A Basic Course in Real Analysis
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Lecture - 31
Uniform Continuity and Absolute Continuity

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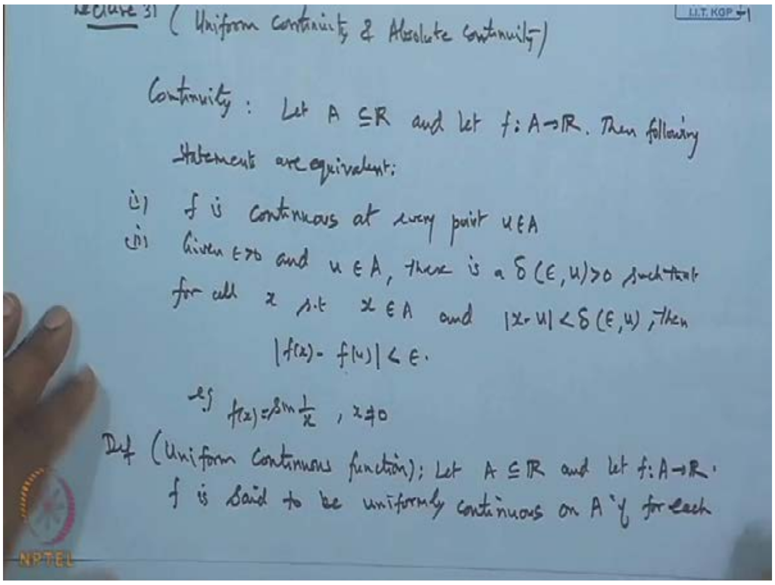

Today, we will discuss uniform continuity and absolute continuity. Let us see the first uniform. We have already seen the continuity definition. Let A, which is a subset of R and let f mapping from A to R – set of real numbers A to R; then the following two conditions, then following two statements are equivalent. That is, the first statement says that, f is continuous at every point u belongs to A; and second is given epsilon greater than 0 and u which is an element of A; there is a delta, which depends on epsilon as well as the point u and greater than 0. Such that for all x such that x belong to A and satisfy the condition mod x minus u is less than delta, which is depends on epsilon, u; then the mod of f x minus f u is less than epsilon. So, this we have discussed already, a function is continuous at a point u if for a given epsilon, there exists a delta such that mod of f x minus f u less than epsilon whenever mod of x minus u less than delta.

And here we have seen that, this depends on the point. If I change the point, correspondingly, delta will change. So, what this shows? This shows the function f changes its behavior when the point changes; maybe we have suppose at some point, the function is very slowly changing their values; and, near to some point, it changes very rapidly. For example, if we take the sin 1 by x when x is not equal to 0; if we look this function, then this function is changing very rapidly when the point is very close to 0. It goes very up and down from minus 1 to plus 1; and, very rapidly it goes. So, we are interested in such type of the functions, where the change is smooth say. Or, we say that, we are interested in the delta, which is independent of u. And, that leads the concept of uniform continuity.

Though the function we say, it is continuous point wise here; but point wise means delta will depend on the u. So, we wanted a definition in which the delta is independent of u. And, that leads to the definition or concept of uniform continuity. So, let us see the definition of uniform continuous function. Let A be a nonempty subset of R and let f is a mapping from A to R. We say f is uniformly continuous.

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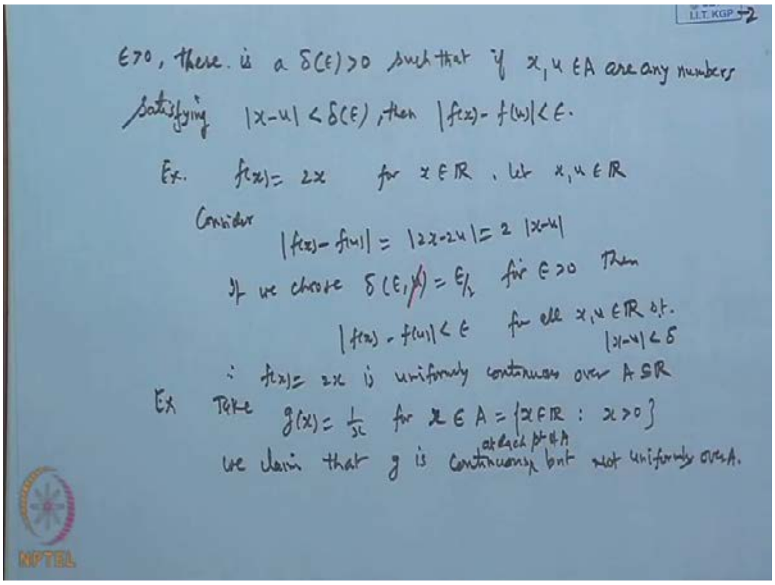

f is said to be uniformly continuous on the set A if for each epsilon greater than 0, there is a delta, which depends only on epsilon – a positive delta, which depends only on epsilon and independent of the point u of A such that if x and u are any two points of A, are any number satisfying the condition – mod of x minus u is less than delta – depends on f singer only; then mod of f x minus f u is less than epsilon. So, what this shows is that, a function is said to be uniformly continuous over the set A. Remember, when we say, the function is continuous, then we can say function is continuous at a point. So, at a point, we can identify a delta, which depends on… But when we say, the function is uniformly continuous, then saying uniform continuous at a point is a meaningless. It will be continuous over a set. So, a function is said to be uniform continuous over the set A, we mean that, if for any epsilon greater than 0, if we are able to get a delta, which is independent of the points of the set A such that whenever we pick up any two arbitrary points say in the delta neighborhood of this, then corresponding fluctuation f x minus f u will remain less than epsilon – the value of this.

For example, if we take the function f x equal to say 2 x; and, this f x equal to 2 x for x belongs to say the real number R. Now, if I consider mod f x minus f u, where the u and v are the point… Let x and u – these are the points in R; and, satisfies that condition. So, consider this. This is equal 2 x minus 2 u, which is equal to 2 times x minus u. So, if we choose delta, which depends on say epsilon u as epsilon by 2; obviously, this delta is independent of u, because this is basically, we are taking delta to be epsilon by 2 whatever the u may be. So, it is independent of this. So, if I take delta this, then obviously, this part for all epsilon greater than 0, then this condition holds – less than epsilon for all x and u belongs to R such that mod of this x minus u less than delta. And, this is independent of… Delta is independent of u and positive quantity. So, this function will be considered – f x equal to 2 x. We can say it is uniformly continuous over the entire real line.

However, there are the functions, which are only continuous, but not uniformly. For example, we take the function. So, f x equal to 2 x is uniformly continuous over any set A, which is subset of R or any subset of R or in (( )). Now, take a function g x say 1 by x for x belonging to the set A, which is the set of those points of real number such that x is positive. Now, we claim that, this function g is continuous, but not uniformly over A; continuous at each point of A, but not uniformly. Let us see how.

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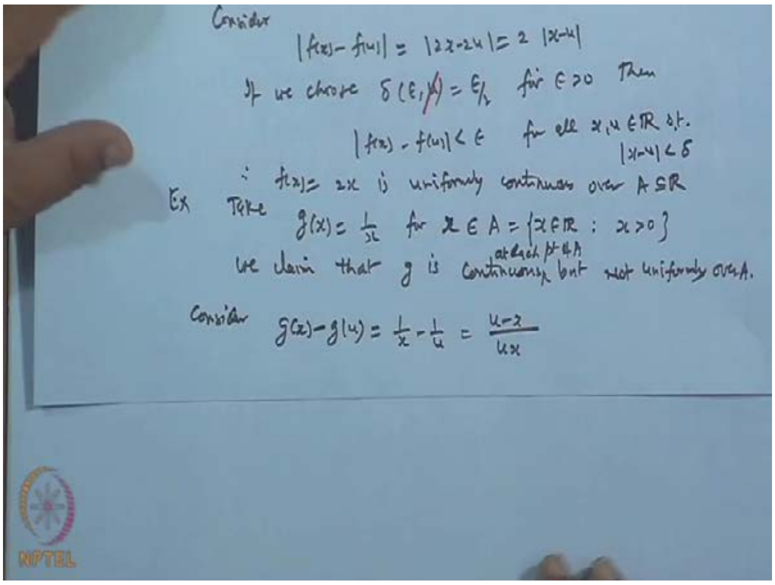

Let us consider g x minus g u. This is equal to what? 1 by x minus 1 by u which is the same as u minus x over u x.

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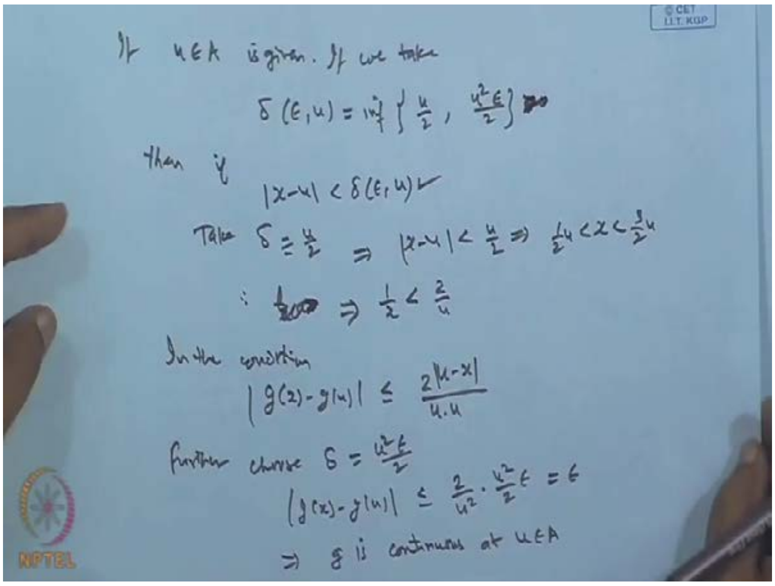

Now, if u belongs to this – if u belongs to A suppose; u belongs to A as given; where, we wanted to test the continuity is given. So, if we take delta, which depends on u as the infimum of u by 2 and u square epsilon by 2. So, let epsilon greater than 0 be given. Here we let us write, let epsilon greater than 0 be given. And, let us pick up the point u at which the continuity is tested. So, not choose the delta as this. So, when you take this delta, then if mod of x minus u is less than delta and delta, which is depending on epsilon u; then I can choose, suppose first it is less than u by 2, so then take delta as u by 2. So, what happens is, this shows that, x minus u is less than u by 2; or, this implies that, x lies between 3 by 2 u and half u, because x minus u is less than u by 2. So, it becomes less than 3 by 2. And then x minus u is u minus u by 2 is greater than this. So, it is greater. So, it lies bound, therefore the bound for this…

Therefore, 1 by x can be x is less than u by 2. So, 1 by x is less than, because it will be x is greater than this. So, 1 by x will remain less than 2 by u from here. Once it is 2 by u, then in the condition, which we have taken as g x minus g u – in this case, what we get? Mod of this; this is less than equal to u minus x mod over u x, so u into x; x means 1 by x. So, it is 2 by u. So, it becomes the 2 by u square into u minus u.

Now, further choose delta to be this thing – u square epsilon by 2. I have taken delta to be this. Another one I am taking this. So, what we get is, from here, this shows that this part – mod of g x minus g u is less than equal to… That means mod of u minus x is less than delta. So, this is less than 2 by u square into u square by 2 epsilon; that is, epsilon. So, this holds that, if x minus u is less than delta, for all such x, then g x minus g u will be less than epsilon. And, this is true. So, this shows, that g is continuous at u belongs to A. But here the delta which you are choosing is coming; which depends on u is positive quantity. But what happens to this? This is not, each individual delta is positive when u is taken to be there. But when you take the infimum value of this – when you take the delta as the infimum of all such, then what happens to that? Infimum of delta…

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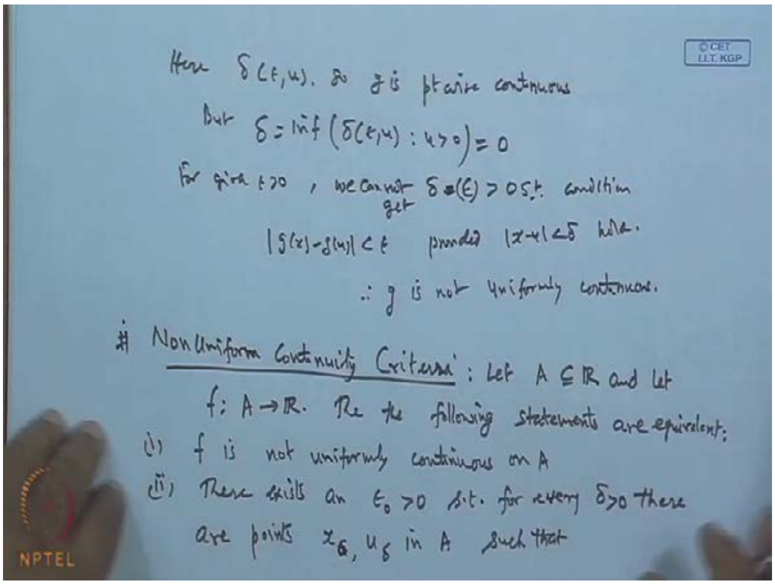

Here delta depends on epsilon u. So, g is point wise continuous. But what is the infimum of all such deltas? Infimum of all such delta, which depends on epsilon u; and, u is greater than 0. The infimum value of this is coming to be 0. Why? Because each delta… Here is nothing but either u by 2 or u square by 2. So, u is greater than 0. So, each delta is greater than 0. But when you take the infimum value of this delta over u, then this infimum will be come out to be 0. So, we are not getting a delta, which is greater than 0. So, for a given epsilon greater than 0, we cannot get delta, which depends only on epsilon and greater than 0 such that condition – mod of g x minus g u is less than epsilon provided mod of x minus u less than delta hold. This we cannot get. Therefore, g is not uniformly continuous. So, we have seen the example where the function is continuous, point wise and the function is uniformly continuous.

Now, to show the uniform continuity, we require the delta, where independent of point over the entire set. So, that is not that easy. So, what we will be… We can develop the (( )) which will give at least sufficient criteria when the function is not uniformly continuous. So, we just state those results without proof – the criteria for the non-uniform continuity. So, the non-uniform continuity criteria – this will be needed. So, proof – we just are dropping. But it can be easily done with the help of previous knowledge. Let A be a nonempty subset of R and let f is a mapping from A to R. Then the following statements are equivalent. The first statement says, f is not uniformly continuous on A.

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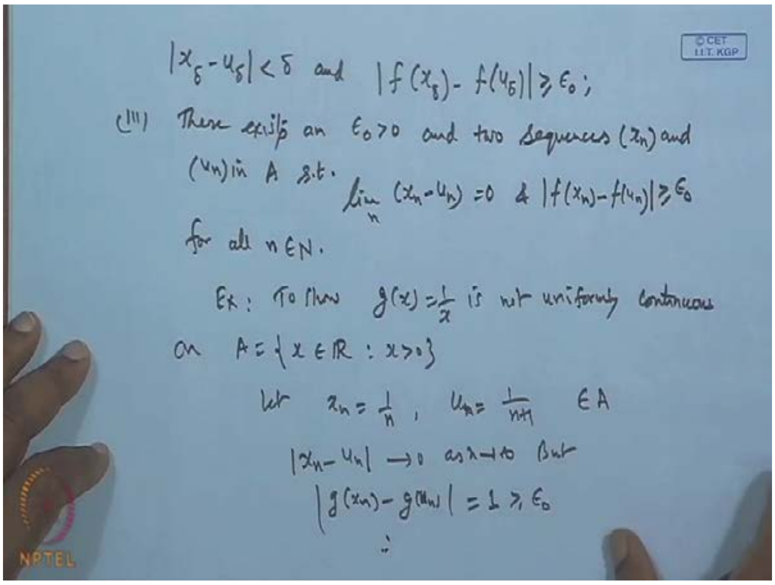

Second statement says that, there exists an epsilon naught greater than 0 such that for every delta greater than 0, there are points say x depends on delta and then u depends on delta in A such that, the mod of x delta minus u delta less than delta and mod of f x delta minus f u delta is greater than or equal to epsilon naught. And, the third statement says, there exists an epsilon naught greater than 0 and two sequences say x n and u n in A such that limit of x n minus u n over n is 0 and mod of f of x n minus f of u n is greater than equal to epsilon naught for all n and belongs to capital N. Let us see what is this?

Uniform continuity criteria says, if suppose function is not uniform; then by the definition of not uniform means a function is said to be uniform continuous over the set A if for each epsilon, there exists a delta, which depends only on epsilon, not on the delta such that the difference of f x minus f u can be made less than epsilon provided the points are in the delta neighborhood. So, if the function is not uniformly continuous, it means this condition will be violated. If we choose the point in the neighborhood of delta, the images of this, the fluctuation may not be less than epsilon; it can exceed to any arbitrary number epsilon naught. So, that is why, what he says is that, if f is not uniformly continuous, then there exists an epsilon naught such that, whenever the point x and u are in the delta neighborhood, the corresponding images exceed that bound epsilon naught greater than…

Similarly, this is Cauchy's definition; this is Heine's definition. Instead of choosing the two arbitrary points, if we picked up the two sequences x n, u n, which are tending to 0, the difference of this is tending 0. This means x n and u n are very close to each other as n is sufficiently large. But the corresponding image is not close, is greater than equal to some positive number epsilon naught. Then we say, the function f is not uniformly continuous. We got this. Now, this criteria can be applied very directly.

Suppose I apply this function to show g x, which is 1 by x, is not uniformly continuous on the set A, where x is greater than 0 – set of all real number n. So, what we do is, we have to pick up the two arbitrary sequences. So, let x n – I choose 1 by n; and, u n – I take to be 1 over n plus 1. Both are in A, and the difference of these two sequences – obviously, x n minus u n. This goes to 0 as n tends to infinity. But what is g of x n minus g of u n? This mod is nothing but what? g of x n is nothing but n plus 1 minus n, that is, 1, which does not go to 0; in fact, it is greater than equal to any number epsilon naught. Therefore, g is not uniformly continuous. That is what. Now, uniform continuity – when its function is defined over a closed interval and the function is continuous, then it will be uniform continuous.

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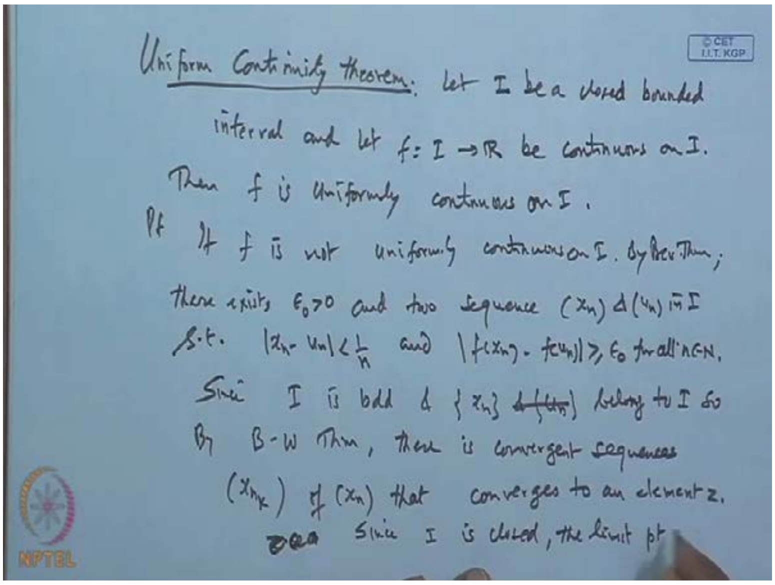

This result is known as the uniform continuity theorem. The theorem says, let I be a closed bounded interval and let f, which is a mapping from I to R I be continuous on I. Then f is uniformly continuous on I. So, what is said, if the function f, which is continuous over a closed and bounded interval; then the function must be uniformly continuous. Suppose f is not uniformly continuous on I. So, I can use one of the criteria, which I listed earlier. I will take in the form of the sequence by now. So, by the previous results or previous theorems, where the criteria are there, we can choose; then there exists an epsilon naught greater than 0 and two sequences x n's and u n's in I such that the difference between these two… that is, the limit of this is going to 0 means difference is very very small – say 1 by n. But the mod of f x n minus f u n – this difference exceeds by this epsilon naught for all n. So, this is by the criteria when non-uniformly continuous criteria. From here we are getting this one.

Now, since I is bounded and the sequence x n and u n's – both are the sequences belong to I. So, by Bolzano-Weierstrass theorem, every bounded sequence has a convergence of sequence. So, there is convergent subsequences say x n k. If there is convergent subsequence, let us take first, this part; then we can take belongs to this. So, there exists a convergent subsequence x n of that converges to an element say z belongs to I; that converges to z. Now, we wanted to show, z is a point in I; which follows, because I is closed. Since I is closed… So, all the limit points of a sequence in I must be pointing there. Also, all these sequences x n k must be lies between the lower and the upper bound of this interval. So, by the theorem, the limit of this sequence x n k must be the point in I. So, this implies, since I is closed, the limit point z belongs to I.

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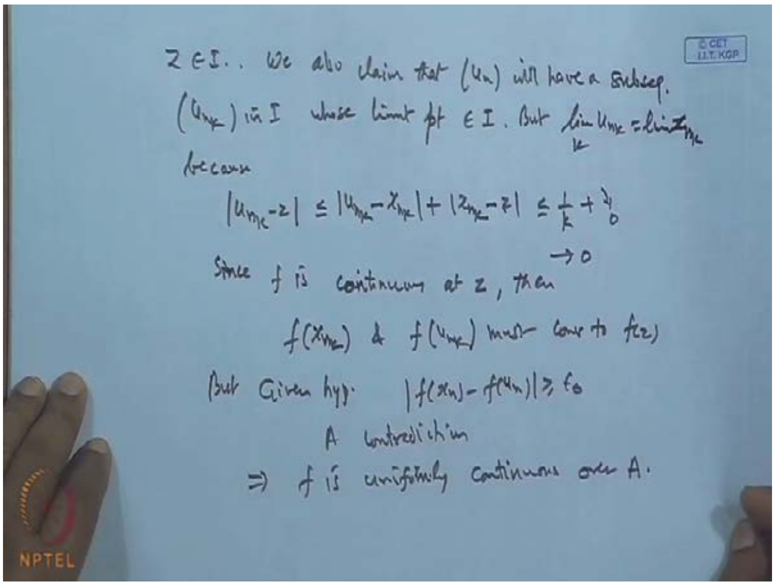

Similarly, we also claim that, the sequence u n will have a subsequence u n k in I, whose limit point belongs to I. But the limit point of u n k and x n k will be the same. The reason is, because if I consider the u n k minus z, then this can be written as the u n k minus x n k plus x n k minus z. Now, this term is less than equal to k by condition, which (( )), because both are in A and we are choosing the interval neighborhood in such a way, so that this is less than equal to k and this converges to z. This is the limit point of this. So, it goes to 0. So, s tends to total tends to 0. Therefore, both will have the same limit point. Once they are having same limit point, f is continuous.

Now, further, since f is continuous at z, then both the sequences: f of x n k and f of u n k must converge to f z, because x n k goes to z. So, f of x n k will go to f of z; u n k goes to z. So, f of u n k will also go to A because f is continuous. So, once they are continuous… But the given hypothesis is that, mod of f x n minus f of u n is greater than equal to epsilon naught. This is given, so the contradiction. And contradiction is because of our wrong assumption that, function is not uniformly continuous. Therefore, f is uniformly continuous over A. That shows the result. So, this proves that uniform continuity. Now, there is one more result, which we say. Then we come to that (( )).

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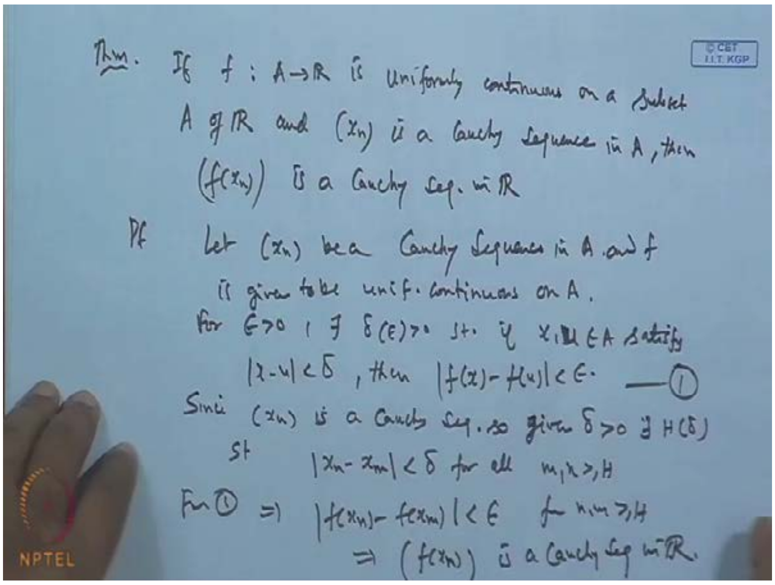

Theorem is, if f is a mapping from A to R is uniformly continuous on a subset A of R on a subset A of R and if x n is a Cauchy sequence in A, then f of x n – this sequence is a Cauchy sequence in R. It means if f is a uniformly continuous function, then it will transfer the Cauchy sequence to the Cauchy sequence. The proof is let x n be a Cauchy sequence in A and f is given to be uniform continuous over A. So, by the definition of continuity, let for a given epsilon greater than 0, we can identify a delta. There exists a delta, which depends only on epsilon greater than 0 such that the mod x minus u is less than delta such that, if x comma u belongs to A satisfies this condition; then f of x minus f of u – this is less than epsilon. So, let it be 1.

Now, it has given the sequence x n is a Cauchy sequence. Since x n is a Cauchy sequence. By definition of Cauchy, for a given delta greater than 0, there exists an H, which depends on delta such that the difference between any two arbitrary terms of the sequence after a certain stage can be made less than delta for all m, n, which are greater than equal to H. This is true. Now, by the same choice of delta, since x n m is less than this, if we take x equal to x n, u equal to x m; from 1, it follows that f of x n minus f of x m – mod of this will be less than epsilon for all n, m greater than equal to H. This shows the sequence f x n is a Cauchy sequence in R. So, that proves the result.

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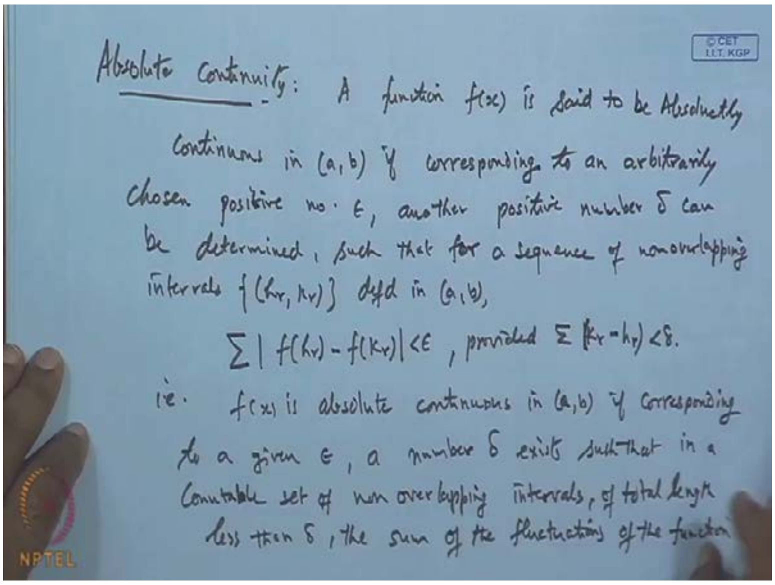

Now, we come to the thing, which is say absolute continuity. That is the new concept. Absolute continuity means a function f x is said to be absolutely continuous in the interval say a, b if corresponding to an arbitrarily chosen positive number epsilon. Another positive number delta can be determined such that, for a sequence of non-overlapping intervals – open interval h r, k r defined in a, b. The sigma of mod f of h r minus f of k r is less than epsilon provided sigma of k r minus h r – this length is less than delta. So, what we say, it means we say, that is, f x is absolutely continuous function. The meaning of this is absolute continuous over the interval a, b if corresponding to a given epsilon a number delta exists such that in a countable set of non-overlapping intervals of total length – I am not using the measure – less than delta, the sum of the fluctuations of the function is less than epsilon. Let us see what the meaning of this is.

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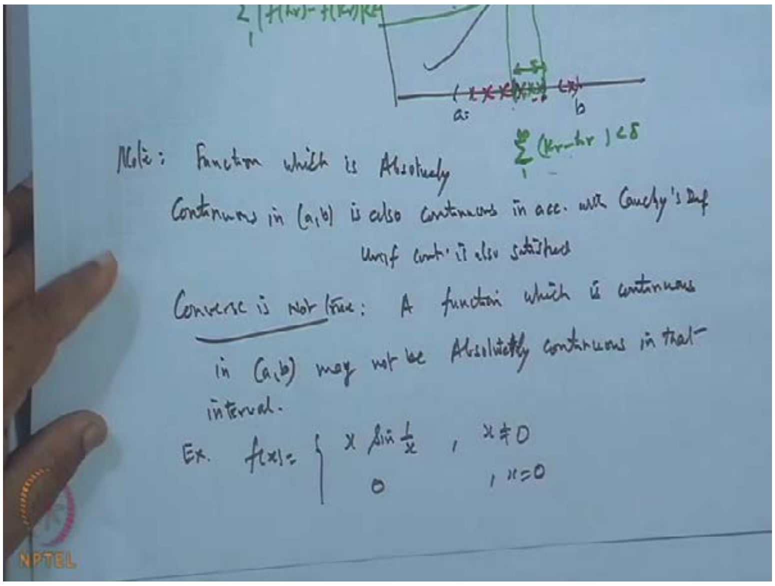

Suppose we have an interval a, b and a function f is u. We say this function is absolutely continuous in the interval a, b if for a given epsilon greater than 0. If we have a non-overlapping intervals means divide this one. Say here this is like this – if I take this non-overlapping intervals, and so on like this. So, if we take a countable number on non-overlapping interval of this, whose length is less than delta. So, we have to take a small portion. Suppose I take this small portion. Now, this small portion total length is delta; total length of this is delta. So, this small length over this small interval – we can find a non-overlapping intervals, such that sigma of this length k r minus h r – this length is less than delta. These are countable – 1 to infinity; R is 1 to infinity. Total length is less than delta.

For given epsilon – say this f singer is there; for this given epsilon, we can find a delta such that whenever we have countable number of non-overlapping intervals, whose length is less than delta, then corresponding fluctuation over these subintervals – the total sum the corresponding fluctuation should not exceed by epsilon. If so then we say, that is, sigma of mod f of h r minus f of k r – this should be less than epsilon 1 to infinity. So, if the total fluctuation of the function over, this subintervals is less than epsilon whenever the points are in the countable number of intervals, whose length total sum is less than delta, then function is said to be an absolutely continuous function. So, obviously, every absolutely continuous is continuous.

As a note, we can say, function, which is absolutely continuous in the interval a, b is also continuous in accordance with Cauchy definition, because what we do, we replace it that, total set k r – we consider single interval; some of these, we can replace by a small interval; total sum is less than delta. And correspondingly, here we get the total fluctuation is less than epsilon. So, this will be obviously true – consists of this; less than that. Then the condition of uniform (( )) Similarly, we can also say that, we may take this interval – single interval delta; and, the definition of uniform continuity is also satisfied over this. So, function, which is absolutely continuous is a continuous. And, for b also, we say it is uniformly continuous.

And, uniform continuity – uniform continuity condition is also satisfied if we choose the k r to consider single interval length delta. Converse is not true; that is, a function, which is continuous, may not be absolutely continuous in that interval; that is, a function, which is continuous in the interval a, b may not be absolutely continuous in that interval. A function, which is continuous, may not be absolutely continuous. For example, if we take the function f x, which is defined as x sin 1 by x when x is different from 0; 0 when x is 0. We know this function is a continuous function. This is a continuous.

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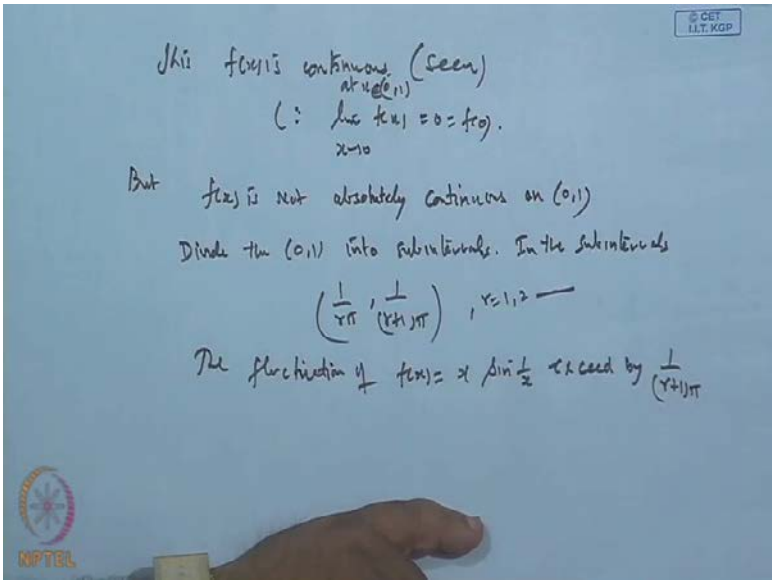

This we have seen. This function f x is continuous. This we have already seen. So, because the limit of f x – continuous at 0. And, otherwise also, it is continuous in the total limit as x tends to 0 is the value of this function coming to 0, which is f 0. And, for other point at x belongs to 0, 1 interval, in fact, entire interval it is continuous, because at the point 0, the value is coming to be this and continuity follows. Now, it is not absolutely continuous we claim. But this function f x is not absolutely continuous on the interval 0, 1. Why? It means the condition of the absolute condition is not satisfied; that is, if we choose the infinite number or countable number of subintervals, whose total length is less than delta, but the fluctuations may not be less than epsilon. So, that is what.

Suppose I divide the interval 0, 1 – in each of these subintervals, we get into subintervals say 1 by r pi, 1 over r plus 1 pi – these subintervals, where r is 1, 2, 3 and so on – into subintervals. And, in each intervals in the subintervals 1 upon r pi and 1 upon r plus 1 pi; r is 1, 2, 3. The fluctuation of the function f x, which is x sin 1 by x exceed by this number 1 by r plus 1 pi. Let us see, why?

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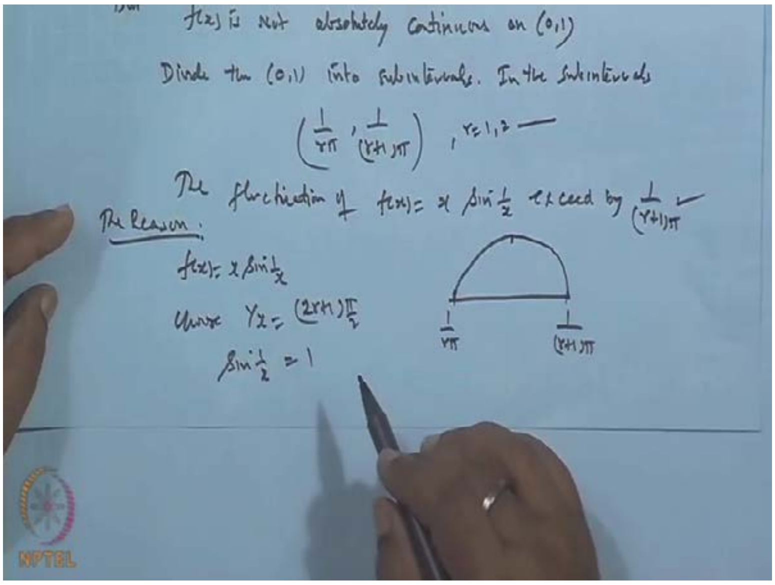

The reason is over this function, Tthis is the interval 1 by r pi; this is the interval 1 by r plus 1 pi. The function x sin 1 by x… If we (( )) the function f x, which is x sin 1 by x, at the end point, the sin of this is 0, because it is an integral multiple of pi. So, basically, the function will go like this, because at this end point, r pi or r plus 1 pi, this value will give the value 0. So, the fluctuation of the function – the value this minus the value this will be something, which is greater than this one. Now, here is the maximum value. Suppose I take x equal to say…

Let us choose 1 by x to be 2 r plus 1 pi by 2. So, x will be 1 upon this lying say here. Then what will be the sin of this value? Sin of 1 by x is 1 (( )) multiple of this 1. And, this value will give the f of x 2 r plus 1 pi by 2 – 1 upon 2 r plus 1 pi by 2. So, this will be greater than the value at this point r plus 1. So, fluctuation will exceed by this number, which is greater than this – by this. So, we get from here is that, the fluctuation is exceeding by this number.

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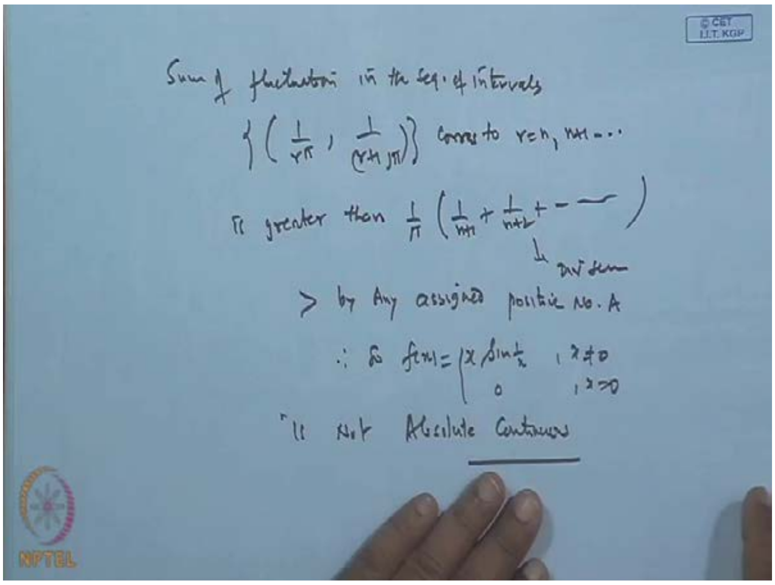

So, sum of the fluctuation in the sequence of intervals say 1 by r pi and 1 over r plus 1 pi – this sequence of intervals corresponding to r is equal to n, n plus 1 and so on, is greater than is greater than – if I take the sum, is greater than 1 by pi 1 over n plus 1 plus 1 over n plus 2 and so on. But this series is a divergent series. So, this cannot be made as small as we please. So, it is greater than by any assigned positive number A for the series. Therefore, this large number cannot be… So, the function f x, which is x sin 1 by x; x is different from 0 and 0 when x is 0, is not absolute continuous. So, this shows (( )).

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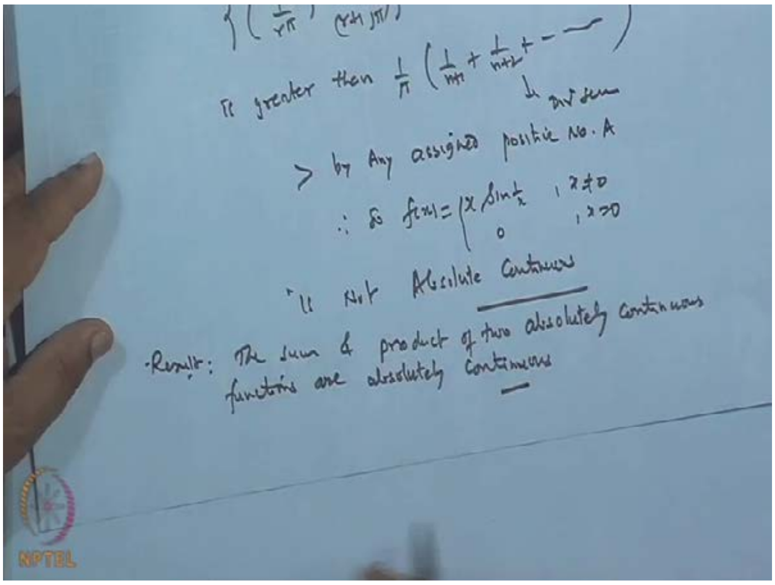

Results are just I state one result. The result states, the sum and the product of two absolutely continuous functions are absolutely continuous. So, this proof follows by the definition; and, one can go with it.

Thank you very much.