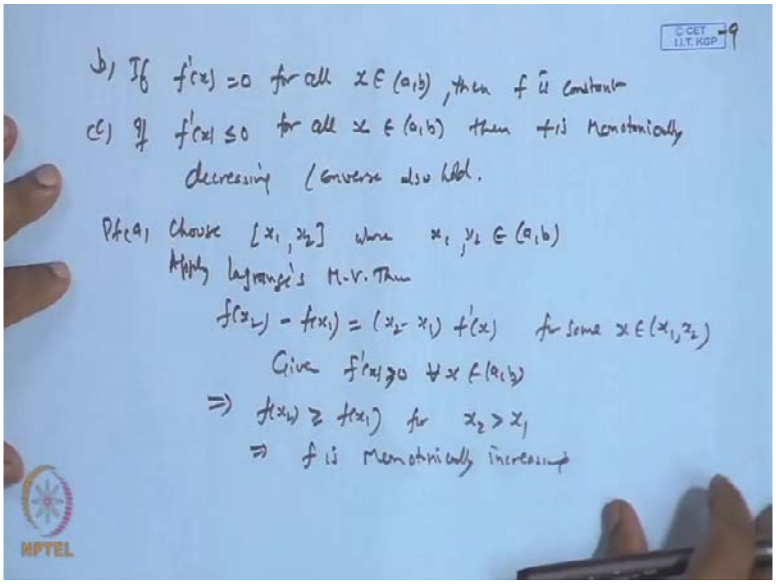

b if derivative f prime x equal to 0 for all x belonging to the a b, then f is constant which is already shown, and c is if the derivative of function is less than equal to 0 for all x belonging to a b, then f is monotonically decreasing, the converse is also true here, if I take here the converse also hold that, if f is monotonically decreasing, then the derivative will be negative. The converse is holds also hold, and here also converse hold.

So, we get proof is very simple, just we take the choose the interval, say x 1 x 2, choose x 1 x 2, where x 1 and x 2 these are the points of the interval a b, then apply Lagrange’s mean value theorem, then f of x 2, this is proof for a; f of x 2 minus f x 1 is equal to x 2 minus x 1 derivative of the function f for some x belonging to x 1, x 2, now it is given the derivative is greater than 0 for all. So, this is given the derivative f prime x is greater than 0 for all x in the interval a b. So, this implies that f of x 2 is greater than or equal to so it is greater than or equal to f of x 1, when for all for x 1 satisfying this condition, x 2 is greater than x 1. So, this shows the function f is monotonically increasing function, the converse of this also true.

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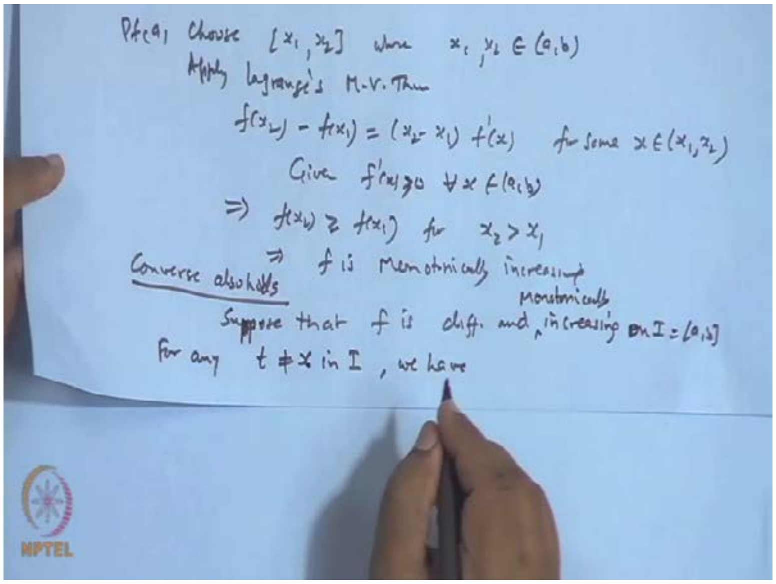

The converse also holds, why because, suppose f is differentiable, and increasing monotonically increasing in the interval i or on the interval i which is say our a b, i is the interval a b on the interval i. Now, take a point for any point t, which is different from x in i.

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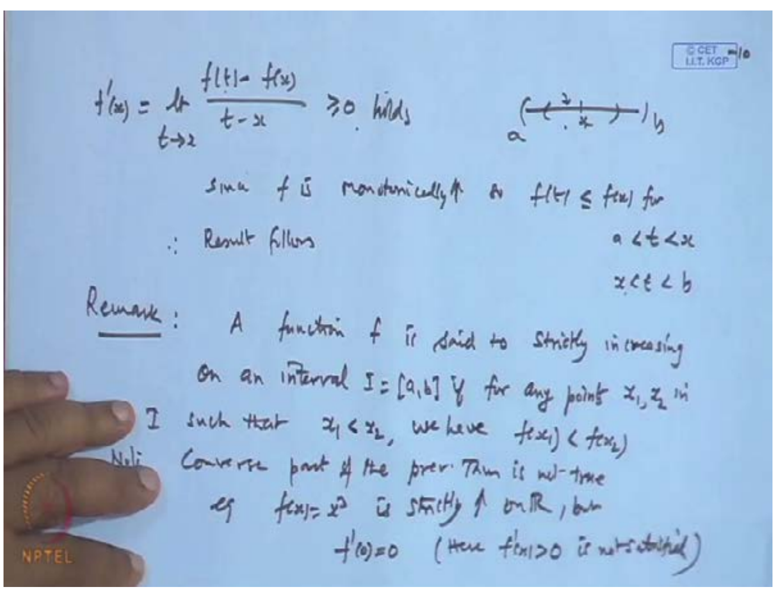

We have f t minus f x over t minus x. Now, if I take this is the point x, here i am taking interval suppose, and t is point somewhere here, if t is in this interval, then x is greater than t. So, t minus x will be negative is a not, t minus x will be the limit of this, the limit of this t tends to x because this limit exist, it is differentiable, and it is the derivative of the function at the point x, and since it is monotonically increasing. So, f t will be less than f x. So since f is monotonically increasing.

So, f of t is less than f of x for t lying between this, say interval a b here. So, this is less than 0; this one will be less than 0. So, this entire thing will be greater than or equal to. So it is greater than equal to 0, if t lies between this interval, then what happened, this will be if when t lies in this interval then already t minus x that is equal to what, this will be greater than equal to 0, because x will be here is it not. So, we can write it f t is a f of x is less than f t. So, this is positive

So, basically this holds therefore, the result follows. Similarly, for the second case we can go so we are not going to them now, here is a remark, we can say a function f is said to be is strictly increasing on an interval say I if for any points for any points x 1 comma x 2 in I, such that x 1 is is strictly less than x 2, then we have f of x 1 is strictly less than f of x 2, then we say the function is strictly increasing now, when we prove the converse part of the previous result.

The note is the converse part of the previous theorem is not true, that is if the function is strictly increasing function that you cannot say the derivative will be strictly greater than 0 for example, if we take the function f x which is x cube from r to r is strictly increasing on r, but the derivative of the function at a point 0 is 0. So, what we say the function will not be (( )). So, here the function f 1 a x is strictly greater than is not satisfied, though the function is strictly increasing function. So, that is the important point which I will second remark which

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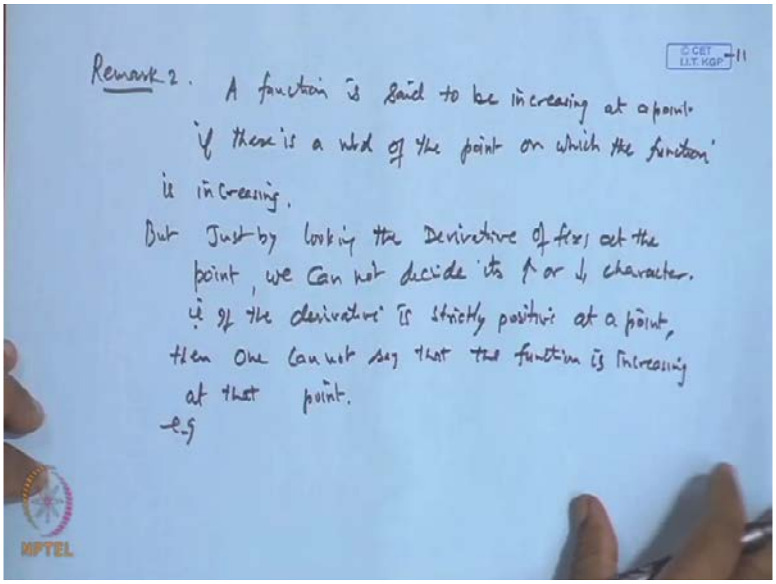

I want to give it, and it is interesting also. The remark says, when we say the function is increasing at a point, and then it has no meaning. Then it means, there exist some neighborhood in which the function is increasing. When, we say a function is said to be increasing at a point, if there is, a neighborhood of the point, on which, the function is increasing.

So, increasing means, the derivative is strictly positive, and then the function is increasing at this point. But just by looking at the derivative of the function f x at the point we cannot decide its increasing or decreasing character. That is, if when we say, if the function is its point, if the derivative is strictly positive, at a point, then the function is increasing at this point. One cannot say that the function is strictly increasing at that point.

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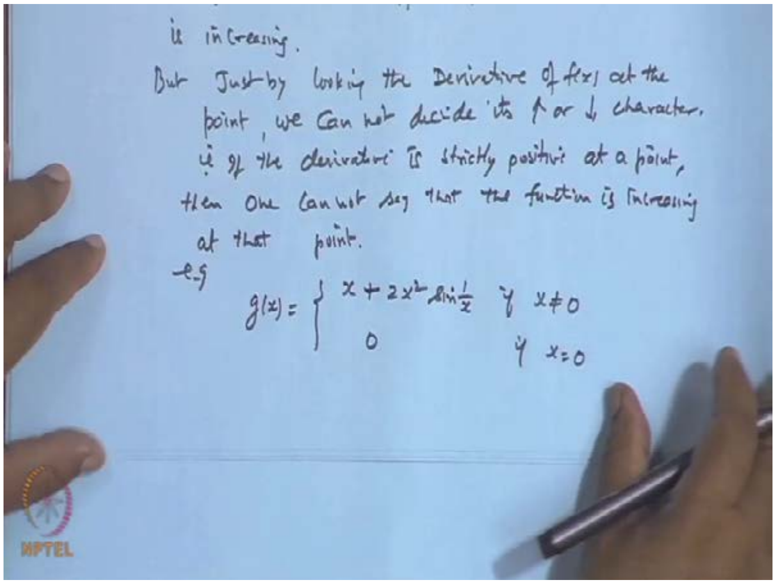

For example, I take a function g x, which is defined as x plus two x square sin of one by x, if x is not equal to 0 and equal to 0, if x is equal to 0.

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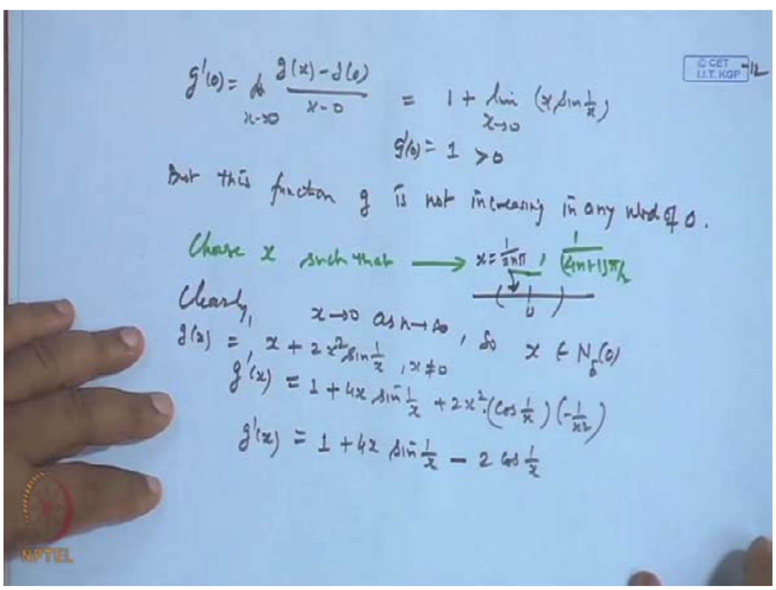

Now the derivative of the function g prime 0 is g x minus g 0, over x minus 0, limit x tends to 0. So, that comes out as 1 plus limit of this x sin 1 by x as x tends to 0 and this limit comes out to 0 as g prime 0. So, g prime 0 is 0 which is strictly positive. But this function, g is not increasing in any neighborhood of 0. This neighborhood consider the point, say here, the point I am choosing as, x equal to 1 by 2 pi and also let us take another point, x is equal to 1 by 4 n plus 1 pi by 2. Now, both this point clearly choose x, such that x is either this or this, and then x tends to 0, as n tends to infinity. So, these are the points in the point neighborhood of the 0. So, x belongs to the neighborhood of 0, with a suitable radius, say, and delta.

The derivative, g prime x what is this value? If we look at the derivative, the function g prime 1 upon 2 and pi. What is the function is this. So, when at this point when you find the derivative of this g dash x is function, because the derivative g is this function x plus 2 x square sin 1 by x when x is not equal to 0, so when we differentiated directly you get 1 plus 4 x sin 1 by x plus 2 x square of cosine 1 by x in to minus 1 by x square. Now, the value is g prime x is comes out to be 1 plus 4 x sin 1 by x and minus 2 times cos 1 by x.

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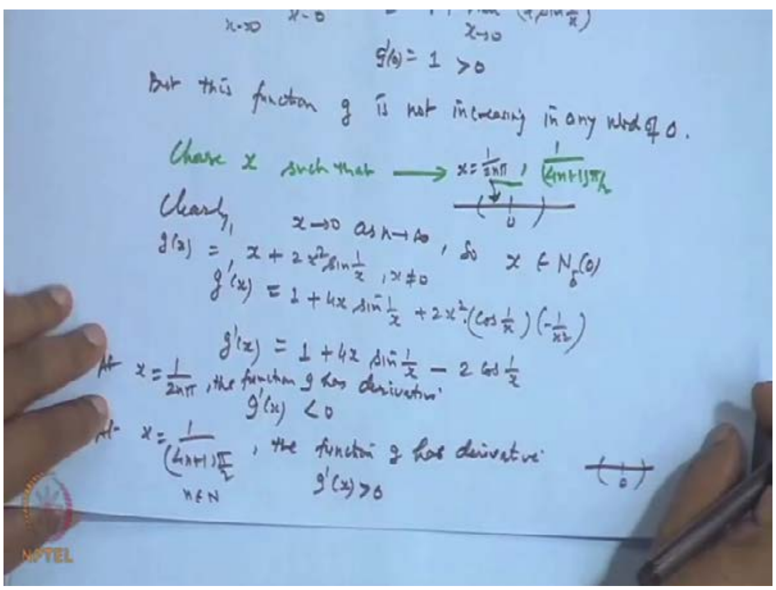

Now, you see, if I take x equal to 1 by 2 n pi, that derivative g prime x, this is any integral multiple sin is 0 sin 1 by x becomes 0. So, this part is not there cos of 1 by x, when we take the multiple of pi 2 pi then it is always be cos 0 is 0 cos 2 pi is 1. So, it is always be 1. So, value will be minus 1, so it will be negative for at this point and if we take x equal to 1 by 4 n plus 1 pi by 2, then in that case, what happens is at this point the function g has a derivative negative. So, at this point, the function g has derivative, that is g prime x will be positive. Why it is positive? Let us see. If we look for the odd multiples pi by 2 with n is 1 2 3. In fact, then n pi by 2 so on for this will be odd multiple of pi by 2, this will go to 0. So, here it will be positive value 1. Here, we are always getting the positive value.

So, this is always positive when a positive integer belongs to N, it is positive, therefore, in the neighborhood of the 0, the derivative is negative as well as positive. So, neither it is increasing, nor it is decreasing function or though the function, which we are taking g prime at the point 0. What is the conclusion if the derivative of the function at certain point is positive or negative? We cannot conclude its increasing or decreasing, until we are sure that in the neighborhood of the function has a criterion of increasing or decreasing.