**A Basic Course in Real Analysis**
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**Lecture - 34**
**Differentiability of real valued function Mean Value theorem**

So, today we will discuss in this lecture the derivative of the function at a point, as well as on the interval and also the mean value theorem.

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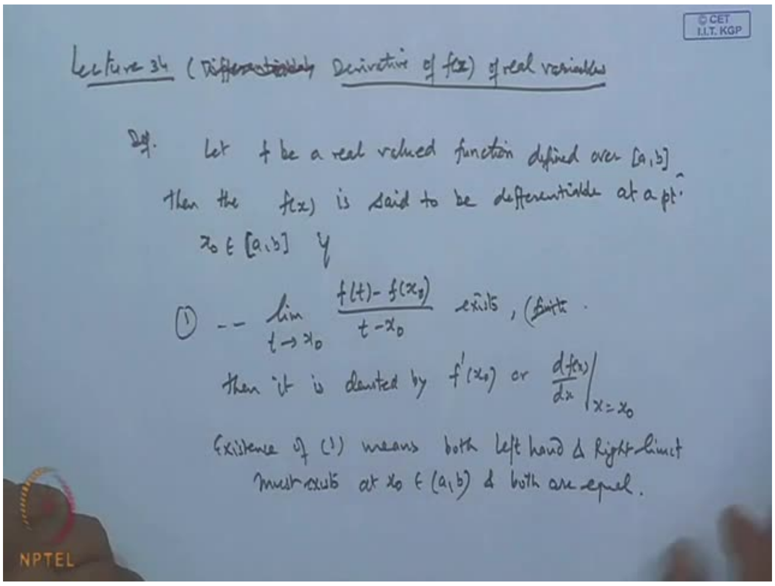


Differentiability, differentiability or derivative of the function, derivative of the function f x of real variables that is let f x be a real valued. Let f be a real valued function, defined over the interval say a, b over the interval say a, b. Then the function f x is said to be differentiable, differentiable at a point say x naught belonging to the interval a, b. belonging to the interval a b

If the limit of this function f t minus f x naught divide by t minus x naught limit t tends to x naught if this limit exist and finite, exist and finite of course, finite. In case infinite we say if the limit is infinite derivative then the differentiability, we can also discuss the case of infinity later on only. So, let us say just exist then it is denoted by, it is denoted by f prime x naught or we also say d over d x f x at a point x equal to x naught that is also a notation to use the derivative at this point.

Now, here if the point is coinciding with a or b then in that case, we have a different concept. Now, when we say the limit of this exist that is 1 if this limit exist means, the existence of 1 means, that both the limit both left hand and right hand limit, limit must exist at a point x naught, which is in the open interval a, b which is in the open interval a b and both are equal, both are equal then we say the derivative of function exist.

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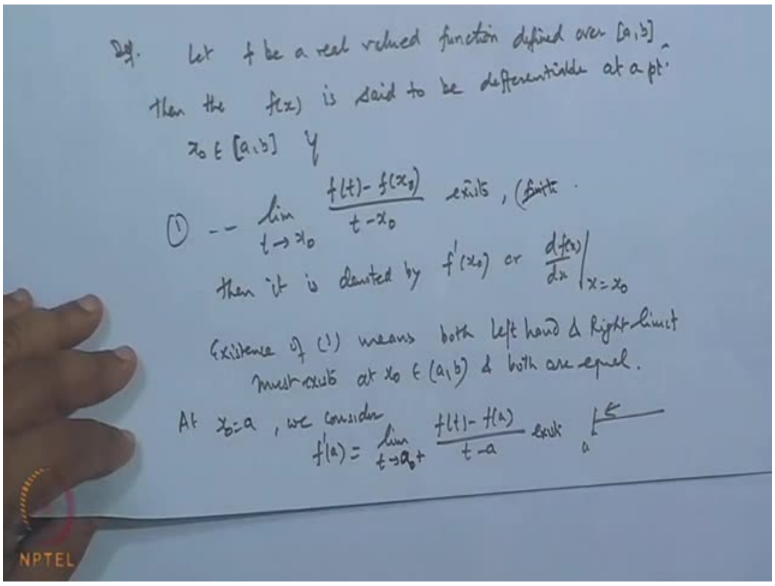


At the point, at the coronal point at x naught is equal to a, we consider only the right hand limit that is we consider the limit t tends to x naught, say x naught is a. So, here t tends to a, a plus f of t minus f of a overt minus a. So, this is an interval we will look only this side the point which are approaching to a from the right hand side, right hand side of a. So, this point t, a plus so right hand limit of this if it exist, if exist then we say the derivative of the function at a point a.

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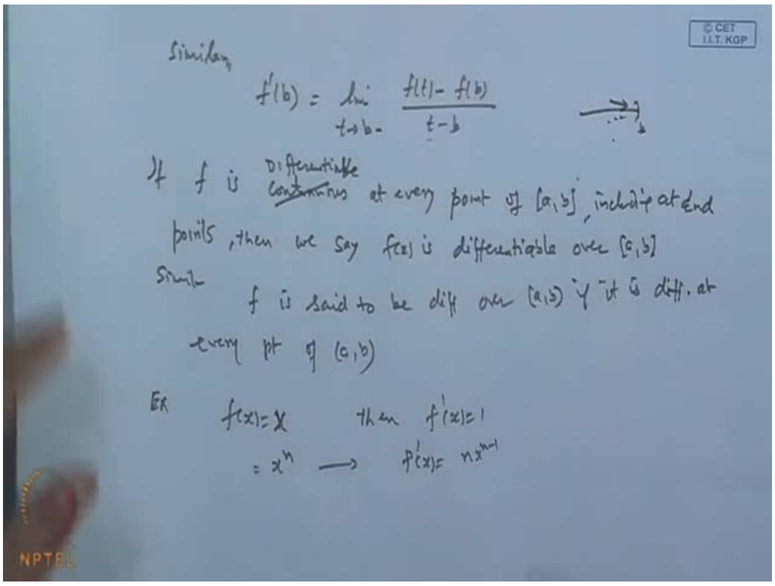


Similarly, the derivative at the point b that is at the end points, this is the end point interval means, limit t tends to b minus f of t minus f of b divide by say t minus b. When b tends to t tends to b minus well. So, if this we are taking the point this is b and we are approaching towards this side left of the b, all the points are taken consideration which are left to b and this point we are taking the image of this when t tends to b. Then we say the function has a limit at the point b.

So, if the function f is continuous, if the function f is continuous at every point of the closed interval a, b including at the end points, then we say the function f x is differentiable over the closed interval a, b. Similarly, we say the function f sorry, if f is differentiable not sorry it is differentiable, differentiable at every point of the interval including end points then similarly, a function f is said to be differentiable over the open interval a, b if it is differentiable at each point it at every point of the interval like this. So, we can extend this definition.

Now, using this definition one can easily show that if f x equal to 1, f x equal to x the derivative will be 0. If f x equal to x then the derivative of this function will be 1 in general if f x is x to the power n the derivative will be equal to n x, n minus 1, which can be used which can be proved directly with the help of this result.

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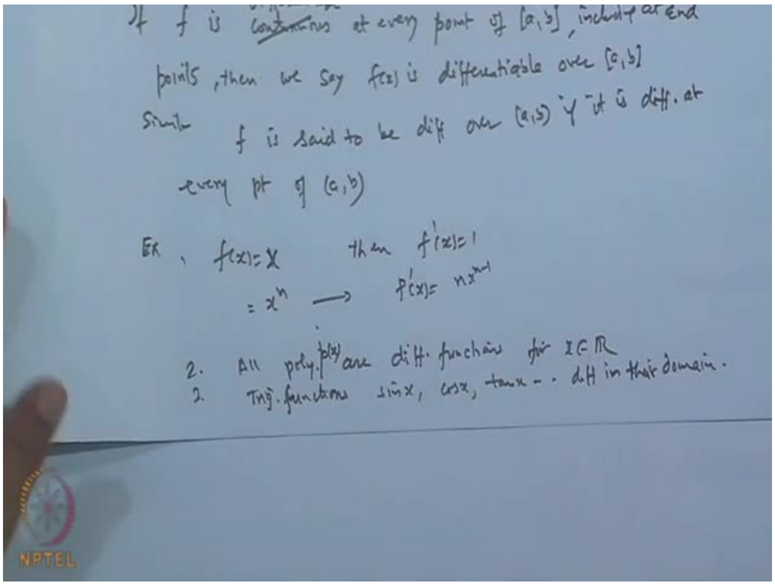


So, all the polynomials functions, all polynomials are differentiable functions are differentiable function for each x belongs to R, then polynomial p x they are all differentiable function. Then trigonometric functions like sin x, cosine x, tan x these are all differentiable functions in the domain, in its their domain differentiable in their domains like this. So, these are all examples which is just results, which we know in a calculus the differentiability of a various functions and formula. So, we are not going in detail for that about this.

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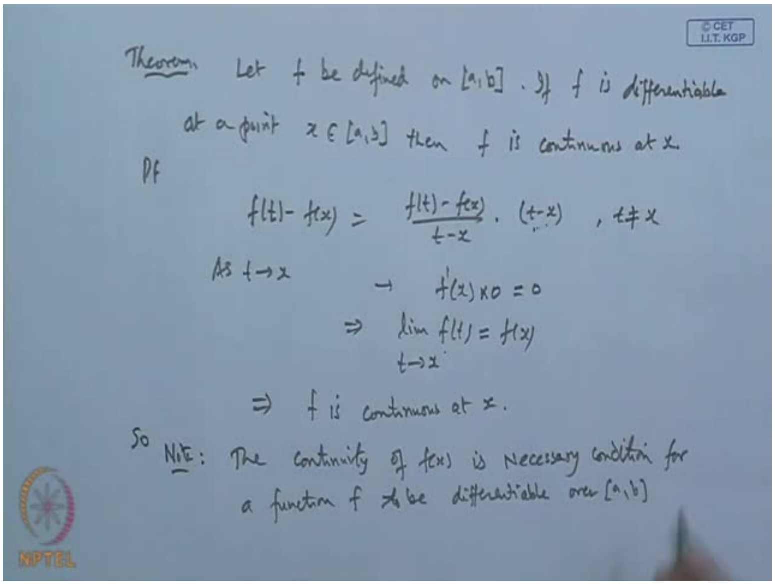

Now, interesting result is that every differentiable function must will be a continuous function that is. So, let f be defined on the closed interval a, b on the closed interval a, b if f is, f is differentiable if f is f is differentiable at a point at a point say x belonging to the interval a, b then f is continuous f is continuous at x, the proof is very simple let us we want the continuity.

So, let us take the point t which goes to as so take f of t minus f of x this can be written as f of t minus f of x over t minus x into t minus x, where the t is different from x. So, obviously this is well defined. Now, as limit t tends to goes to x this gives the derivative f prime x by definition and this part will give the 0. So, total will be 0 therefore, limit of f t when t tends to x either from the left hand side or from the right hand side, if this limit exist means if both will be equal. So, it will always give the derivative ad this go to 0 whether we approach t from the t to x from the left or right.

So, this will always be when t tends to x the f t will be equal to f x that is the limit exist and equal to the value of the function at the point, where the limit is required this implies the function f is continuous at the point x. So, we have discussed these thing the differentiability, and for the differentiability just we will need only the functions to be well defined in this end. So, what is required now if we look that that this example shows or this result shows, that continuity is a necessary condition for a function to be differentiable because every differentiable function, we are getting continuous.

So, continuity comes automatically when the function is differentiable. So, it is a necessary condition for the functions to be differentiable is it not then we say, the converse of this so as a remark or note the continuity of the function f x is necessary condition, condition for a function f to be differentiable, differentiable over the a, b or at any point, or at any point.

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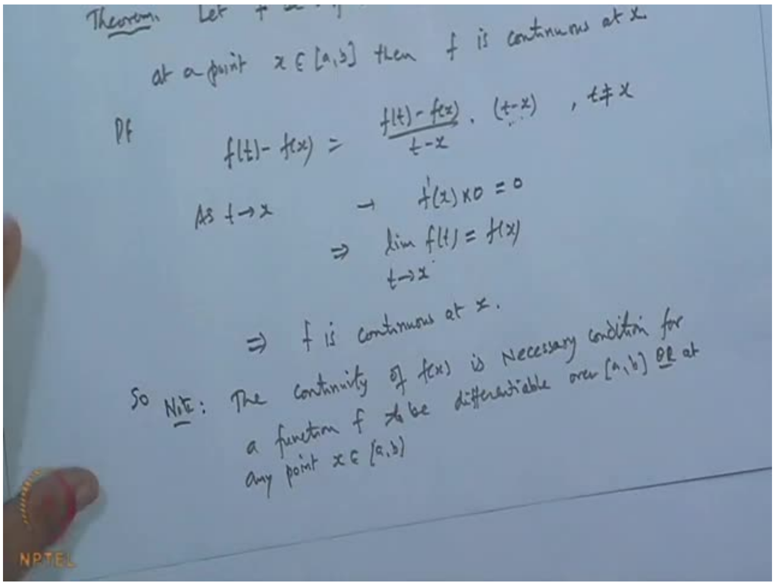


Any point x belongs to a, b. However, this condition…

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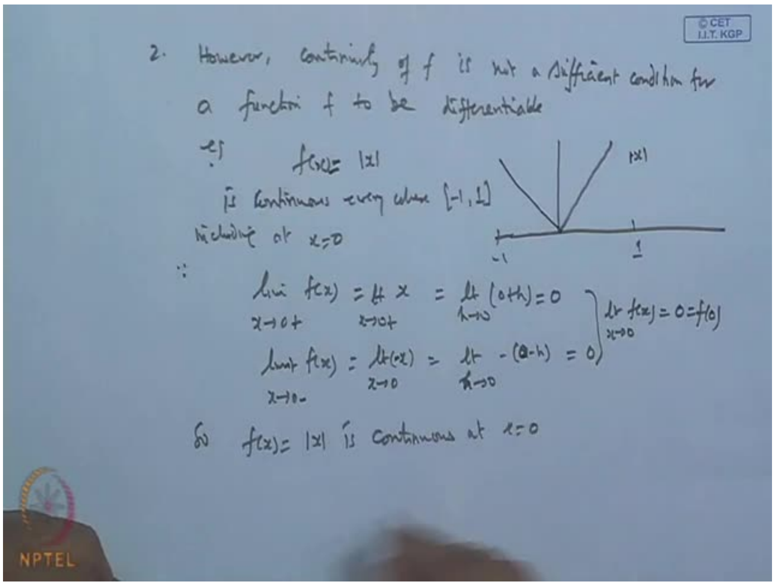


However, continuity of the function f is not a sufficient condition for a function f to be differentiable. For example, if you look the function f x which is equal to mod x, the function the curve of this function is like this x is 0 it is 0, when x is positive then it y equal to x, when x negative y equal to minus x. So, we are getting again this way so, this is the graph for mod x. Now, it is very smooth curve is a continuous curve, in fact the function f x is continuous everywhere, continuous say everywhere I just take the interval for my convenience is 1 to 1 including at the point x equal to 0.

The reason is because suppose I want to test at the point 0, what is the limit of the function f x when x approach to 0 from positive side, the limit of this is nothing but what f x is x. So, it is x, f x is x and then limit x tends to 0 from positive side this is the same as limit h tends to 0, 0 plus h which is 0 and if we look the limit of this function f x, when x tends to 0 from the negative side, then we say it is the limit x, x tends to 0 because all negative. So, it is x is a minus mod of x is minus x so, this is minus x, x tends to 0 that is x equal to x minus h. So, we can write limit x minus h, a negative quantity.

And then h tends to 0, x is 0 so we get again it is 0 and the value of the function is equal to a 0. So, limit exist limit of the function f x when x tends to 0 is 0, which is the same as f of 0. So, function is a continuous function at 0 so, function f of x which is mod x is continuous at x equal to 0, but is not differentiable.

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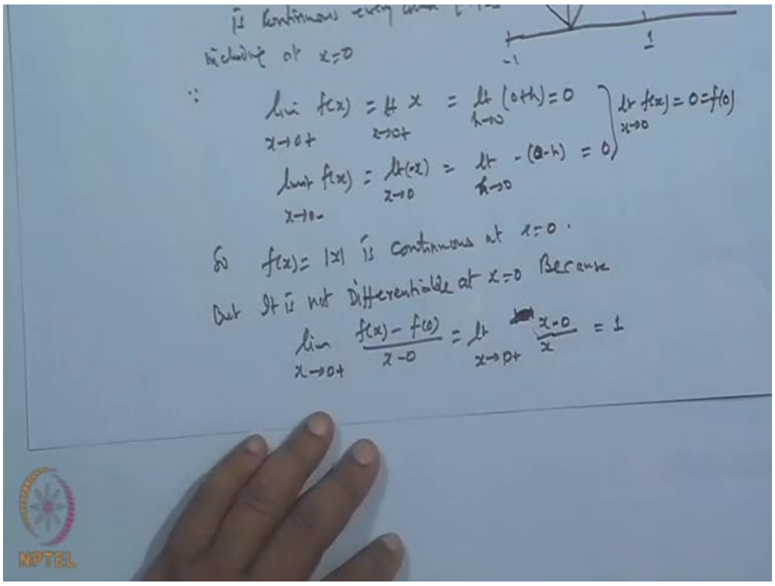


But it is not differentiable at x equal to 0 because the reason is, if we find the limit of this function f x minus f 0 divide by x minus 0 as x tends to 0, from the positive side what we get limit f x when if positive side is mod x minus mod x means, simply x only. So, it is a nothing but the x only because it is positive minus 0 and then divide by x, and x tends to 0 plus 0 plus. So, this is 0 plus means when x is positive mod x equal to x. So, limit comes out to be 1.

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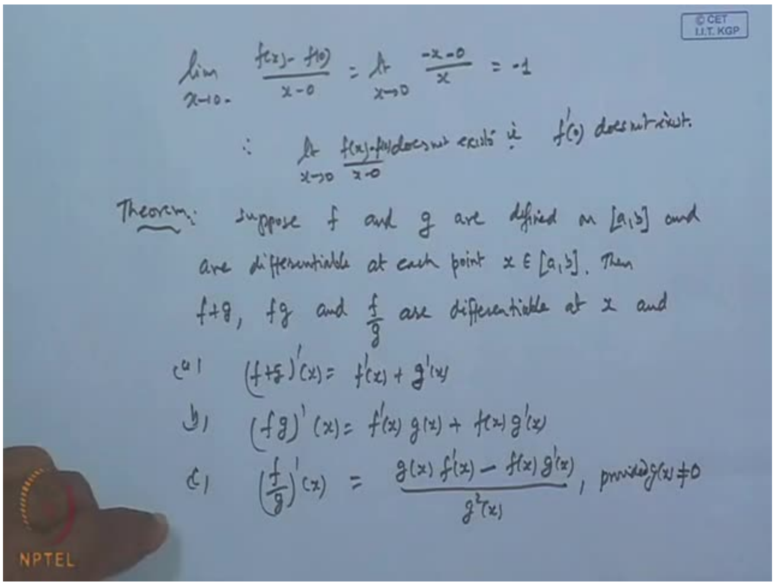


When x tends to 0 minus, 0 minus the f x minus f 0 over x minus 0, this is the same as x minus. So, f x equal to mod x when x is negative so, it is minus x minus 0 divided by x when x tends to 0, but that comes out to be minus 1. So, what you are getting is the right hand limit comes out to b plus 1, left hand limit comes out to minus 1. Therefore, limit of this function f x when x tends to 0 does not exist. Sorry f x f minus f 0 over x minus 0 this does not exist, that is the derivative of the function at a point 0 does not exist. So, function is not differentiable.

It means, the continuity is no longer the sufficient condition just by looking the function is continuous, you cannot say the function must be a differentiable function no, but it is a necessary that is if a function is not continuous, we cannot talk about the differentiability of the function. Function cannot be a differentiable if it is not at all continuous. So, that is a very important result for that then just like a limit case, we have proved that in case of the limit if or continuity if f and g are continuous, the addition of the two function is also continuous subtraction is continuous, multiplication of the two continuous function is continuous like this.

So, similar results here also are for differentiability functions, the results are in the form let suppose f and g are defined, are defined f and g are defined on the interval say a, b closed interval a, b and are differentiable at each point x belongs to a, b then f plus g, f minus g, f g, f g and f by g are differentiable are differentiable at x, and the values f the derivative this x is the same as f prime x plus g prime x, b f g derivative at a point x is the derivative of the first function f x multiply by the second, this is the product of the derivative of two function. So, derivative of the first function multiply by second plus the first function into the derivative of the second.

Similarly, when we say the derivative of the ratio of the two differentiable function at a point x, x is the denominator g x into the derivative of the numerator f prime x minus numerator into the derivative of the denominator divide by the square of the denominator, provided the function g is not 0 in at a new point, where the differentiability is tested in this interval. So, let us see the proof just 1 or 2 proof.

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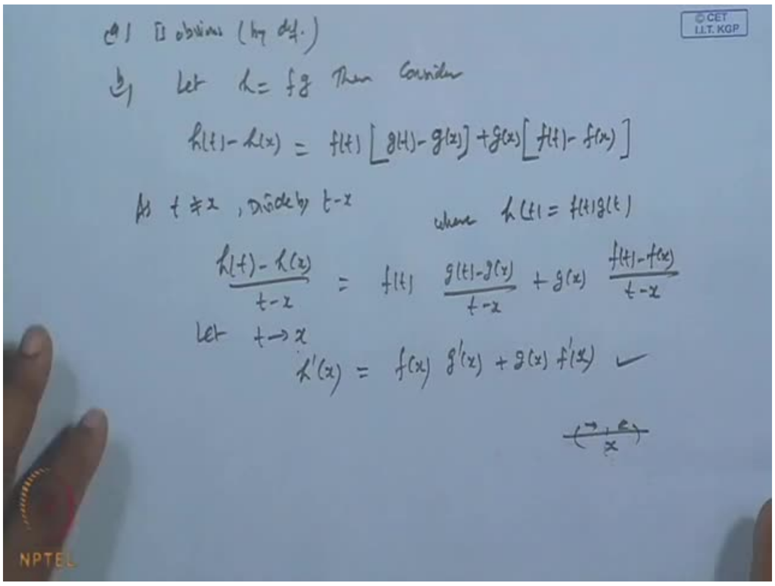

Let us see a is obvious that follows by definition, definition we can just use it the part b you just construct let h equal to f g. Suppose, this function then consider this difference h t minus h x. Now, this can be written as f t then g t minus f x, g x and then plus g x into f t minus f x. Now, you just see that g x, f t, g x an f t, g x is subtracted so subtracted. So, this will be 0 here and rest will be we are just taking h t means, f t g t here here h of t means f of t g of t. So, h of x means, f of x g of where this means like so just as well as now divide by as t is different from x.

So, divide by t minus x so when you divide by t minus x, what you are getting is f t, g t minus g x over t minus x plus g x, f of f minus f of x divided by t minus x and then let t tends to x, let t tends to x. Now, what is the right hand side, the right hand side is when t tends to x f of t will go to f of x because f of x is already given to be f is given to be a differentiable function. So, it must be a continuous at the point x so when you take the t x is here and this is the interval.

So, either t is this side or may be t is this side function is continuous. So, f of t limit of f of t will be f of x so, this will and this ratio g t minus g x over t x is also given the function g is differentiable function at any point. So, at a arbitrary point x it is differentiable so, this limit will be equal to g prime x because t tends to x and this is g x and this will be f prime x. And this will give the because of the function at a point x so this proves the b part.

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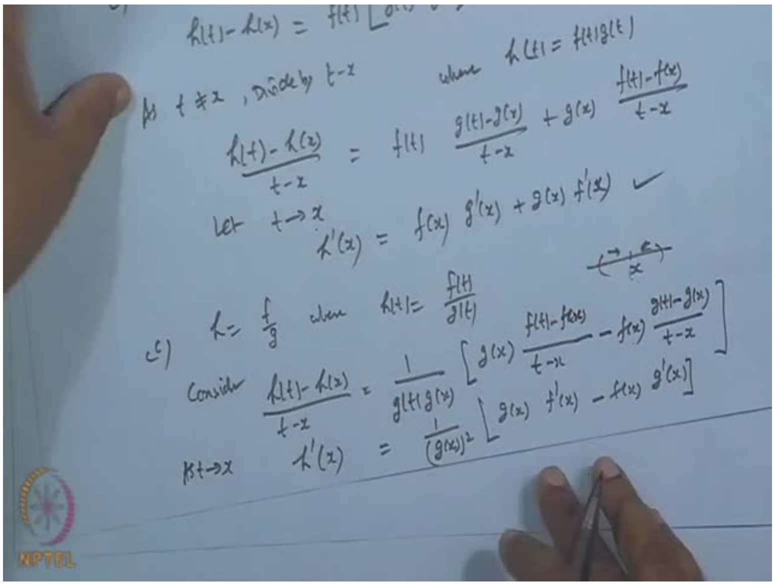


C portion to prove the c, let us consider h to be f by g where h of t means, f t by g t this is the meaning. Now, consider this h t minus h x divide by t minus x just a and then manipulate the term. So, what you are getting is h t means g t and g x will come so it is basically f t g t minus f x by g x. So, when you take the L C M then you are getting g t, g x outside and inside we are getting as g x, g x into f t minus f x, f t minus f x and this t minus x take it here, then plus sorry minus f x into g t minus g x divide by t minus x.

Now, you see t minus x suppose outside then all these things are adjusted, you can just say this g x, g x, f x and f x, g x both get cancelled and rest of the things will come from here. So, s t goes to x as t tends to x, the this will give as g x. So, g x square and this one will be g x and then, this is the limit of the f at the point x derivative of f at the point x because this limit ratio of this limit will be derivative minus f x into the derivative of g at a point x and this. And this will give the derivative of the function x h means, derivative of so that result follows. So, these all of fundamental things and we you can use it for getting the derivative of various type of functions.

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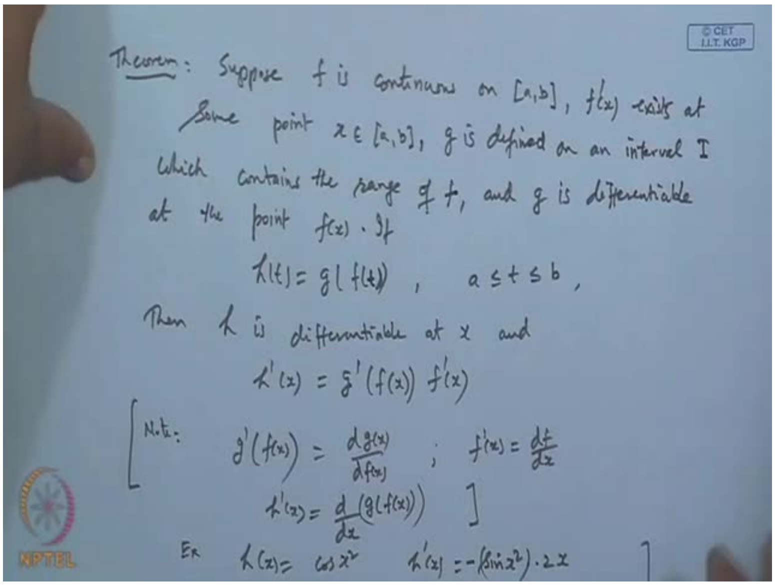


Now, for the composite functions also the similar result for as we have in case of continuity, just like a function f is continuous at certain point and g is also continuous at the point f of t, where the image of point where the function is continuous then f composition g is continuous at a. So, just like a continuity the same result follows for differentiability. So, suppose f is continuous on the closed interval a, b and let f prime x, f prime x exist, f prime x exist at some point at some point say x of a, b belongs to a, b.

G is defined, g is defined on an interval I, interval I which contains, the range of f the range of f and g is and g is differentiable at the point f x at the point f x. Now, if h of t equal to g of f t, g of f t for t lying between this a to b then. Then g h is differentiable at x and the derivative of this is equal to derivative of g with respect to this f x, and then f with respect to x.

Now, remember here I will just explain here we say note, when we say g dash of f x means, we are differentiating this g with respect to f with respect to f and then f with respect to x, then f with respect to x. So, an f dash x is d f over d x so when we are saying the h dash x it means, we want to differentiate this composite function g of f x this composite function, we want to differentiate with respect to x. So, since it is a composite function g is a function of f so first we have to differentiate with respect to that composite function and that composite function with respect to, x that will be the idea so, that is the value for this. So, we get.

Suppose for example, let me see just one thing suppose for example, I say the h function h of x equal to cosine of x square say suppose I take then f x is x square, g is cosine function. So, when you differentiate this h with respect to x you will define cos with as if x square is t a function of t then cos of as a function t is the sine, minus sine derivative of cos is minus sine x square and then, x square we will differentiate it with respect to x. So, that will get so that is the composition so this is all.

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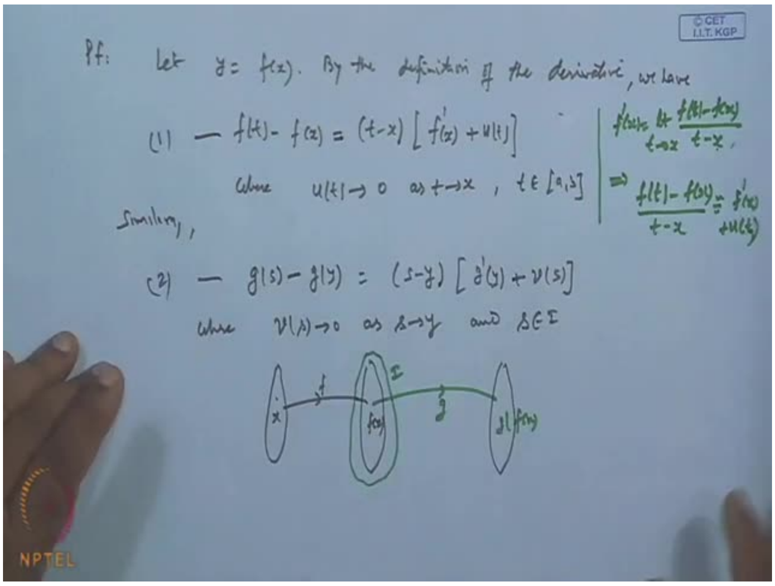


Now, let us see the proof for it let y equal to f x, y equal to f x. Now, since f x is continuous as well as differentiable, it is given f is continuous on a and derivative exist at this point. So, by definition of the differentiability of the function f x, we can say so by definition of the derivatives, of the derivatives we have f t minus f x this is equal to t minus x into f prime x plus u t, plus u t. Where u t tends to 0 as t tends to x u t tends to 0 as tends to 0 and t belongs to a, b. Let us see what is the meaning of this?

When we say the function is differentiable that is the meaning of this is, the function f prime x exist it means, it is the limit of this f of t minus f of x, f of t minus f of x divided by t minus x when t tends to x is it not. So, when t is approaching to x then this ratio, the limit of this ratio exist and we are denoting by a prime it means, the difference between this ratio and a prime is very, very small when t is sufficiently close to x, or in other words if you say that if I multiply this t minus x to this or we can say, that this ratio f of t sorry this one f of t minus of this over t minus x over t minus x is this.

So, if I remove the epsilon t minus x then this is equivalent to the derivative of the function at a point x plus some number say u t, I am taking u t it may be epsilon and this u has a property it goes to 0 as soon a, t approaches to x. So, when you take the t approaches to x if this part is tending to 0. So, this will coincide with the limit of function will coincide with the derivative. So, the same thing I am rewriting in this way that f of t minus f x is t minus x times this term so, this is what we get.

Now, similar thing we can write it for our function g. Now, g function is defined over a interval i, in this the range of f lies. So, similarly, so let it be equation one and similarly, for g t, g of s minus say g of y because a function g is continuous at the f x, f is f is differentiable at x. So, g f x is differentiable g is differentiable with the f x so, y is equal to f x am writing g of f x here, which is s minus y into g prime of y plus a term v s and the v s tends to 0, where the v s goes to 0 as s tends to y, s tends to y and s belongs to the interval i in which the function f lies range of the f lies mean, this is our say here.

So, the function f is defined over this domain this is the domain. So, x is here we are getting f x, the function g is defined over an interval, I am in place of interval I am just writing this I, where the range of this f is contained. So, when you take the g, g then g is defined as g of f x is it not this define, and function f is differentiable at x and g is also differentiable at f x. So, we can write it this as this form that is what so, this can be where t belongs to where t is this. So, let it be equation 2.

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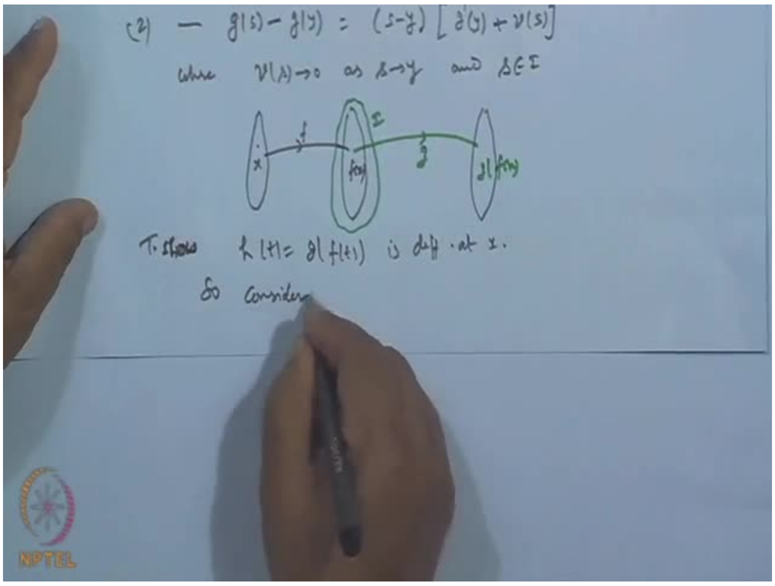


Now, suppose in this now we wanted the function h, t which is we wanted to show to show that h function, which is a composite function h of t which is g of f t is differentiable is differentiable, is differentiable at x this we wanted. So, write down the relation h t minus h x.

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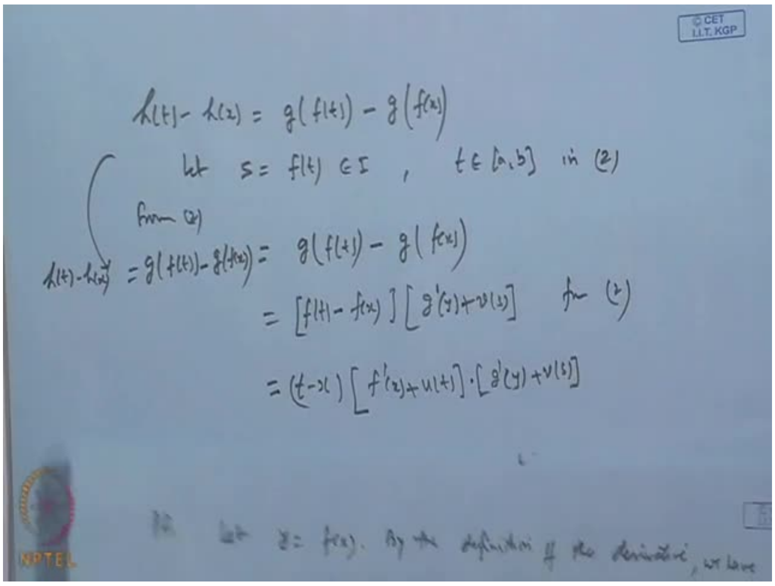


So consider h t minus h x now, h t minus h x means h is this function h of f t. So, we can write this as g of f t this is h minus h x means, h t and h x means g of f x this thing. Now, replace this s by f t because s is a point in the interval I, t f is a function defined over the domain and t is a point in the interval a, b. So, replace s by f t so, when you replace s by f t, what you are getting is this g of f t minus this. Now, let us see g of this if I replace this in the in equation 2, if we replace let this be taken in equation 2.

So, from the equation 2 what we get 2 is g x minus g y is it not so g s minus g y, g of f t minus g y, g of y means, g of f x this is our which is nothing but this part, s t minus f x which is the same as h t minus h x. So, if I replace this s by f t then from here we are the this one, but the right hand side of this gives you s minus y. So, s will be equal to what g of f t so this is equal to g of f t minus, minus g if f x that is the same thing, which why I have written now apply this here f t minus f is continuous differentiable.

So, we can use this thing now so when we use this thing then this become g is differentiable so, it is f t minus f x, f t minus f x into g prime at some point y where it is continuous plus v s that is what we getting, use the this one condition. Then similarly, for this g of f x so, when you write this thing the g of this difference, g of this difference is this and this one s minus y s is our f t, y equal to f x g prime y plus v s this one we are getting this get it from 2, from 2. Now, apply the first use first f t minus f x is this so, use first so it will be equal to t minus x f t minus f x is t minus x into f prime x plus u t this is our f t minus f x, we want this value this value I am replacing by this, and here this remains as it is say g prime y plus v s that is as it is it not v s. Now, let us take this now.

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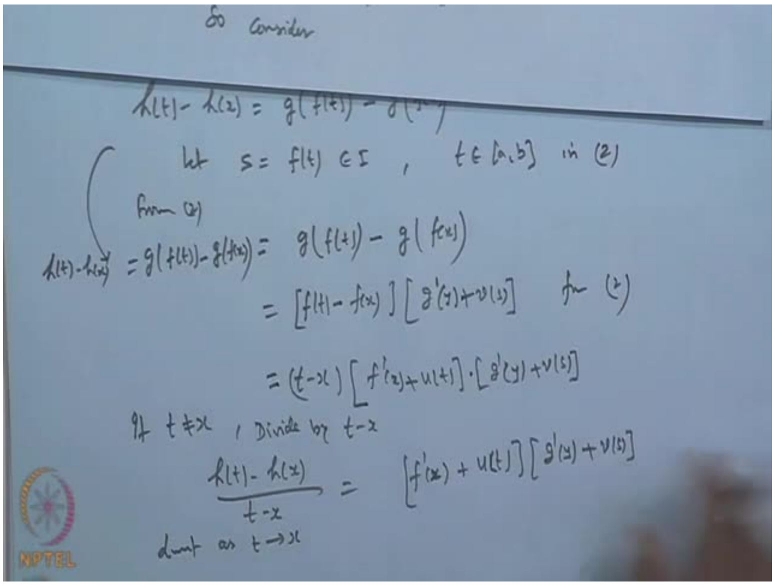

So, if we if t is not equal to x then we divide by t divide by t minus x. So, when we divide by t minus x you are getting h t minus h x over t minus x is nothing but what f prime x plus u t, u t into g prime y plus v s. Now, this will be equal to what taking the limit as t tends to x so, when you got for the limit as t tends to x.

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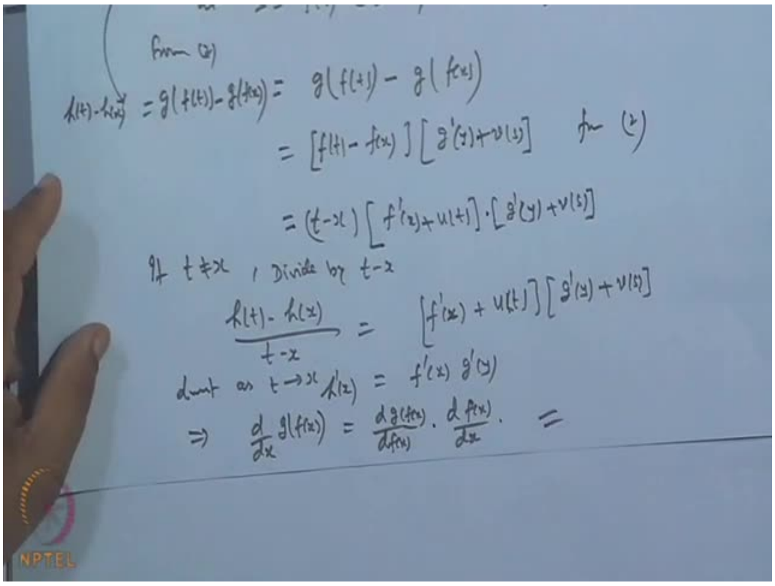

The right hand side if I look as t tends to x u is tending to 0 because of this condition is already given f is differentiable at x. So, u t will go to 0 as t tends to so, this part will go to 0 similarly, this part will go to 0 the right hand side will be f prime x g prime y and the left hand side is nothing but the derivative of the function at a point x. So, this implies that the derivative of this composite function g of f x with respect to x is the derivative of g, with respect to f x then derivative of f x with respect to x that is what is so, this proves the result so, that is what. So, this is composite function which is very much helpful in getting. Now, let us take few examples, where the differentiability is there.

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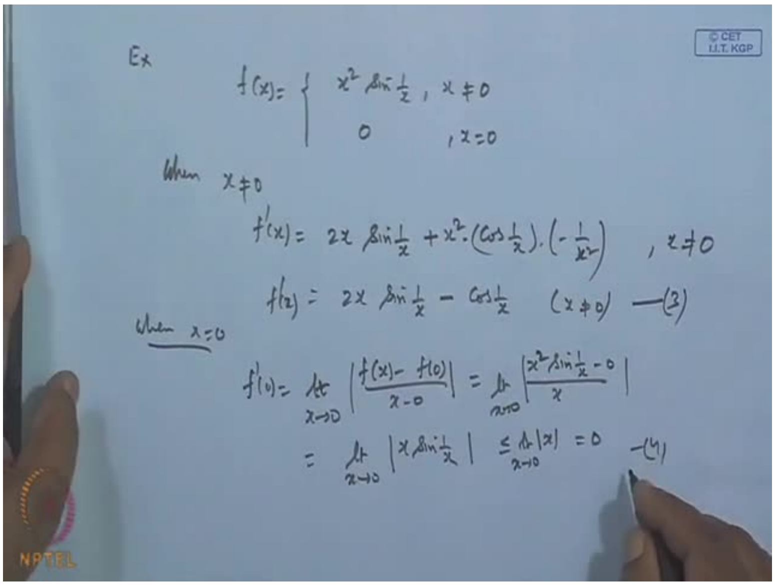


Suppose I take the function f x equal to x square sine 1 by x when x is not equal to 0 and 0 when x equal to 0. Now, when x is not equal to 0, what you are getting is the function is defined in this fashion. Now, x square is continuous function it is also differentiable function sine is a continuous and differentiable so for x is not equal to 0. So, we can directly apply the formula and when we take x is not equal to 0, when x is not equal to 0 you can directly apply, the formula product of the two functions.

And product of the two function will give the derivatives of the first two x sine 1 by x and then the second function x square, x square and the derivative of this is cos 1 by x into minus 1 by x square because this is a composite function now, derivative of sine as a function 1 by x is cos 1 by x, but 1 by x derivative is minus 1 by x square. So, it is a composite function so, this is this hold this hold when x is not equal to 0 that is the derivative of the function comes out to be 2 x sine 1 by x minus cos of 1 by x for x, which is different from 0, but when x is 0, when x is 0 what happens?

The derivative of the function, we have to compute it why because at the point x is 0 when you are looking for the derivative then the value at this point apply the value, which are different from 0 is given by this form. So, find the derivative of this function at the 0 by using the definition that is limit of this that is f x minus f 0 over x minus 0 x tends to 0 and that comes out to be what, f x is x square sine 1 by x minus 0 divide by x when x tends to 0. So, that is equal to basically limit of his x tends to 0, x sine 1 by x.

Now, if we look that term x sine 1 by x is always be dominated by mod x by mod x. So, basically when you are taking the mod of this then this is mod of this which is less than equal to mod x limit x tends to 0 and this limit will always comes out to be 0 this limit comes out to be 0. Therefore, the limit of this exist and the derivative at the point 0 will be 0, but what about the equation, let it be this equation 3 this is 4, the function is differentiable.

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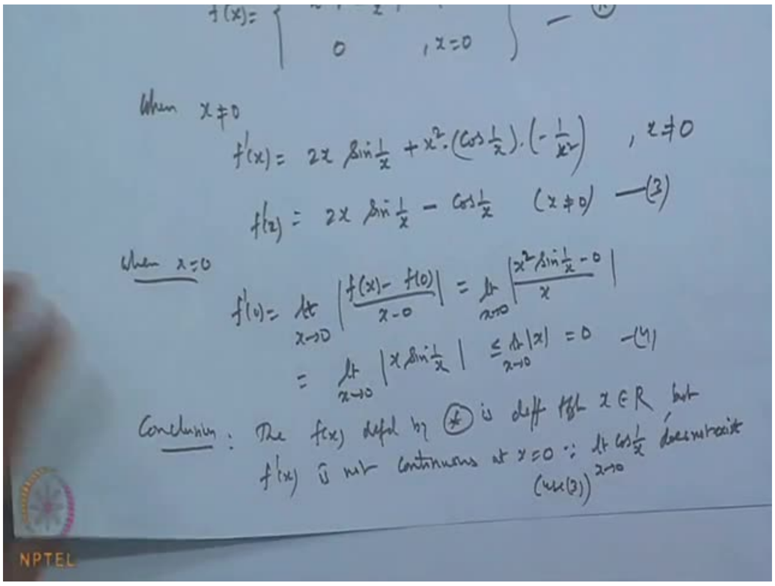

So, conclusion is the function f x defined by say here star defined by star is derivative throughout x belongs to R including x is 0, but the derivative of the function f x if you look the derivative is not continuous at x equal to 0. Why? Because when you take the derivative the derivative is involving is in a involving cosine of 1 by x. So, when you take the limit as x tends to 0, the limit does not exist because limit of the cos 1 by x as x tends to 0 does not exist, which is available in 3 because use 3 then we are getting it.

So, when the function is differentiable you cannot say that derived function remain continuous, it may not be continuous if it is continuous then there is a possibility of going further is it not, but if it is not continuous we cannot talk about the. So, in this case we cannot talk about the differentiability, second derivative of the function at a point 0 if we look the f x is similarly, sine x, sine 1 by x we will see in a similar way, the limit does not exist.

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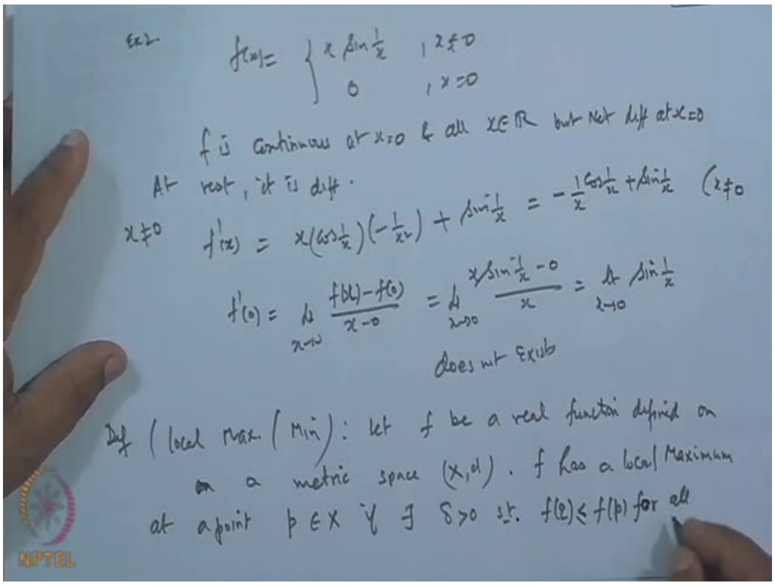


Similarly, example 2 if we take the function f x, x sine one by x, x is not equal to 0 and 0 then function is continuous f is continuous because at 0 and also for all x belongs to R continuity follows, but not differentiable at x equal to 0 it is difference at rest of the point, rest it is differentiable because when you take the derivative of x for x is different from 0, you can just apply the formula first function x derivative of this is cos 1 by x into minus 1 by x square then plus sine 1 by x.

So, the derivative will come out to be minus cosine 1 by x divide by 1 by x and then plus sine 1 by x when x is different from 0, but when consider a derivative at a point 0 then as per the limit f x minus f 0 divide by x minus 0, x tends to 0 this comes out to be sine 1 by x minus 0 by x, x sine 1 by x minus 0 limit x tends to 0 this turns out to be limit of 1 by x as x tends to 0 is does not exist. So, this function is throughout continuous function, but is not differentiable at the point 0 it is differentiable at rest of the point, but not at the point 0 so, we can get.

Now, there are certain mean value theorems, which we will make use for that so before that let me see the definition again for the maxima minima, local maxima, local minima this is already we discussed local maxima or local minima. Let f be, let f be a real function defined on defined on a metric space on a metric space x, t we say then f has a local maxima or local maximum at a point p belongs to capital X, if there exist a delta greater than 0, such that f of q is less than or equal to f of p for all.

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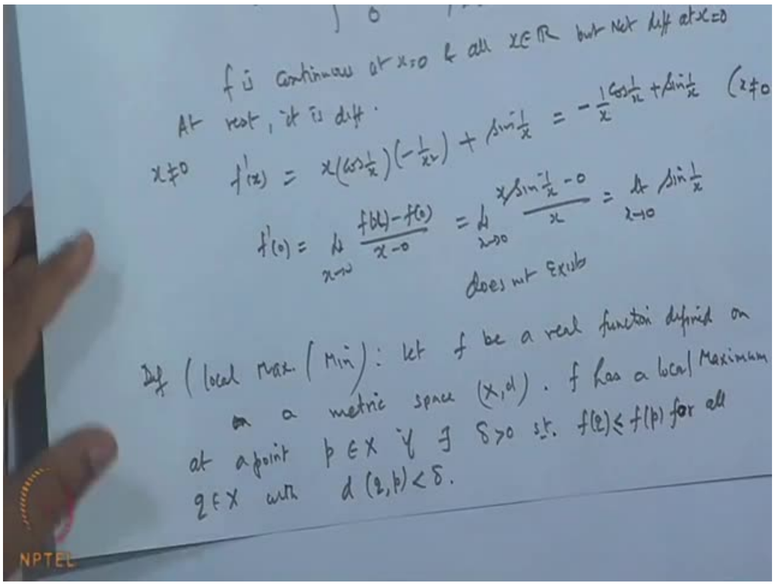


For all q belongs to x satisfies, the condition with the condition that distance from p distance of q and p is less than delta distance of this is less than delta. So, this is it means, what is the local.

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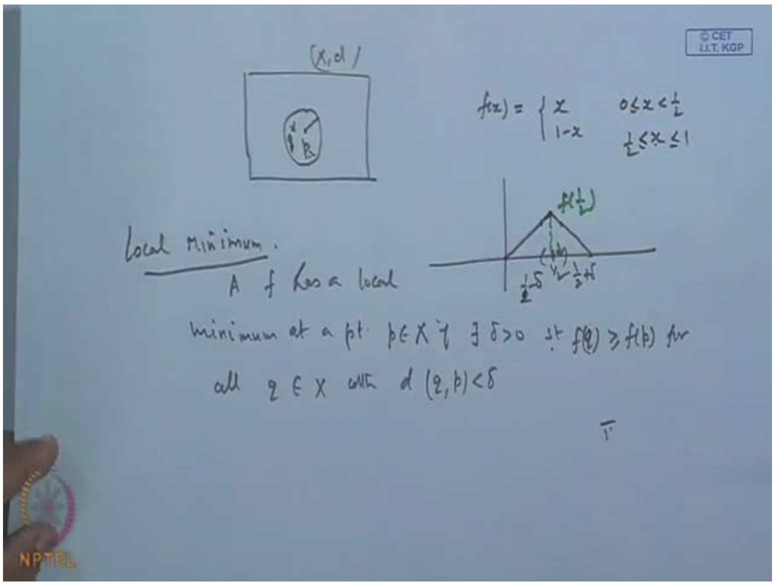


Suppose, this is our x t a metric space and p is a point we say the function attains a local maxima at this point, if there exist a some positive number delta and a neighbourhood around the point p. Such that, if we pick up any arbitrary point q here, then value of the function at this point q should remain is remain less than equal to p, then we say the function a attains a local maxima at this point. Just for example, if we take the function f x equal to say suppose this function, I will say x when lying between 0 less than equal to x less than half and say equal to 1 minus x, when x is say half and say 1, less than 1. So, what this function is? If we look the function f x equal to x between 0 and half the function is defined like this, at the point half the function is again half so, it attains the value here and then one it is again 0.

Now, if I look the point half what is the character of the half, if we take any neighbourhood around this point with a positive delta, say half minus delta, half plus delta this interval, if I choose then the value of this of any point here in this neighbourhood will be somewhere here, or somewhere here which is less than the value at the point half, this is the value of the function at a point half. So, we say half is a point corresponding to a local maxima of this function f.

The same definition continuous for a local minima, local minimum we say a function f has a local minimum at a point, at a point say p belongs to x, if there exist a delta greater than 0, such that f of q is greater than or equal to f of p for all q belongs to x, with the condition that distance from q and p is less than delta. So, this is the local minima for this point. Now, we will continue with the mean value theorem next time, this over.

Thank you.