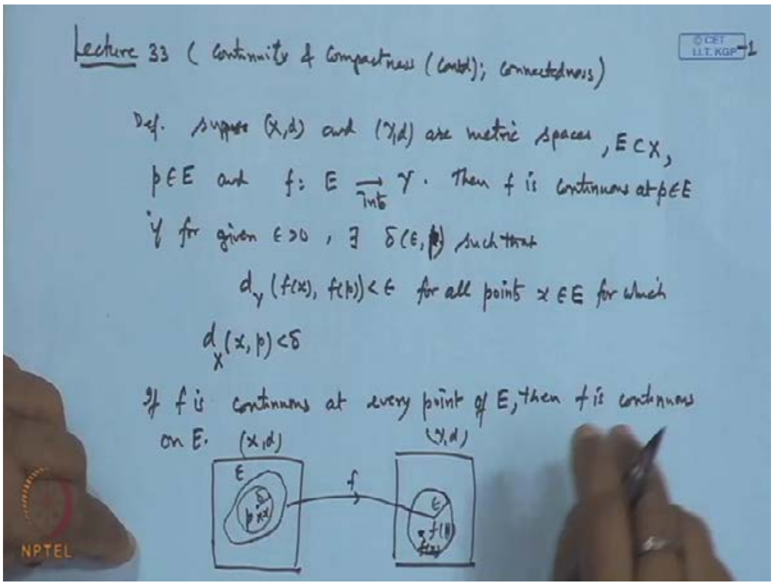
A Basic Course in Real Analysis

Prof. P. D. Srivastava
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 33
Continuity and Compactness (Contd.), Connectedness

So far, we have discussed the definition of continuity over the interval at the point and also for an arbitrary sets. We can extend the definition or we can define the continuity of function in an arbitrary metric space, in a similar way which we have defined in case of a point on the real line by choosing the in terms of the neighborhoods.

(Refer Slide Time: 00:46)



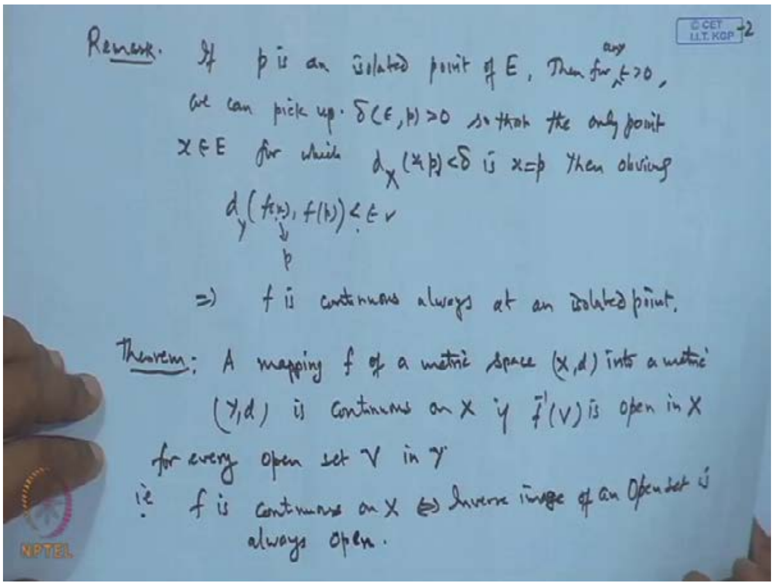
So, we here just for the sake, we define the continuity of the function in a arbitrary metric space at a some point in the arbitrary metric space.

So, suppose X d and Y d are two metric spaces are metric spaces, and E is a non-empty subset of x, p is a point belonging to E, p is a point belonging to E, and f is a mapping from E to E into y into y. Then we say f is continuous at a point p belongs to E, if for given epsilon greater than 0, there exist, there exist a delta which depends on epsilon as well as the point say p, as well as the point p, p such that d of f x comma f p in the metric of y is less than epsilon for all points, all points x belongs to E for which the distance from p in a metric x is less than delta.

So, and if the, if f is continuous, if f is continuous at every point, every point of E then we say f is continuous, continuous on E. So, the meaning is like this, suppose this is our set a metric space x d and here is say, metric space y d; E be a non-empty subsets of x and p is any arbitrary point and p is a point in E; f is a mapping which maps E to E into y.

So, image of p will go to f of p in y; the function f is said to be continuous at the point p, if we draw a neighborhood for a given epsilon greater than 0 or if we draw a neighborhood of the point epsilon, neighborhood of the point f p in the metric space y d, then corresponding to this neighborhood, there exist a delta neighborhood of p in x, such that image of any point x inside this neighborhood, delta neighborhood will fall within the epsilon neighborhood of a p, then we say f is continuous at p. So, it just like a similar definition which we have used in a epsilon delta definition over the real line, over the point in a, on the real line in a interval at p. So, that is a general form and if p suppose an isolated point, then the function will remain continuous.

(Refer Slide Time: 04:56)



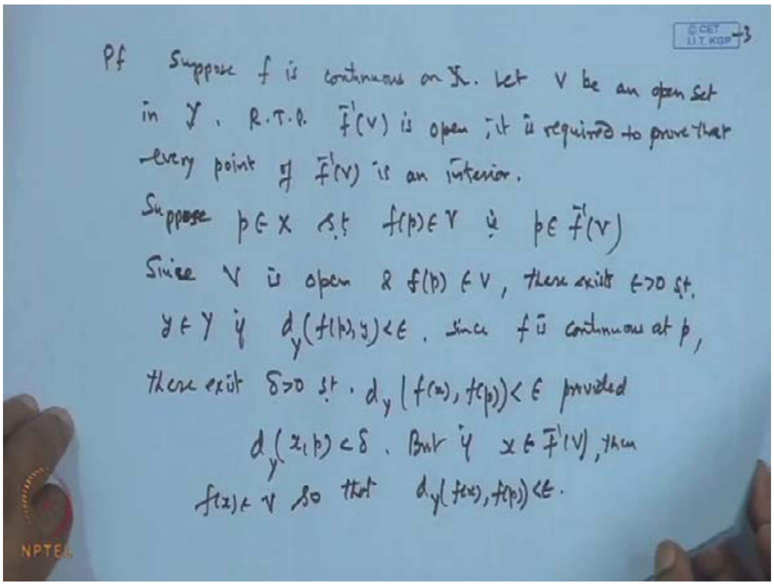
So, this is very just a remark, we can say that if p is an isolated point, if p is an isolated point, isolated point of E, if p is a isolated point of E, then for any epsilon greater than 0, then for any epsilon greater than 0, for any epsilon greater than 0, we choose, we can picked up, we can pick up a delta great depends on say epsilon and the point p, of course greater than 0, greater than 0, so that, the point x, So that, the only point the only point p x belongs to E, p belongs to E is basically for which, for which this results holds, d by d x x p is less than delta is nothing but x equal to p, because p is an isolated point. So, basically in the neighborhood of the full point p within arbitrary small delta, there we cannot get any other point except p. So, for this point, we can assume, any epsilon, the image will fall within always this reason; then clearly obviously, the image of this f of x f of p under y will remain less than epsilon, because there is no such point, only the point x will be p itself, x will be p itself. So, the image will be 0.

So, for any arbitrary epsilon greater than 0, this conditions always holds; because if p is a isolated point, then there is no such x different from p available, only the x will be p itself, and as soon as, you take x is equal to p the distance between f p t f p becomes 0, and this will always be less than any positive quantity; therefore, the function is automatically continuous; this implies f is continuous always at an isolated point, this is two.

Now, in this lecture, we wanted to give the some relation between the continuity and compactness, continuity and connectivity. So, those results are interesting, that is we have seen that image of a open set or a image of a close set under a continuous function need not be closed or open set need not be an open set. So, this will give a very good gap that we cannot assure that image of a open set under a continuous function will always remain open, or image of the closed set will always remain closed; that is not a, that we have seen counter example for it. However, in case of the compact set, this is very interesting result, if x is compact the f of x will be compact, that we will saw. So, the relation between the continuity and compactness if f is continuous, it is a very interesting one.

But prior to this, we make a definition of the continuity in terms of the inverse image of the open sets. So that gives a result which can also be treated as a definition for a continuity. A mapping f, a mapping f of a metric space x d, x d say, here d x of course, we write into a metric into a metric y d, a mapping f of a metric space x d into a metric y d is continuous on x, is continuous on x if f inverse, if f inverse v is open, is open in x, in x for every x d in x for every open set v in y, in y, that is the meaning is that f is continuous, f is continuous on x if and only if inverse image of the open set is open, inverse image of an open set is open is always open. So that gives you another criteria to judge whether the function is a continuous or not. So, that is an interesting, particularly it is used when the topology, and topology, normally, the definition of the continuity is taken is this way; the inverse image of the open set is open then the function will be set to be a continuous.

(Refer Slide Time: 10:34)



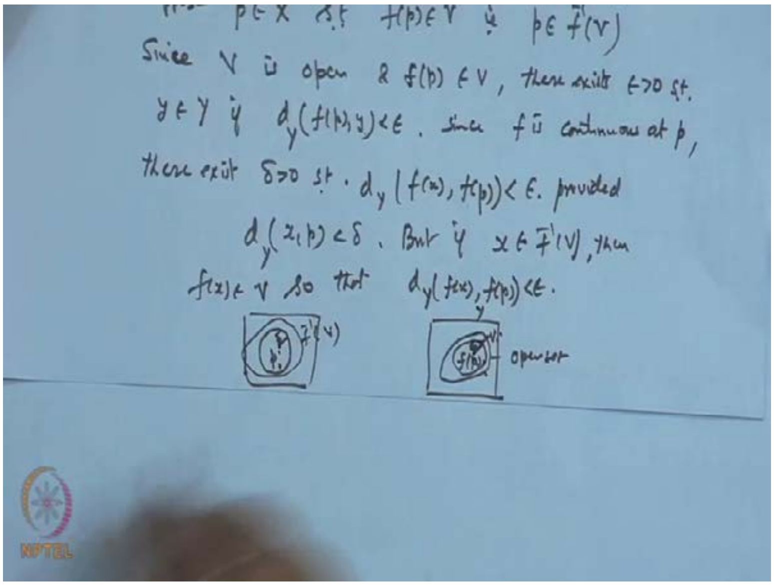
Let us see the proof of this theorem, because it will need like. So, what is a… Suppose f is continuous, suppose f is continuous, f is continuous on x, f is continuous on x. We wanted the inverse image of the open set is open. So, let us take v be an open set, v be is an open set in y; in order to show the inverse image of v is open, what is required? It is required to prove, to prove that every point, every point of this set f inverse v is an interior point. So, if it is interior point then f inverse v will be open. So, this we wanted to…

Let us picked up a point, so suppose, p is a point belonging to x such that f of p is in v. So, what do, we it means that is f p is taken in the set of f inverse v that is we know. We wanted to show there exist a neighborhood around the p which is totally contained in f inverse v. So, let us start with this, what is v? Since, v is given to be, since v is open, it is already given and f p is a point in v, this is a point in v. So, we can find an epsilon neighborhood around the p, f p which is totally contained in b. So, so there exist an epsilon greater than 0 such that, such that y belongs to capital Y, if the distance of f p from y, under the metric y is less than epsilon, is it not? Distance of this is less than…

So, this y belongs to Y, since we (( )) there exist this y. And since f is continuous, and since f is continuous at p. So, for a given epsilon greater than 0, there exist a delta neighborhood in the domain, here all the point in delta neighborhood, the image will fall here. So, because it is continuous. So, there exist delta greater than 0, such that the image of this d of y f x f p this less than epsilon, provided the distance between x and p under y is less than delta, this by definition.

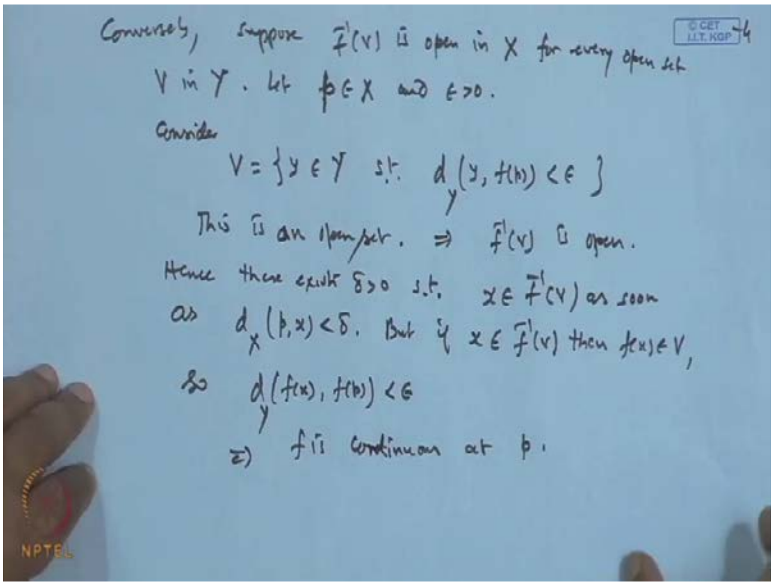
But all the points which lies here, the point y belongs to capital Y, if this is, all the points with this x 1 less than delta, this condition holds. So, this condition means f x is in the epsilon neighborhood of f p. So, x must be the point of the inverse, but no sorry, but if x belongs to f inverse p f inverse v, that is f x belongs to v then that is f x belongs v, then what happens there? If f x is in v, f p is also in v, it lies in the epsilon neighborhood of v and f is continuous; therefore, this continuity. So, for this epsilon, this holds; it means the continuity of f follows. So, so because of the continuity, we get that, so that distance between f x f p will remain less than, less than epsilon.

(Refer Slide Time: 15:21)



Now, I will just explain what we did. In fact, what we have taken is that this is… First we have taken a v as a open set in y, then we have fixed up the, since p is already taken the image will be f of p belongs to f v. So, f p is here, now since this is an, f is continuous at p. So, what we get? There exist a delta greater than 0, such that whenever the point x lies between the delta neighborhood, the corresponding image f x and f y less than epsilon. So, what this shows? This shows the point x, if it is in f inverse v the f x must be in v. So, once it effects in v then the distance of this we got less than epsilon; this shows there exist a neighborhood around the point f p which is totally contained inside v. So, v is (( )) So, this contain, therefore, this will be an open set. So, f inverse v an interior point p is an … So, p will be an interior point because this is our f inverse v.

(Refer Slide Time: 16:55)

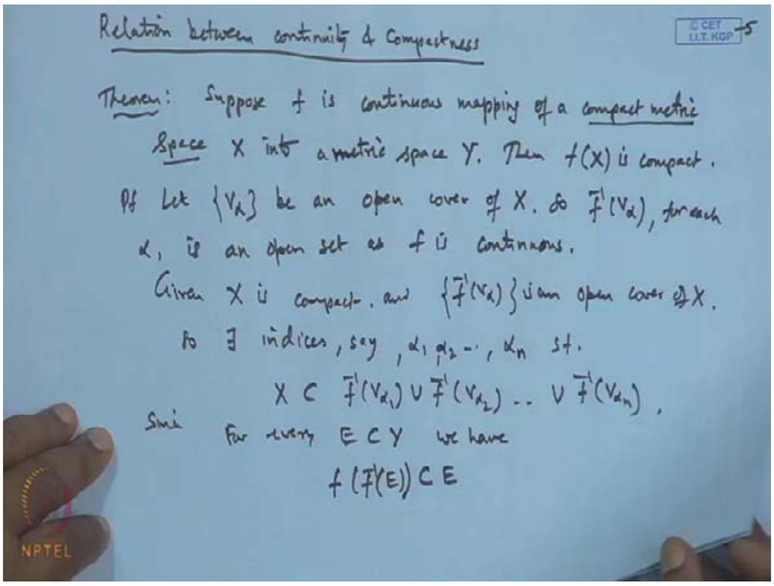


So, this will be an interior. Now conversely, conversely, suppose f inverse v is open, is open in x, open in x for every open set, for every open set v in y then we wanted to show f is continuous. So, let f belongs to sorry p belongs to p, belongs to X and let epsilon greater than 0, continuity at the point. Let us consider the set v at the set of those points y belongs to capital Y, such that the distance of this point y with f p in the metric device is less than epsilon, consider this set. So, this is basically now an open ball centroid at f b with the radius epsilon, so obviously, this will be an open set.

So, this is an open set. So, there is nothing to prove it. So, once it is open set, according to our assumption the inverse is also open for every… So, this implies that f inverse v, this will be an open set is open in x; hence, once it is open. So, there exist a delta neighborhood. So, every point is an interior point; hence, there exist, hence there exist delta greater than 0, such that, such that x belongs to f inverse v. As soon, as soon as the distance from p of x under the metric is less than delta, it means, because it is open. So, we can find a neighborhood around the p which is totally contained inside this, contained inside this neighborhood. So, this x will be the point inside this neighborhood. So, it belongs to, as soon as this.

But, if x is an element of f inverse v, then the image f x will be in v a definition f inverse. So, then f x in v. So, what we get? So, the distance of f x from f b under the metric y will remain less than epsilon, will remain less than epsilon, because of this, because as soon this point is in v and for all y which are in v contains those y whose distance is less than epsilon. So, this point f x minus f x comma f b under the metric y is less than epsilon. So, this shows. Now, this shows that f is continuous at p and that proves the result. So, this particular result will be used for estimating the relation between the compactness and continuity.

(Refer Slide Time: 20:42)



So, let us see the result is relation between continuity and compactness, between continuity and compactness theorem. Suppose, f is continuous, f is continuous mapping, if it is continuous mapping of a compact metric space, of a compact metric space X; X is given to be compact, this is important, X is given to be a compact metric space; X into a metric, into a metric y, metric space Y, Y then the result says f of X is compact. So, a very interesting result, the image of a compact set under a continuous function will always be compact; and in fact, as a particular case, we have seen that if you take any close interval, the image of close and bounding interval in r 1, because close and bounded interval in r 1 is a compact set.

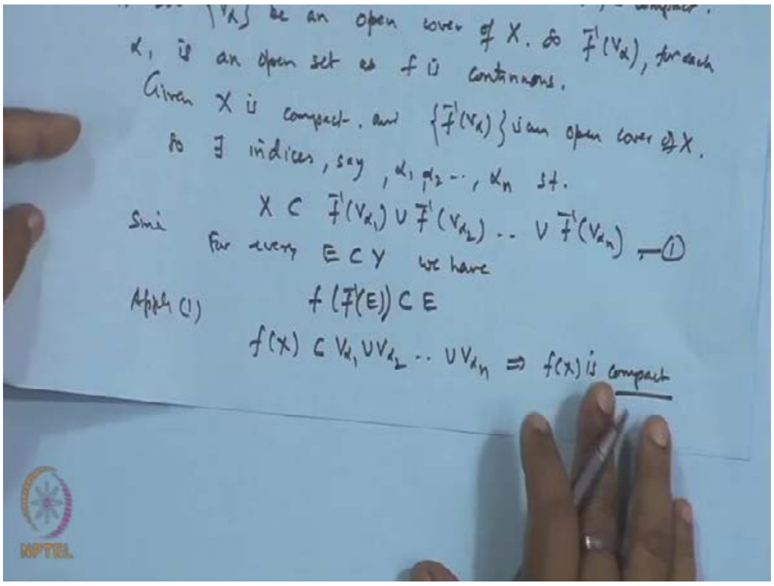
So, image of the close and bounded interval under f is coming to be compact; and even the r, a cell, k cell in r n space or in r k space is a compact set, k cell in r k space is a compact set, because r k is spaced the k cell is a compact set. So, close and bounded interval will be compact, but if we take only the close set and need not bounded, it may not be a, image may not be close set; it may be different, that we have various counter examples, we have seen, where this, but this so.

In order to show it is a compact set, what we want to prove is that every open cover of this has a finite sub cover, and it is already given x is compact. So, with the help of this, we will establish this result. So, let us suppose, v alpha, v an open cover of x, open cover of x; now, once it is open cover, it means each element, each point v alpha, when alpha is a index set, it is an open set, and since f is continuous. So, by the previous result, the inverse image of the open set must be open. So, f inverse of v alpha for each alpha is a compact, is a open set as f is continuous function. So, this is open, that one shows.

So, what is given is, given that x is compact. So, any open cover of x will have a finite sub cover; f inverse v alpha is an open set in x. So, the and f v alpha is an open cover. So, correspondingly we can say, this will be the alpha belongs to a, is if we choose this an open cover for x, then since x is compact there must be a finite sub cover for it. So compact and the sequence of this is an open cover of x.

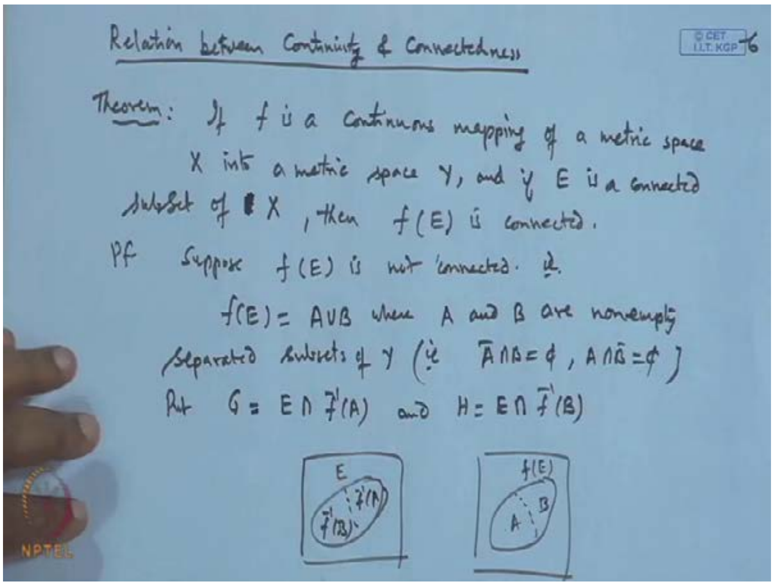
So, there exist the finite sub cover, there exist indices say alpha 1, alpha 2 say alpha n, alpha n such that the finite union of this, f inverse v alpha 1 union f inverse v alpha 2 union f inverse v alpha n will cover x, because x is compact. Now since our set E, since for every E which is a subset of Y, for every which is a subset of Y, we have this f of f inverse E, f of f inverse E is contained in E, this is true; if E is a subset of Y this may not be true, it is in opposite direction, but this is true, when E is a subset of Y.

(Refer Slide Time: 26:01)



Then using this, you on one, so apply on one, what we get? f of x is contained in v alpha 1 union v alpha 2 union v alpha n. So, f x is covered by a finite union of the open interval open sets. So, from this open cover, we can identify a finite sub cover which covers the f x; therefore, f x is compact. So, that is a very interesting, clear?

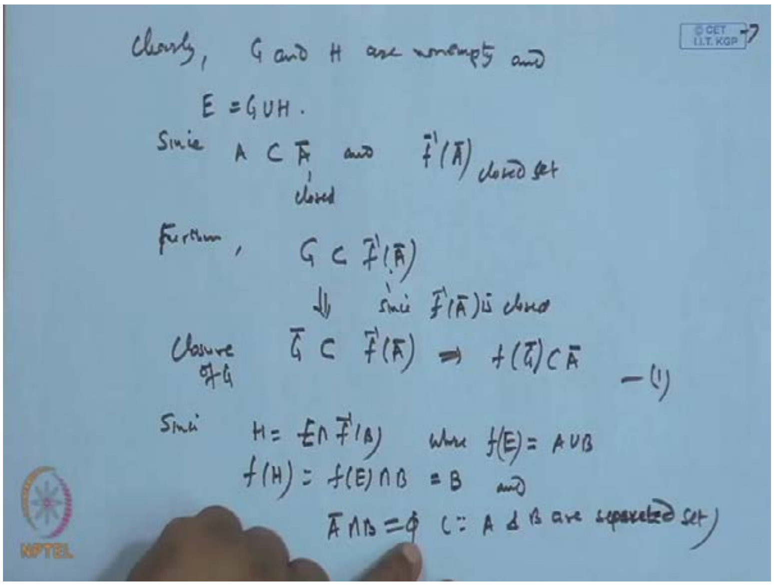
(Refer Slide Time: 26:50)



Now, for the next result, we will show it the relation between the continuity and the connectedness. So, the relation is, relation between continuity and connectedness, this is also we proved and write in the form of theorem. The theorem is if f is, if f is a continuous mapping, f is a continuous mapping of a metric space capital X, of a metric space of X, into a metric space Y, into metric space Y and if, and if E is a connected set, E is a connected subset of E, and E is a connected subset of X sorry subset of X, is a connected subset of X, then f of E is connected; if f is a continuous metric of a metric space X into a metric space Y and if E is a connected subset of X, then image of this connective subset will also be connected. Let us see the proof. Assume that contrarily, suppose f E is not connected, then we will reach a contradiction.

So, if it is not connected means, that is f E can be expressed as the union of the true sets A and B, where A and B are, A and B are non-empty, are non-empty separated subsets of y, that is A bar intersection B is empty, A intersection B bar is empty, that is why definition. So, that is of this form. So, when A bar intersection of this, and this is… Now, let us take, then G, we can write it, put G as the, put G as E intersection f inverse A and H as E intersection f inverse B; now, if we take this, what is our f E? This is set E, this is set E, here this is f E, this is f E; we have assumed f E is not connected, it means there are the two sets say A and B such that this condition is satisfied. Now, find out the inverse of this. So, we are getting f inverse A here, f inverse B here; now, if I find the intersection with a E intersection of inverse G, A and E intersection f inverse G, then obviously, G and H will be non-empty set because f E, we have already assumed is a not connected sets. So, there are the non-empty sets A and B, A and B are non-empty, whose union is non empty and they are separated. So, inverse is, intersection of this is non empty.

(Refer Slide Time: 31:12)



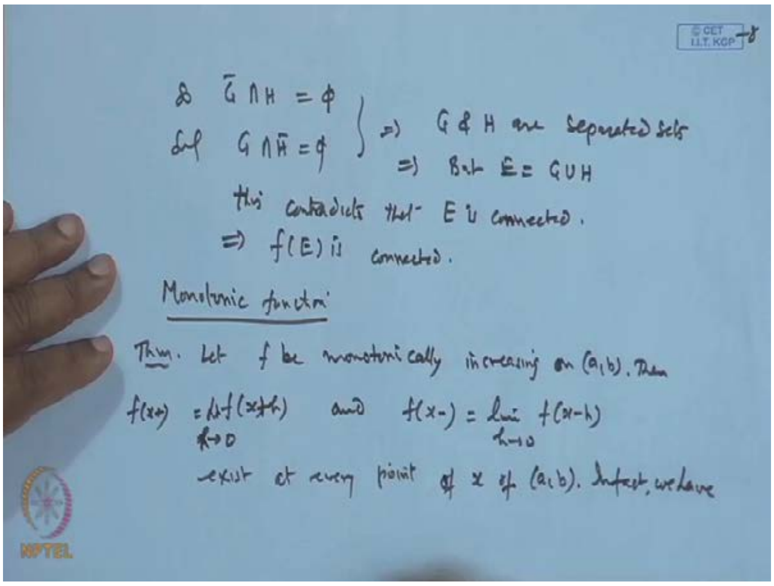
So, clearly G and H, clearly G and H are non-empty, first thing. So, there no problem, and also, the union of G H is nothing but E, union of this is nothing but E; then non empty neither G is empty nor H is empty. Now since our A is always contained its closures, is always contained its closure, and f inverse image of this A closure, which is a close set A bar, A bar which is a close set. So, inverse image of the open set is open, it can be extended it to the closed set, inverse image of the closed set is closed if f is continuous. So, f inverse A G will be a, inverse image will be a closed set. So, that is not. So, if it is closed and G is there, so we can say G is contained in, because G is already contained, G is a subset of this f inverse A, from here G is a subset of f inverse A.

So, if you replace this A by A bigger closure of A; obviously, G will remain as a subset of this. So, we can say G is, so further G is contained in f inverse A closure, thats it. Since this is close set, since this is close set is closed, so obviously, all the limit point of this must be here. So, the limit point if I take, then of G, closure of G, closure of G if I take all the point set, point of G including the limit point, then obviously, it will also be contained inside it. So, this is correct. So, once it follow, then what does it mean? It means f of G closure is contained in a closure, this is one thing. So, let it be one, f of G.

Now what is our H? The H is this E intersection f inverse B. So, f of H, since H is f intersection f inverse B. So, f of H, sorry E, this is E intersection. So, f of H will be, what and what is our E? Where E is the, E is we have taken this one, E is connected f E is and f E is sorry f of E is this, A union B, and A union B, they are separated set satisfy this condition. So, if you find the f of H, then it becomes f E intersection B. So, that is equal to f E intersection B and then when you find the intersection with this, obviously it comes out to be B. So, f of H will be B.

Now, f of G inverse, G closure is contained in A bar, A bar intersection B is empty, A bar intersection B is empty, and A bar intersection B is empty, because A and B are separated set, are separated set, these are separated set. So, this is empty set; therefore, when you find the intersection of G bar, G bar with our H, what happens? Intersection of G bar with H that is nothing but an empty set. So, intersection of G bar and H.

(Refer Slide Time: 35:31)

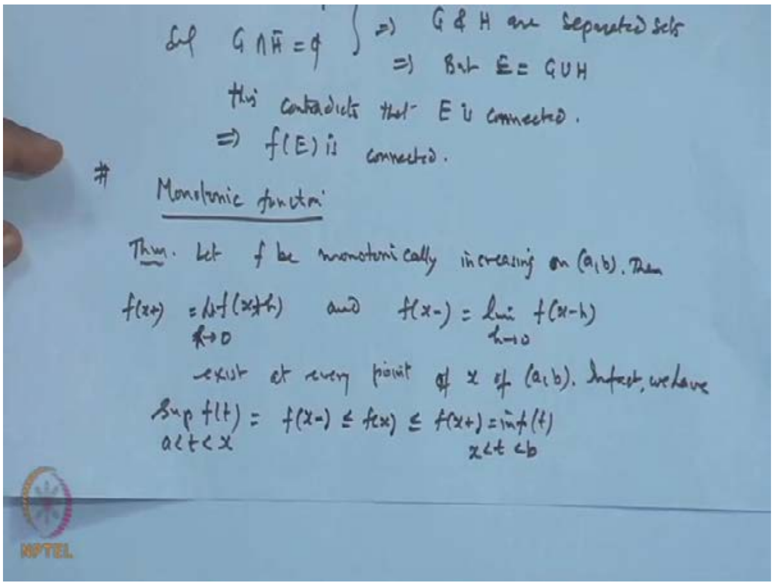


So, intersection of G bar intersection H is empty, why? Because G bar is contained in this f inverse A, H is B, f inverse A is contained in A bar, this G bar is contained is a inverse A and this H is E intersection of this. So, when you find the G bar intersection H, they will be disjoin, they will be an empty. So, we conclude this. Similarly, we can say G intersection H bar is empty. So, this shows G and H are, G and H they are. So, G and H are separated set. So, E, but E is the union of G and H, is it not? So, but E is the union of G and H. So, a contradiction. So, it is not possible; therefore, this, so this contradicts that E is, E is given to be connected set, E is connected; and it is contradiction is because our wrong assumption, that we assumed, that f is not connected. So, this implies f of E is connected. Is it okay or not? Because why it is pi? Because G bar is contained in this and

H is B, where this two are disjoint, this two are. So, therefore, the intersection will be empty set. So, that is what. So, this shows our relation between the, this one. Now as similar relation, we can also establish for monotonic functions. So, the relations between some monotonic, they conditions.

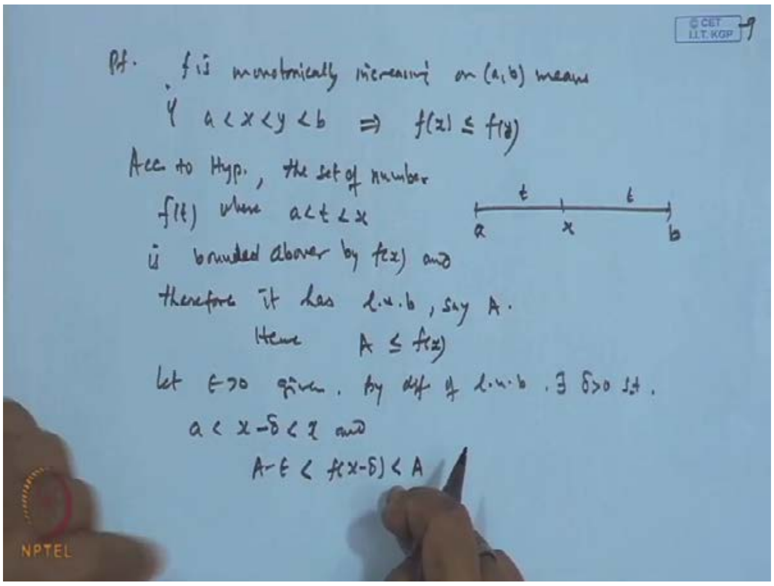
So, let us see the monotonic functions if f is continuous then what. So, the first result, we have seen that result which is, I will not draw the results, let just see, let f be monotonically increasing function f B, monotonically increasing on the interval say A B then the left, right hand limit of this f x, when x tends to plus side, x tends to plus, that is f x plus, f x plus, that is, this is the same as x plus H, x tends to 0. So, f x plus and f x minus left hand limit of this that is equal to limit f of x minus H, when H tends to 0; and then this exist, if f is a monotonic increasing function left hand limit and right limit will always exist.

(Refer Slide Time: 39:35)



At every point, at every point of x of the interval a b, and in fact, we have this inequality that supremum of f of t, when t lies, when t lies between a and x, a and x will be equal to the left hand limit, which is less than equal to f x, which is less than equal to the upper limit and which is equal to infimum of f t, when t lies between x and v. So, over the interval, this scenario is there, means the value of the function f x always lies between the lower limit, and the left hand limit and right hand limit; now, if both are equal then we say the continuity follows, if they are not equal, the point will be point of discontinuity. The question is, how many such points is possible over in the interval a b if the function is monotonic increasing function or monitoring decreasing function? Will this set of points, where the monotonically increasing function is exist, there in that will it be countable or uncountable? The answer is that set of all points where the monotonically increasing or decreasing function is not continuous forms a countable set. So, that is the main result which I wanted to show. So, let us see what is that monotonically? Because this will help in getting the result.

(Refer Slide Time: 41:15)



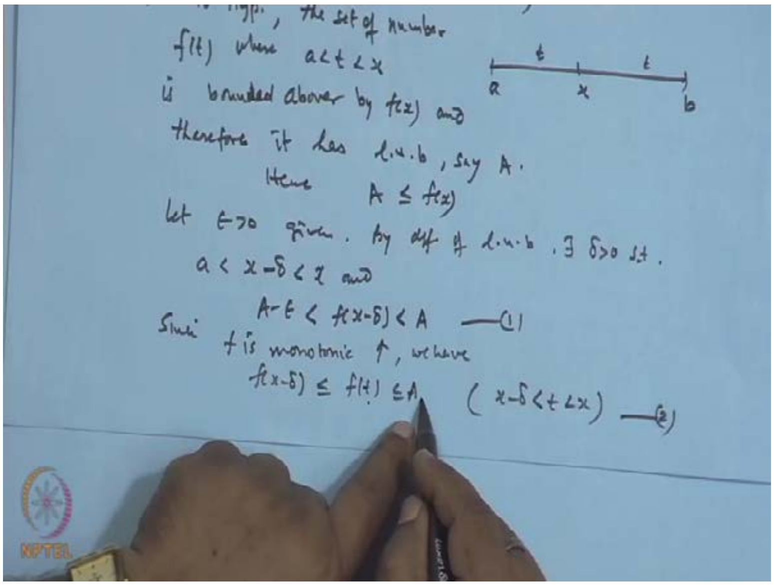
So, I hope that this result, we can prove it or let me see just proof of this result very fastly, and then we can go for it. Suppose, what is given is, this is given that f is a monotonically increasing function, what it mean? If f is monotonically increasing, it means what? That is if x and y are the two points and if x is less than y then the corresponding image that is a increasing on the interval a b means that if a is less than x less than y less than b implies that f of x is less than or equal to f y, if we say it is strictly increasing, then the sign is strictly will follow, otherwise.

Similarly, for a decreasing the order reverses. So, now, if we take this f to be monotonically increasing function, then this interval a b is this; now, here is the point x. So, first I am taking this point t which lies between a and x; and then here I am taking the point when it lies here. So, over this interval, the function is an increasing function. So, what will be the upper bound? Upper bound will be x.

So, in fact the least upper bound will be there. So, the set of according to the hypothesis, according to hypothesis or this hypothesis, the set of numbers f of t, f of t for t lying between, where a is less than t less than x, this number is bounded above, is bounded above by f x; because it is monotonically increasing sequence; and therefore, and therefore, it has a least upper bound, therefore, it has least upper bound say A, A is the least upper bound. So, clearly there, hence A will be less than equal to f x, this is true. Now, we have to show this. Now let epsilon greater than 0 is given. So, if I choose a number, it is slightly lower than a, say a minus epsilon, then there exist a delta so that a minus x.

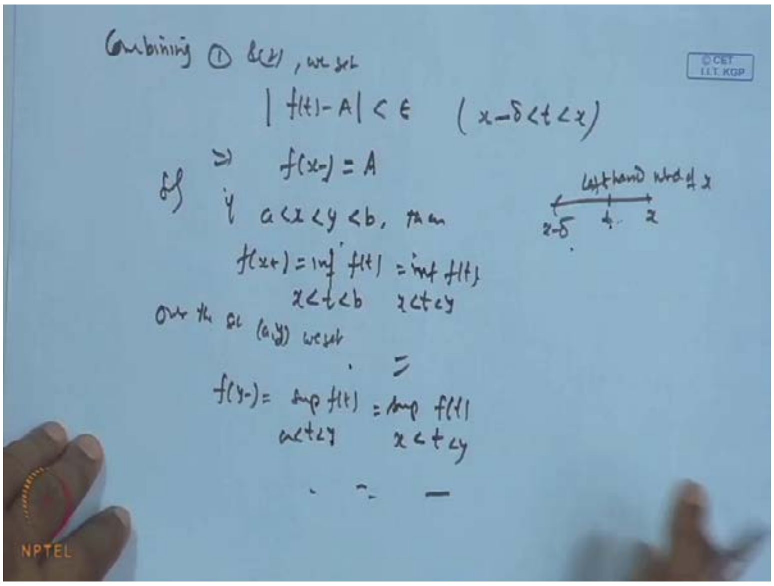
So, that a will not be average in, a minus epsilon will not average in upper bound. So by definition of the upper bound, by definition of least upper bound there exist a delta greater than 0 such that for all x lying between a less than x minus delta less than x and this condition, and this a minus epsilon will remain less than f of x minus delta which is less than a, because a is a least upper bound. So, if I take a number exactly lower than this, then there is a point lying x minus delta, a point will be available such that the functional value will lie between that.

(Refer Slide Time: 45:03)



Now further, since f is monotonically increasing function, let it be equation one, let it be equation one, since f is monotonic increasing function. So, we have that f of x minus delta is less than equal to f of t is less than equal to A, whenever x minus delta is less than t less than x; this is by definition monotonically, let it be equation two. So, if we combine one and two, what happen? a minus x is less than this, which is less than f t which is less than equal to A. So, combine one and two, can you not say like this?

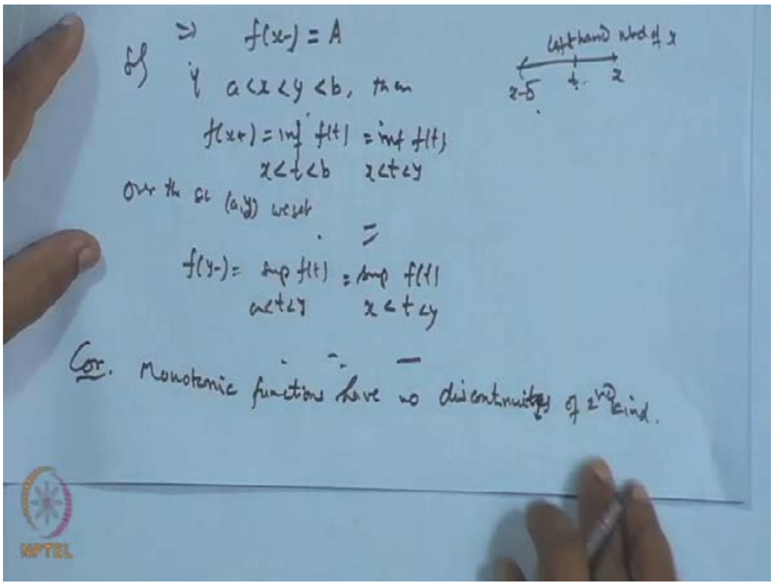
(Refer Slide Time: 45:55)



So, combining one and two, we get, combining one and two what we get is the mod of, we get modulus of f at t minus A remains less than, less than say epsilon, whenever this less than (( )) whenever x minus x minus delta is less than t less than x. So, from the x, x is point here, and here this is the neighbor, left hand neighborhood, left hand neighborhood of x. So, we are approaching from this side, the t is somewhere here. So, what you are doing is the image is f t under a, with a is less than epsilon, it means the left hand limit of the function when x approaches approach from the left hand side (( )). So, this shows, this imply the left hand limit of the function is a, that is one thing.

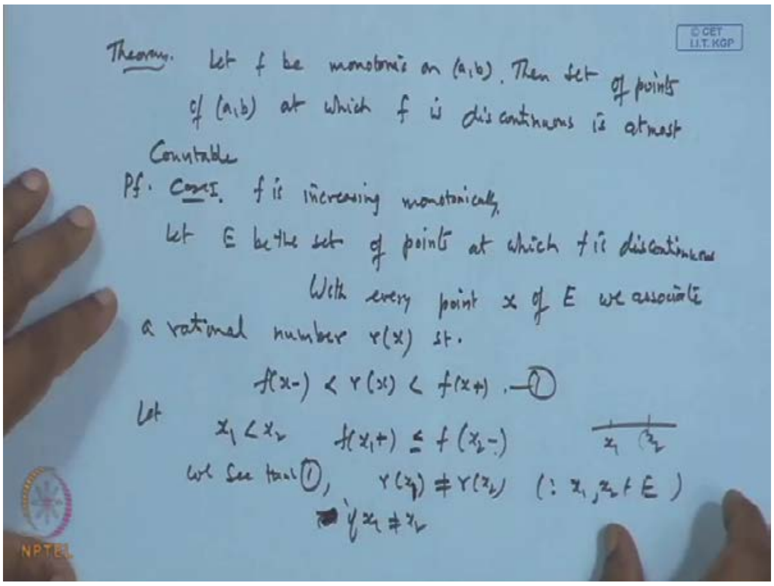
Similarly, we can prove the other side. So similarly, we can show, similarly we can show if a is less than x less than y less than b, then we get, we can see the right hand limit of this is the infimum of the f t, when t lying between x and b and this is the infimum of f of t and t lies between x and y. So, using this, another we get the result. So, the last week we obtained by combining this two over the set a and then if we take over the set over the set, say a y, then we get, then we get the, this point f of y minus is the supremum of f t, t lying between a and y which is equal to supremum of f t when t lies between x and y. So, this follows, comparing we get the results for this.

(Refer Slide Time: 48:29)



So but this result is interesting because the corollary of this; the corollary says, monotonic functions, monotonic functions have no discontinuity, discontinuity, have no discontinuities, no discontinuities of the second kind. Here the limit does not exist, but the limit will always exist, lower limit right hand limit or left hand limit always exist, they may not be equal. So, the point of discontinuity may be there, but it is not of the second kind, this is clear from here.

(Refer Slide Time: 49:18)



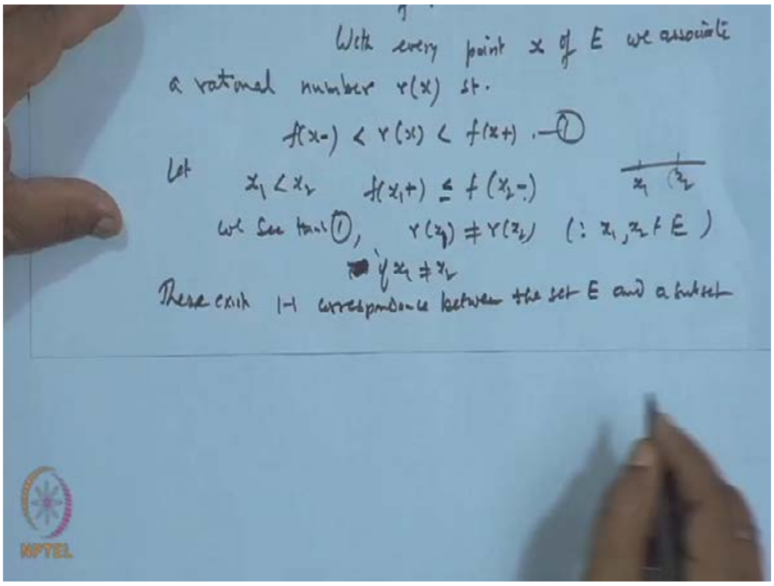
Now, if the limit does not exist then there will be a point of discontinuity and how many points do they have? This can be shown by the following theorem, the theorem says let f be a monotonic, f be monotonic on the interval a b, monotonic on the interval, then the set, then the set of points, set of points of a b, set of points of a b at which f is discontinuous, discontinuous is at most, is at most countable, is at most countable. Let us see the proof of it.

Suppose for the sake of a, suppose f is, case one. When f is a increasing function, monotonically increasing function, increasing monotonically, monotonically, f is a monotonic increasing in function and let E be the set of all points, E be the set of points at which, at which f is discontinuous. Now, this point cannot discontinuous, cannot be of second type, only first time discontinuity there, or removable discontinuities. So, with every point x of a, now with every point x of E, where the point, discontinuous point, we associate a rational number, a rational number say r x, r x such that the lower limit left hand limit of the function at the point x is strictly less than r x is strictly less than, means it is of the first kind, where the limit exist, but both are not equal. So, we can identify ration number in between. So, corresponding to each point, we can identify rational number which is two, Now this is a, there is only one point. So, let us, let us x, since x is less than, let x 1 and x 2 are the two points.

Now, x 1 x 2 are the two points then if the image of this, the right hand limit of the x 1 will, if x 1 is less than x 2 then the right hand limit of this will be either less than or equal to the left hand limit of this; because x 1 lies here, x 2 is here. So, when you get the points x 2 will always be greater than equal to value of this. So, left hand limit of x 2 will at the most coincide with the right hand limit; but if x 1 is different from x 2, and x 1 and x 2 both are the point of discontinuity, so we can identify the rational number r x 1 and r x 2 which are different, but we see here, we see that from first that the rational number r x 1 is not equal to rational number r x 2; because this x 1 and x 2, these are the point of E. So, what (( )).

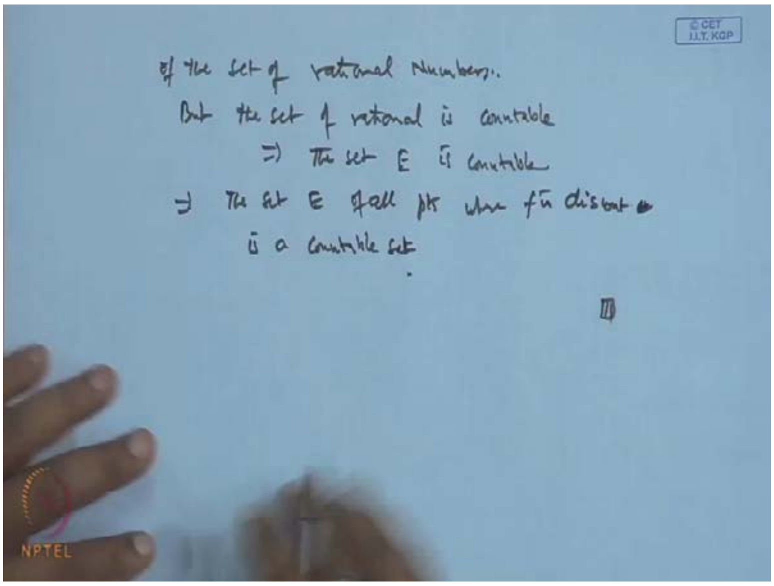
This implies if f x 1 is, this imply x 1 is different from x 2. So, if x 1 is strictly less than x 2 and if, sorry if this is r x 1 is different from, if x 1 is different from x 2 then x 1 is also point of discontinuity. So, we can get the r x 1, corresponding to x 2 we get a point x 2, when x 1 and x 2 at different points, then in that case we do not have this; it means that corresponding to each points into x 1, there is a rational number and vice versa. If the rational numbers are there, this satisfy this condition r x 1 is not with them, then with point x 1 x 2, we can identify which are the points at which the lower limit and the upper limit do not coincide, that is it will be a point of this. So, there is a one to one correspondence.

(Refer Slide Time: 54:19)



So, there is, there exist one to one correspondence, one to one correspondence between the set E, between the set E and a subset, a subset of the set of rational numbers, rational numbers, but set of rational numbers.

(Refer Slide Time: 54:42)



But the set of rational numbers is a countable set, and there is a one to one correspondence between this, this imply the set E is countable. So, what this shows? E is the set of those (( )). So, this means that f continuous that is, that is the set E of all points, where all points, where f is discontinuous, discontinuous on is a countable set and that we wanted to show. So, this proves that it.

Thank you very much, thanks.