A Basic Course in Real Analysis
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Lecture - 32
Types of Discontinuities, Continuity and Compactness

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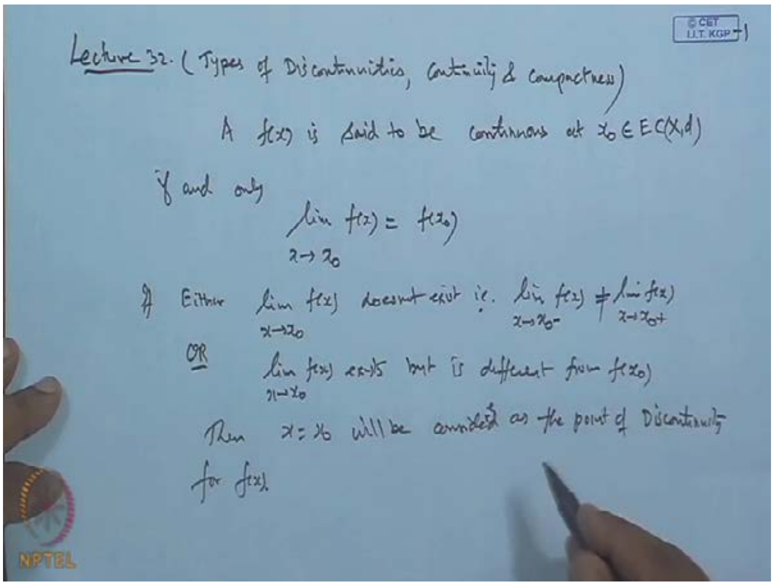

So, today we will discuss types, various types of discontinuities, and then we will see the relation between continuity and compact compactness, continuity and compact, compactness. So, let us see, first we know a function f x is said to be continuous at a point x naught belonging to the domain say E, which is sub set of say x d, if and only if limit of this function f x, when x approaches to x naught exists, and equal to the value of the function, at a point x naught.

Now, if this condition, when we are saying limit of f x tends to x naught equal to f x naught, it means the function must have a limit, when x supposed to x naught, that is the left hand limit, right hand limit, both exist, they are equal, and the value of the function at the point x naught must coincide the value of the limit f x, when x approach to x naught, Then only the function will be said to be a continuous function.

If any one of the condition fails, that is if either the limit does not exist, does not exist, that is, that is the left hand limit of this function when x x goes to x naught from the left hand side different from limit of the function f x when x tends to x naught from positive side, then we say the limit does not exist at that point or may be if the limit comes out to be infinity for that. In that case one or limit exist, limit exist, but is different from, different from the value of the function f x naught.

So, when the limit does not exist, left hand right limit right hand limit does not exist, then also the function will not be continuous, or if the limit exist, but it is value is different from f x naught, then also the point x naught will be said to be continuous, or it may sometimes in the limit of this function f x, when x tends to x naught and y tend to y naught, both does not go to left hand limit also does not exist, right hand also does not exist; then also we say the point x naught is a point of discontinuity.

So, if this case is, there then x equal to x naught will be considered as the point of discontinuity for the function f x; so obviously, the point of discontinuity depends on the cases whether limit exist, but different from x naught, then we name this point of discontinuity to some like a removable discontinuity, when the limit does not exist then we give some other name to the discontinuity, or when the limit does not exist, left hand and right hand limit does not exist, we also give some other name, or some time the limit tends to infinity, then also we give some times different name. So, according to situation, the type of discontinuities are characterized.

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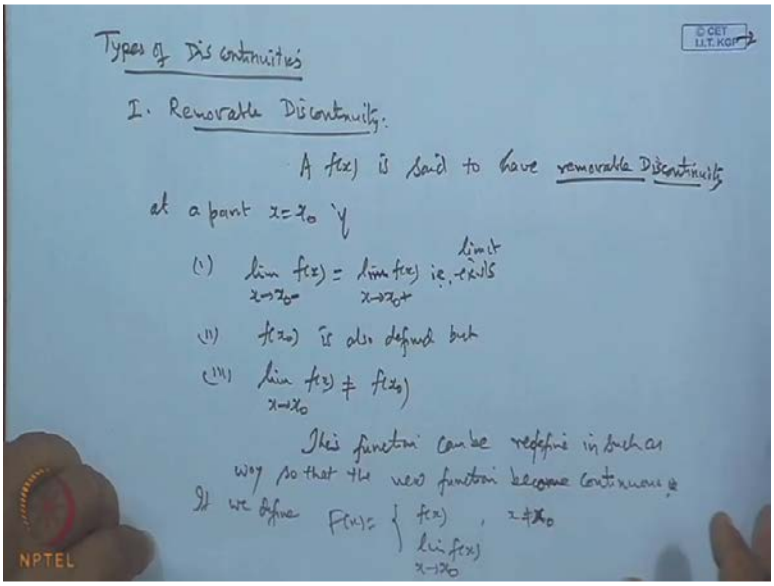

So, first is, let us see types of discontinuity. First is removable discontinuity, removable discontinuity. So, removable discontinuity, if the function, A function f x, f x is said to have, is said to have, is said to have a removable discontinuity, discontinuity at a point say x equal to x naught at a point x equal to x naught in the domain, if the functional limit, if limit of f x, when x tends to x naught from negative side, equal to the limit of this function f x, when x tends x naught from the positive side, there exist and both are equal; means, left hand limit is same as the right hand limit, so limit exist, that is the limit exist that is the limit exist; and second one is the functional value f of x naught also define, is also defined, but the third is the limit, that is limit exist.

Third is limit of this function f x, when x tends to x naught is not equal to the value of the function at a point x naught; then we say x naught is a point of discontinuity, and we call it as a removable discontinuity. Why removable discontinuity? Because we can redefine the function again, so that it can be the new function will be, will be continuous, because the reason is, the reason is the this function can be redefined, redefined, can be redefined, redefined in such a way, in such a way, such a way, so that the new function becomes continuous; that is, if we define capital F x as our small f x, when x is different from a x naught, and equal to the limit of the function f x when x tends to x naught, then the new function f x. So obtained will be a continuous, then f will be continuous; f x has x naught as a continuous point. So, this function f in this, where the x naught was not, it was a discontinuous point, can be converted to a continuous point by simply removing or redefining the function.

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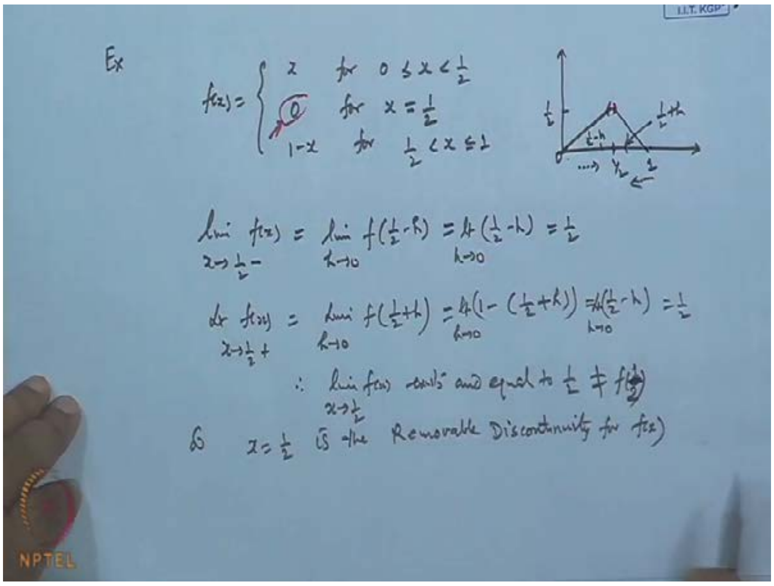

For example, if suppose, we take the function f x is defined as, say x for 0 less than equal to x less than half equal to 0 for x is equal to half, and equal to say 1 minus x for x lying between half open at half close at 1. So, over the interval 0 1, we are defining the function. So, the function is of this type, this is 0.

So, between 0 to half, the function is y equal to f x. So, it will be something like this, this is half, this is half, here is say half; and then at the point half, it is coming to be 0. So, basically this half, this point is excluded, here we are not touching half here, it is coming down, is it not? And then from this point, when x is not adjust near to half, it start from here. So, we are getting this thing and then at the point one, it is coming to be 0, this is one. So, what we say here, x shift at the point one, if the function is very smooth function.

So, here if we look, the limit of this function f x, when x approaches to half minus, then this is the same as the limit of the function f half minus h, h tends to 0 from the left hand side you are approaching, from here you are approaching, so take the point x is a half minus h. So, this is the point half minus h; so consider this, so now when the point is lying between 0 and half the function is define as x.

So, this equal to half minus h and then limit of this, h tends to 0, which comes out to be half. So, from the left hand side, the value of this limit is coming to be half; now, from the right hand side, if we look limit f x, when x tends to half plus, then this is the same as to right limit h tends to 0 have f of half plus h, choose the point here, which is half plus h and then approach about this side.

So, when the point is lying between half and one, this interval is taken consideration, where the function is defined like 1 minus x. So, 1 minus this and then take the limit as h tends to 0. So, this comes out to be half minus h and then h tends to 0, which is equal to half. So, the right hand limit and left hand limit, both are coming to be same as half. So, limit exist; therefore, limit exists, limit of this function, when x tends to say half exist, and equal to, and equal to half; but the value of the function at the point half is 0, which is different from the value of the function.

At a point half, sorry, at the point half, because the value at the point half is given to be 0; here, we are limiting the value coming to be half, it means this function has such a function were the limit exist, but the value of the function at that point is different from the limit. So, half becomes the point of dis removable discontinuity. So, x equal to half is the point is the removable discontinuity for this function f x. So, if we redefine the function instead of 0, if I redefine the function here, if I define redefine the function here, the value at x equal to half equal to half, then our function becomes continuous.

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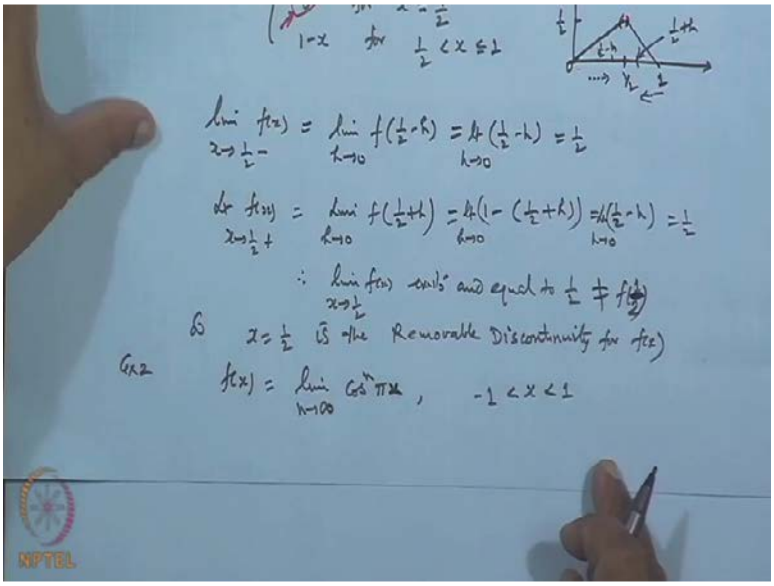

So, that is why removable. Take another example, suppose another one, two; let us define the function f x as f x equal to, f x equal to limit of this as n tends to infinity cos 2 to the power n pi x, here x lying pi x, here the x lies between minus 1 and one, it is not covering end points, open interval minus 1 to 1.

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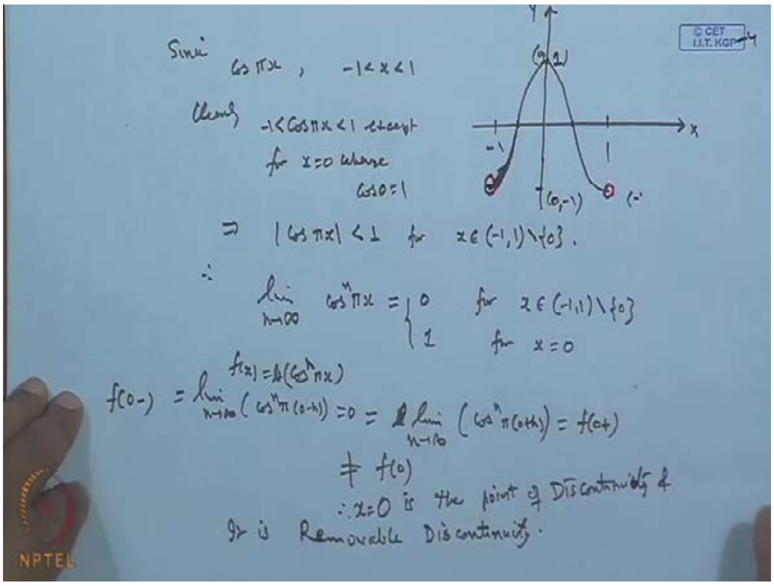

Now if we look the function, the function cosine of this function, cosine pi x, when x lying between minus 1 to 1, the function pi x has the graph like this, minus 1 to 1; when x is 0, when x is 0, cosine 0 is 1. So, when x is half pi by 2, the cos 90 is 0, similarly, minus pi by 2, it is 0. So, it will be something like this, something like this, and when x equal to minus 1, but it is not touching minus 1, is not touching, in fact this point is not attend; similarly, this point is not attend.

So, but when it is minus 1 and plus 1, the value is coming to minus 1. So, this is the graph of this function, x axis and this y axis. So, value of this is 1, x by is 1, 0, 1 in this is the 0, 1 point and here this point is, say minus 1, minus 1, here this point minus 0, minus 1. So, cosine of this thing, for this function, clearly cosine of pi x always lies between minus 1 to plus 1, except for x equal to 0 where the value is cosine 0 is 1.

So, when x is different from 0, the value is always minus 1 to plus 1, that is mod of cosine will be less than 1 and there. So, this shows that mod in mod of cosine pi x will remain less than 1 for all x belonging to the interval minus 1 to 1 minus singleton set 0. This is our concept. Hence, limit of this function cosine pi x 2 to the power n, when n tends to infinity, this limit will be 0, because this will be always less than 1. So, n is infinity it goes to 0 for all x belongs to minus 1 to 1 minus singleton set 0 and equal to 1 for x equal to 0. So, what then? Throughout this function attends the value 0, function is very smooth function, it is continuous except at the point 0, because what happened is, when you take the left hand limit of this, left hand limit of this function f x, f x is cosine n to the power pi x limit of this .

So, when you take the f x is 0 minus point, the left hand side point, it means the limit of this thing, limit of cos n pi 0 minus h, just you take the 0 minus h and then here. So, this number will be less than 1, mod of this, so this will be 0; when you go from the outside the limit of this, sorry, the function limit of this function as n tends to infinity, cos of n pi 0 plus h is also 0, that is f of 0 plus, but the value is entirely different from f of 0, because f of 0 is 1; therefore, 0 their function 0, x equal to 0 is the point of discontinuity, discontinuity, and it is are removable discontinuity, removable discontinuity, that can be easily seen from it at x equal to 0.

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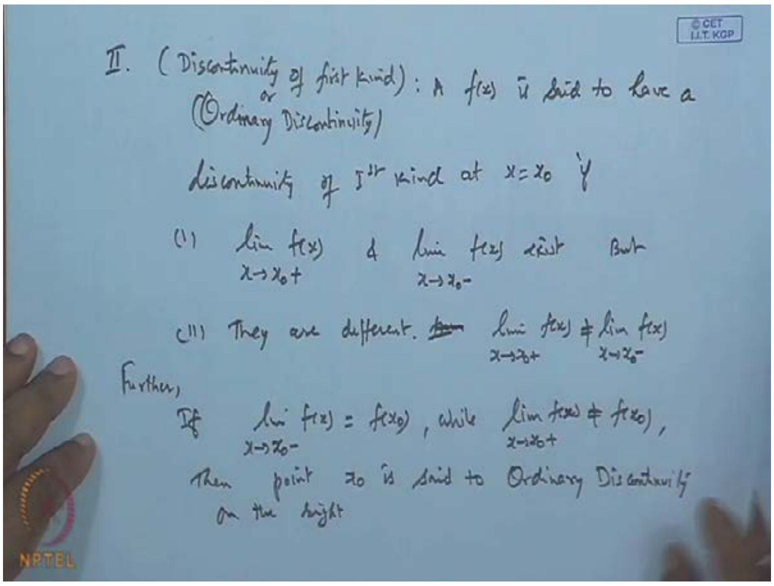

So, this is the first kind of discontinuity that is known. The second type of discontinuity, we call it as a discontinuity of first kind, of first kind, discontinuity of first kind, or sometimes, we also called the ordinary discontinuity, or ordinary, ordinary discontinuity, continuity, ordinary discontinuity at this point. So, what we say is, if the limit exist both the left hand and right limit exist, but they are different, if a function f x is said to have, is said to have, is said to have discontinuity of first kind, of first kind at the point x equal to x naught, if the limit of this f x when x tends to x naught plus and limit of this f x, when x tends to x naught minus, that is right hand limit and left hand limit exist, both the limit exist. But but they are, but they are different from, but different from, have different values, both exist, both exist, but they are different, they are different, they are different, means they are not equal; that is left hand limit f x, when x tends to x naught plus is different from limit of this function f x, when x tends to x naught minus.

So, both are different, then x naught will be said to be a discontinuity of first kind. Now, since we are in the discontinuity, we also test the value of the function at a point x naught. So, suppose the value at a function x naught is well defined and if it coincide with one of the, either left hand limit or right limit, then the further classify it; if the value of the function f x naught coincide with the right hand limit of this; then we say.

So, if further, if the left hand limit of the function f x, when x tends to x naught minus coincide with the f x naught, while the right hand limit of this function f x, when x tends to x naught plus adjust, but different from the value of the function at f x naught, then point x naught, then point x naught is said to be, is said to be ordinary discontinuity ordinary or discontinuity of the first kind, discontinuity ordinary discontinuity on the right, because left hand limit coincide with the value. So, it is continuous from the left hand side, but it is not continuous from the right side. So, what we say is x naught is a point discontinuity of are kind one, but from the right hand side. So, its ordinary discontinuous from the right hand side. So, its ordinary discontinuous from the right hand side.

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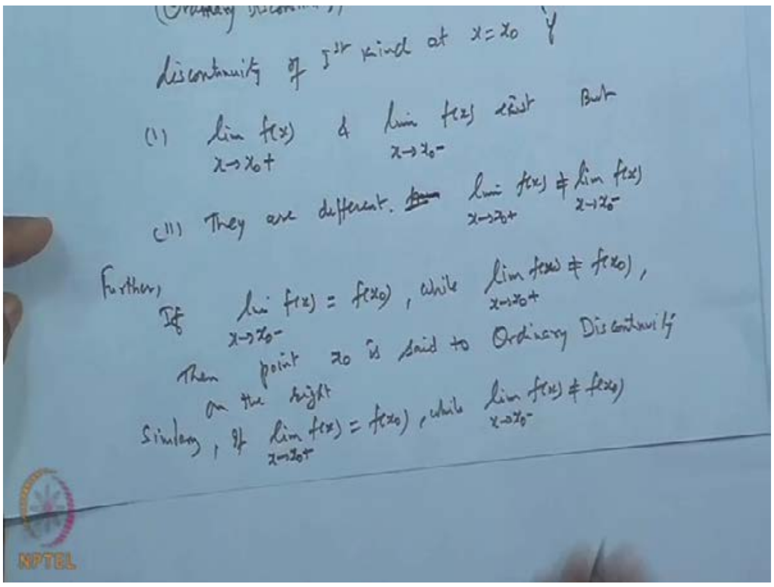

Similarly, if suppose, similarly, similarly, if limit of this function f x, when x tends to x naught plus coincide with the functional value f x naught, while the right left hand limit exist, but is different from the value of the function f x naught.

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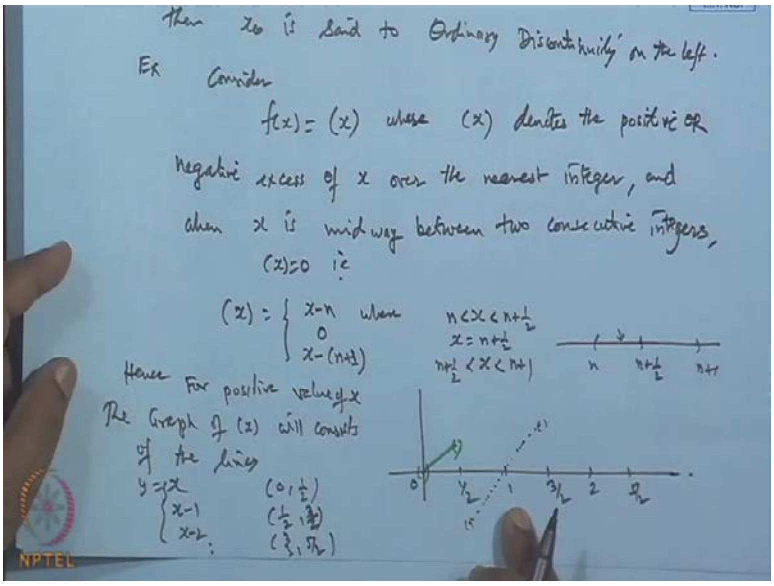

Then we say, it is then x naught is said to be ordinary discontinuity, ordinary discontinuity on the left, on the left. So, that is for example, let us consider the function f x, f x equal to x within a small bracket, where x within small bracket denotes the positive, positive or negative, or negative axis, negative axis of x over, over the nearest integer, nearest integer, over the nearest integer, and when, and when, and when x is mid-way between the two, when x is mid-way between two consecutive integers, consecutive integers, then the value of this is 0, then the value of this 0, it means what?

So, it means like, suppose we have the function, say here is n, here is n plus 1; let us take these two consecutive integer; then this point is n plus half, it is the middle point for basic interval; if the x lies here, the value of this x will be denoted by x minus n. So, x minus n, if x lies here, then in that case, the value will be denoted by x minus n plus. So, that is the x is defined as, x is defined as x minus n when x lies between n and n plus half, when x lies between n and n plus half the nearest the integer is n; so in that case it will be in subtracted from n and we get x minus n, but when x lies between n plus half to n plus 1, then the nearest integer will be n plus 1. So, in that case, the, this will be defined by x minus n plus 1. So, that will be the difference for this, and when x is exactly equal to n plus half, then what happened? If x is exactly n plus half, the value of this is given to be 0. So, this is that graph.

So, if we look the graph, what is that? Suppose I am taking the positive side. So, let us take the interval 0, half, 1 and then 3 by 2, then 2, 2, 5 by 2, like this and so on. Now between the intervals 0 to half, 0 to half, what is this? This curve will be like, if x lying between 0 and half, so here n is 0. So, it will be defined as x only that is the curve will be something like this; but it does not cover contain this point, and as well as it does not contain this point, because x is lying between 0 and is strictly is 0. Now, take the, as soon as the point comes over here, what is this value? This is coming to be x minus 1, x minus 1. So, it is like this, when x minus 1 and x equal to half, the value will be come out to be below. So, below minus 1 and when it is one, one, then it is coming to be 0. So, basically this line will be by minus x.

So, here between the x minus the curve, the function f x the... So, hence we can say the graph of the function, the positive value of x for positive values of x for positive values of x the graph, the graph of x will consist of, consist of the lines y is equal to x, y equal to x in between 0 to 1, when in, over the interval 0 and half, then it will consist x minus 1 over the interval half to 1 and continuous like this, x minus 2 over the interval, say this interval is 3 by 2 half to 3 by 2 sorry will go like this, because here it will start from this, and then it will go from here directly one.

So, it will go something like, by minus x and then 3 by 2, three by 2 something. So, we get this one, and then 3 by 2 to 5 by 2 like this, 3 by 2 to 5 by 2 and so on, continue this one. So, this is the graph for this and remember this end points are not available here. At the point, integers points, the value is coming to be what? When it is integers, say n, n plus 1 and n, n this value is again defined as say integers positive or negative, is it not? So, x minus a integer, it becomes 0 for that, so that is what point. Now, what we see here is, whenever the point is suppose I take the point n to n plus 1.

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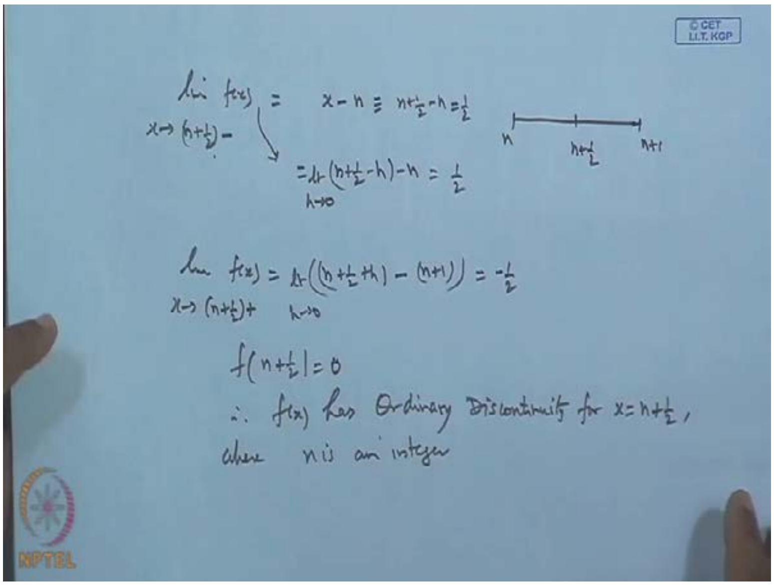

So, let us take this n, here is n plus 1, and here is n plus half. So, what will be the left hand limit of this function when x is n plus? So, the value limit of this function f x when x tends to n, n plus half, this is the point from negative side, from the negative side when you say, then what will be this? The function will have the, when this side, it will be x minus n type. So, we get the x minus n, n type, n type, but x is n plus half. So, basically this is equal to, equivalent to, this is n plus half minus n, so basically this is half. So, when you are taking the function, this function is this one, is it not?

So, limit of this means x equal to x plus h. So, here you write x n plus half plus minus h. So, we can say this will be equal to that is, here we can write it, that x equal to n plus half minus h and then minus n and limit of this thing, is it not? When h tends to 0. So, basically, it comes out to be half. So, when you are choosing the limit from the negative side, the value is coming to be plus 1; then when you are taking limit of this function f x when x tends to n plus half positive side, then this will be equal to what? n plus half plus h minus next term will be n plus 1, this term will be subtracted and taking the limit, x tends to 0. So, minus n will go and basically it will come out be minus 1. So, right hand limits comes out to be minus 1. And what is the f of n plus half? Is exactly coming to be 0, because on the middle point, we are taking already assuming to 0. So, the function therefore, therefore, the function f x has ordinary discontinuity, discontinuity, discontinuities for x equal to n plus half, n plus half, where n is any integer; similarly, for the negative side, we can get for this.

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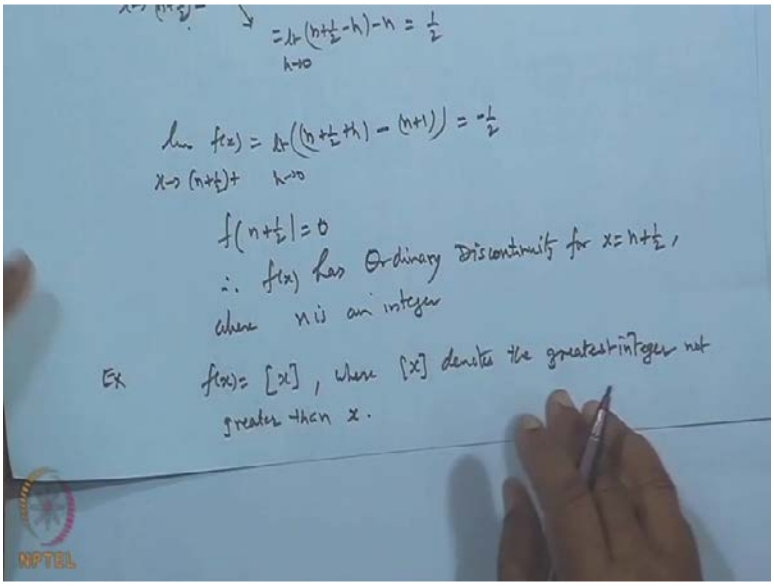

So, this is the point of inter ordinary discontinuity or discontinuity of the first kind. Let us take another example. Suppose I take very interesting and famous one, always we used a box function. Here f x equal to box x, where box x mean that get denotes the, denotes the greatest integer, greatest integer, greatest integer, not greater than, not greater than, not greater than x naught greater than x. So, greatest integer not greater than x.

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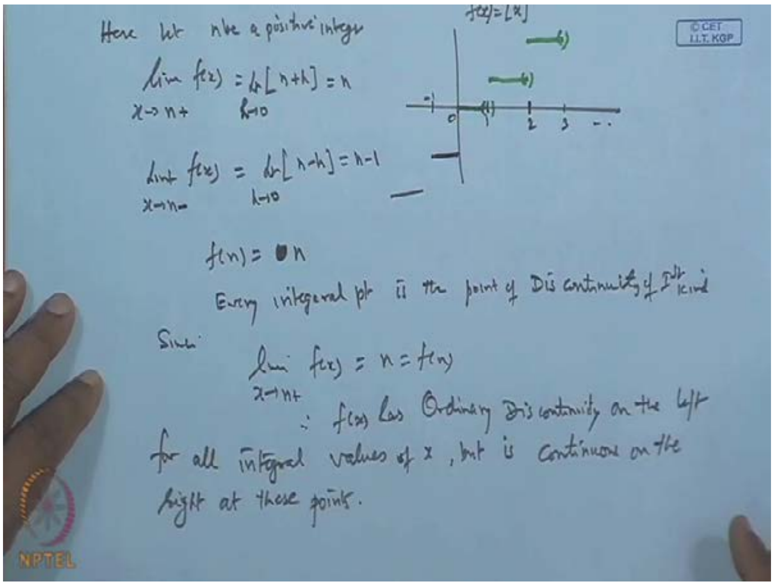

So, let us see, what is the graph of this, if you look the mod x graph, the graph of this function f x is this, f x is this one. Now, when x lying between 0 and 1, the greatest, as x denotes greatest integer not greater than 1. So, what is the integer, is only 0? It means the curve will be like this, it will start from the 0, but it is not touch one, it will not touch one. So, we have this gap; now, from 1 to 2, when you take 1 to 2, then the greatest x denotes the, greatest integer means integer not exceeding by x. So, one is the integer. So, here we are getting the graph like this, start it will here up to go this, here again this point is not, is missing, then three. So, when you got the three, you get again this one, again this point is missing here, this is three and continuous; similarly, when you take minus 1, you are getting something like this, is it not? And so on and so for.

This is like, this is like a step function, is say step function. Obviously, integral points are the point of this continuity. So, the graph function then we say here. So, what is the graph of the function for a positive axis of consist of segment line in the left hand limit. Now clearly here, when you take the limit of this function f x when x tends to n is any integer say 0 n plus, then this value will be what? This is equal to n plus h and limit h tends to 0, but n plus h lies between n and n plus n. So basically, this is n, h tends to 0, is it not? Because the greatest integer is only n. So, limit of this, when you take the n plus 0, it is n, the limit of this function from left hand side, x tends to n minus n is integer.

Let n be positive integer, I am just taking any integer, but positive integer, let us take, and then get for the negative one; then the negative side, left hand limit of this is the right hand limit. So, left hand limit will be n minus h and limit h tends to 0, but n minus as is strictly less than n. So, it lies integer the interval n minus 1 to n. So, this limit will be come out to be n minus 1, is it not? And what is the f of n? f of n is 0, f of n that is sorry, f of n will be what? An integer so it, sorry n. So, f of n is. So, what will we say the limit? Left hand limit exist right limit exist, but they are not equal, they are not equal.

So, every integral point, every integers, integral point is the point of, is the point of discontinuity of first kind. Here now, one more thing is, since the limit of f x, when x tends to n plus is n which coincide with the value of the function, so it has a continuity in the right hand side, discontinuity in the left hand side. So, we say it is a discontinuity on the left. Therefore, therefore, f x has ordinary discontinuity, discontinuity, discontinuity on the left, on the left for all integral values, integral values for all integral values of x, but is continuous on the right at these points, at these points.

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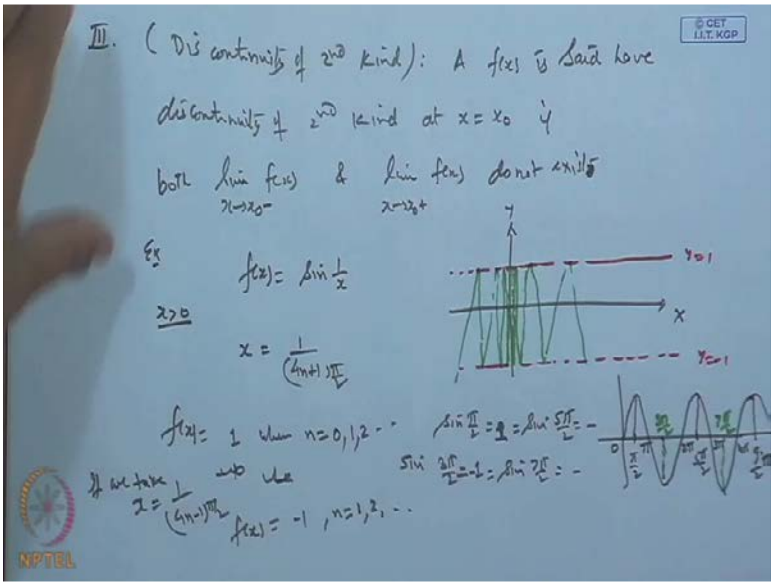

This is the third type of the continuity, discontinuity will be, we call it as discontinuity of second kind, second kind. Now, what is this discontinuity of the second kind? When the limit of this left hand limit, right hand limit, both does not exist, that is a function f x is said to have a discontinuity of second kind at a point x equal to x naught, if the left hand limit and right hand limit, if left hand limit, or right hand limit, both, left hand, or right hand limit, both do not exist, both do not exist. Definitely, and left hand limit do not exist, right hand limit do not exist. In fact, so neither left hand exist nor right hand exist.

So, what happened then? For example, if we take the f x equal to say sin 1 by x, this we have seen the already integer, this function, what is the behavior of the function? If I look this function, here x, this is y and just I am taking, say here y is equal to 1, y is equal to say minus 1. Now, graph of this function, if you look the graph, the graph will be something like that, say here, something like this, is it not? As soon as it comes here is very, very closed; very, very closed and like this, but it never touches it here; similarly here, this is very, very closed, very very closed for this and something like this, something like this. So, only at this is, around the point 0, it gets so much congested and frequently it going up and down very fast, that is x near to x equal to 0, it has a sudden jump, at this point it has a value one, immediately in the nearby point, it has value minus 1. So, it has a sudden jump near the point x equal to 0, in the neighborhood of 0. So, let us see the, we claim that this x equal to 0 is the point of discontinuity of the second kind, let us see.

Suppose, I take positive side value, first you take x is greater than 0, then if I take the number x equal to say 1 by 4 n plus 1 pi by 2. So, what happened? This sin function, we know, sin pi by 2 is 0 then because if we take the graph of the sine function, this is the graph of the sin function, sin 0 is 0 like this, sin 0 is 0 sin pi is 0 sin 3 pi is 0 sin 3 pi is 0 and so on; but sin pi by 2, this is the point pi by 2, here it has a positive value one, sin of this thing, 2 pi plus pi by 2, that is 5 pi by 2 it has a one value then again this 3 pi it 4 pi, so it is 9 pi by 4 pi plus pi plus 9 9 9 by 2 pi; then this value the function is a value one. So, when it is pi by 2 sin, sin of 3 pi by and sin 5 pi by 2 and so on; the value of this is a is one, is one and then if we look the sin 3 pi by 2, sin 3 pi by 2, this is our 3 pi by 2, this value, 3 pi by 2, here it is 7, is it not? 7 pi by 2; so 3 pi by 2 sin 7 pi by 2 and so on; the value is coming to be minus 1; it means, when the point x. So, 1 by x is this.

So, x is very very near, n is sufficiently large the point is very close to 0. So, when it has the value n is the function. So, this function f x has the value one, when n is equal to what? n is 0, n is 1, n is say, two and like this, and equal to minus 1, when n is what? n is here say, 3 pi by 2. So, we are getting n is equal to say, if I choose the function f x, if we take, if we take x is equal to 1 by 4 n minus 1 pi by 2, then the value of the function f x is minus 1 for n is equal to 1, n equal to 2, 8 minus 7 pi by 2 and like this, continued. So, we have a sequence of the point approaching towards 0, where the function has a sudden jump, plus 1 to minus 1 and like this. Similarly, in the left hand side.

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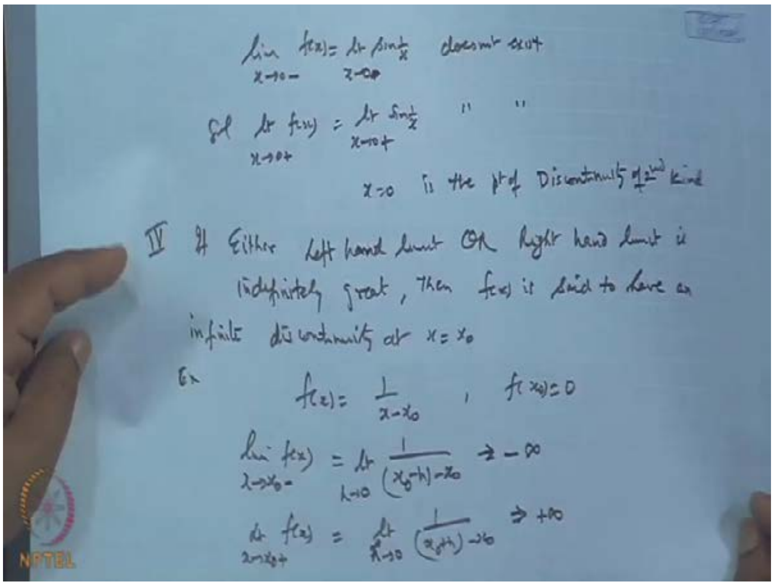

So, this function, we say the limit of this function f x which is sin x, when x tends 0 minus, that is the limit of this sin 1 by x when x tends to 0 minus does not exist, similarly limit of this function, similarly limit of this function f x, when x tends to 0 plus is does not exist, 0 plus, sorry, 0 minus, here 0 plus does not exist. So, x equal to 0 is the point of discontinuity of second type kind second kind.

So, this is what we have. There some other also functions we can go ahead. Now, one more function, we say we say that, of course this is not very, very important, but it still we, if one or the more function at alpha is indefinitely great. If either left hand limit, left hand limit or right hand limit, right hand limit is indefinitely great, indefinitely great means very large, it goes to infinity when these left hand or right hand limit goes to infinity, then in that case the function f, then f x is said to have, is said to have an infinite discontinuity, discontinuity at the point x equal to x naught. For example, suppose I take the function f x which is 1 minus x minus x naught and the value of this at the point x naught I am choosing to be 0; now, what happens?

If you take the limit of this function f x when x approaches to x naught from the negative side, then this is the same as 1 over x naught minus h minus x naught, limit h tends to 0, is it not? Now, this is equal to what? This is nothing but tends to minus infinity, tends to minus infinity, because this value will go to basically x naught cancel and h is sufficiently is small, so it goes to minus infinity; and when we go to the plus side, f x when limit x tends to x naught plus sign, where x naught, then what happen is, limit x naught plus h minus x naught and then h tends to 0. So, this will be tending to plus infinity. So, what the left hand limit or right limit or any both of them, it goes to very large infinity, then point x equal to x naught is the point of discontinuity for this, infinite discontinuity of this, x equal to 0, then.

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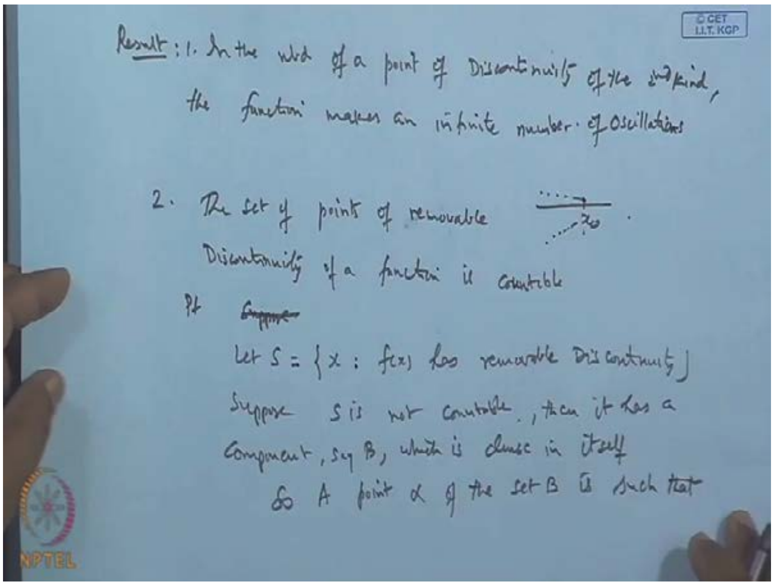

So, almost this, we have covered. One is more example, which we can prove the results, of course without proof, we can just say… In the neighborhood of, in the neighborhood of a point of discontinuity, discontinuity in the neighborhood of the point of discontinuity of the second kind, of the second kind, the function, the function makes an infinite, makes an infinite number of oscillations, because the reason is very simple, when you take the limit of the function either from the left hand side or right hand side, then you get a sequence approaching towards that function where along a different sequence, we have a different values. So, in fact, if x naught is the point of discontinuity of second kind, then we can get one sequence where the limit of the function will be say, alpha 1, then another sequence will be obtained where the limit of sequence is minus something, say alpha 2 something; so there is a sudden jump, makes an infinite number of oscillation. So, that is the case of this.

Second is, of course proof is you can go through, the set of points, the set of points of removable discontinuity, removable discontinuity of a function, of a function is countable, is countable; the reason is, proof is very like this, suppose that if the set s, suppose the set s which is the function, set of those function x, where f x has removable discontinuity, has removable discontinuity. Let s be this; suppose s is not countable, s is not countable, then it has a component, then it has a component, say b; b which is dancing itself, in itself that we have; a point alpha b such that... So, a point alpha we can find, a point alpha of the set b, b is such, is such that in any…

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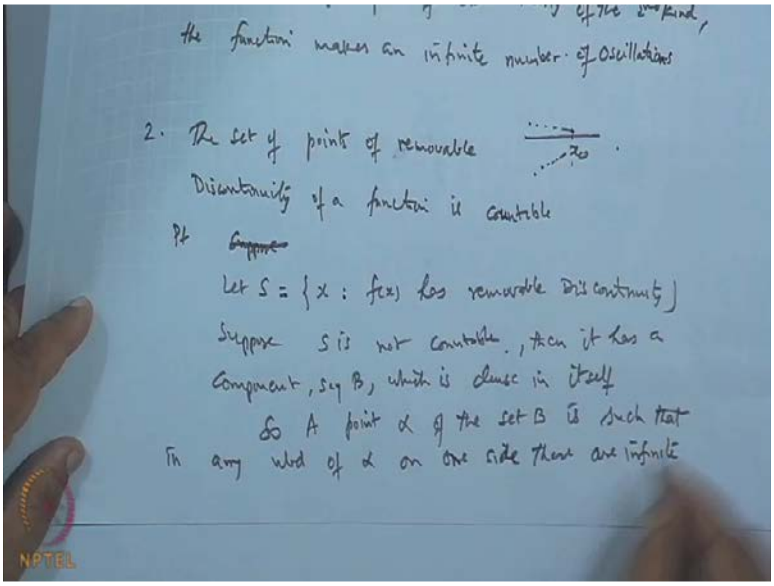

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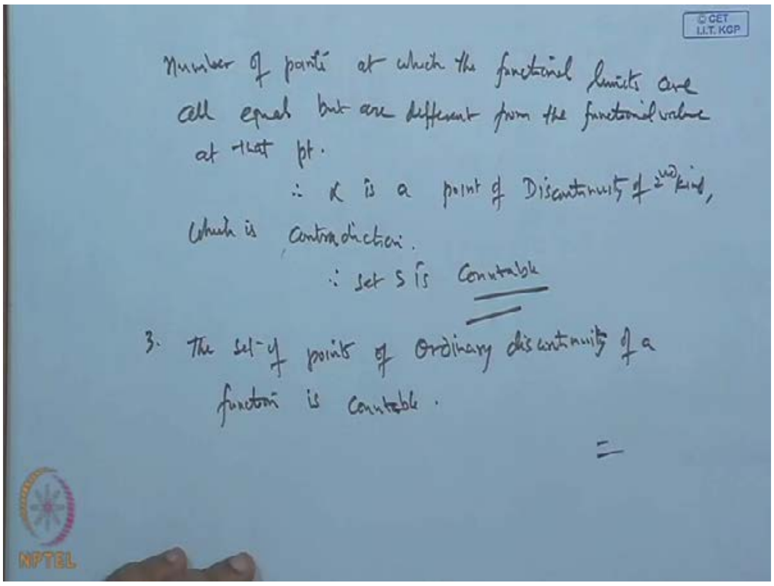

So, a point alpha integer, such that in any neighborhood, in any neighborhood of alpha, in any neighborhood of alpha on one side, on one side, there are there are infinite, there are infinite number of points, number of points, infinite number of points at which, at which the functional values, limits, at which the functional limits, functional limits are all equal, are all equal, but are different from, but are different from functional value, from the functional value at that point, at that point; therefore, alpha is a, is a point of discontinuity of second kind, point of discontinuity of second kind, which is a contradiction, which is a contradiction, because we have assumed that alpha as a point of removable continuity as, discontinuity point; therefore, our assumption is wrong. So, the set s is countable. Similarly, another results is the set of points, set of points of ordinary discontinuity that is discontinuity of the first kind, discontinuities of a function is countable. That is all.

Thank you very much clear.

So, proof is just, thank you.