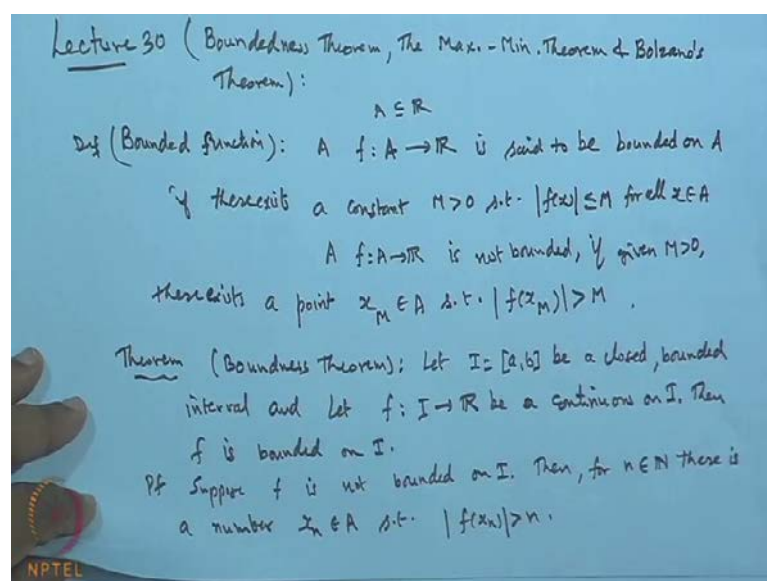


A Basic Course in Real Analysis
Prof. P. D. Srivastava
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 30
Boundedness Theorem, Max-Min Theorem and Bolzano's Theorem

This is in continuation of our previous lecture. We wanted to discuss the various properties of the continuous functions. And in any consequence of that we see that, if function is continuous over the closed bounded intervals, then we have some results, which is known as the boundedness theorem, maximum-minimum theorem and the Bolzano's theorem. And this gives you also a criterion to find out and approximate solution for the root of the function; that is the root location with the help of them.

(Refer Slide Time: 00:54)

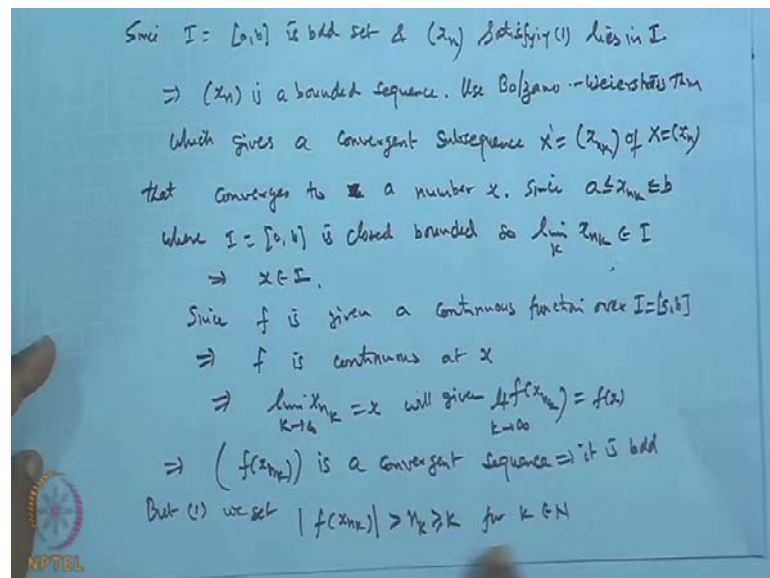


Let us see, first what is the bounded... Before going for the boundedness theorem, I will revise; recall the definition of a bounded function. We define like this, a function f from a set A to \mathbb{R} ; where A is a non-subset of \mathbb{R} is said to be bounded on A . If there exist a constant M greater than 0 such that the mod of $f x$ is less than or equal to M for all x belongs to A . It means a function is said to be bounded if the corresponding set is a bounded set. So, when we say the f is not bounded, it means, that is, so a function f from A to \mathbb{R} is not bounded. It means we are unable get such an M for which this holds; or we can say that if given any M , the function f is said to be unbounded; then if given any M

greater than 0, there exist a point x , which depends on this bound $M \times M$; but M belongs to A such that the value of the functions at these points will be greater than the given number M . So, whatever the number you choose, you can always find a corresponding a point in A for which the function value will exceed that number M . Then we say f is unbounded or not bounded on the set M . So, this we have already discussed.

We wanted now the result, the theorem which is known as the boundedness theorem. The theorem says, let I be a closed bounded interval and let f be a function from this closed bounded interval to \mathbb{R} be continuous on I . Then this theorem says f is bounded on I . So, every continuous function on a closed bounded interval will be a bounded function. That is what he says. Proof – we will prove by contradiction. Suppose f is not bounded on I . So, by definition, we cannot find an M such that the mod of $f(x)$ is less than or equal to M . So, we can get then for any n belongs to natural number, there is a number x_n in the set A such that the absolute value of this functional value f of x_n will exceed by n . So, corresponding to 1, we get 1 point x_1 in A , so that $f(x_1)$ is greater than 1. So, 2 – we get $f(x_2)$. So, we get a sequence of the points in A . This will satisfy this condition f of x_n is greater than n .

(Refer Slide Time: 05:48)



Since I , which is given to be the closed bounded interval is a bounded set. And all the sequence is x_n , which you are getting satisfying the condition say 1 satisfying 1 lies in I , because these are all the sequences belonging to I . And I is a bounded set. So, the

sequence x_n is a bounded sequence, because x_n lies between a and b , so that all the terms of the sequence have a lower bound say a , upper bound say b ; it is a bounded sequence. And we know by Bolzano's theorem, every bounded sequence has a convergence sequence. So, use the Bolzano-Weierstass theorem. The Bolzano-Weierstass theorem, which gives a convergent subsequence say x_{n_k} of this x_n that converges to a number say x , because by definition, x_n is a bounded sequence. So, Bolzano-Weierstass theorem says by Bolzano-Weierstass theorem, we can get a subsequence, which is convergent and converges to a number x .

Now, this x officially belongs to I . Why? Since x is x_{n_k} . All these terms of the sequence x_{n_k} lies between a and b . This is a close interval. I is close interval. All the terms of the sequence lies between this. So, the limit of this sequence x_{n_k} cannot exceed between these; will always lies between to these two bonds. So, since this is there, where I is a closed bounded interval, the limit point of this sequence x_{n_k} over k – this limit point obviously belongs to I . But this limit point is x . So, this shows that, this x belongs to I . So, what we get is that, a sequence x_{n_k} has a subsequence, which is convergent and the limit point belongs to I .

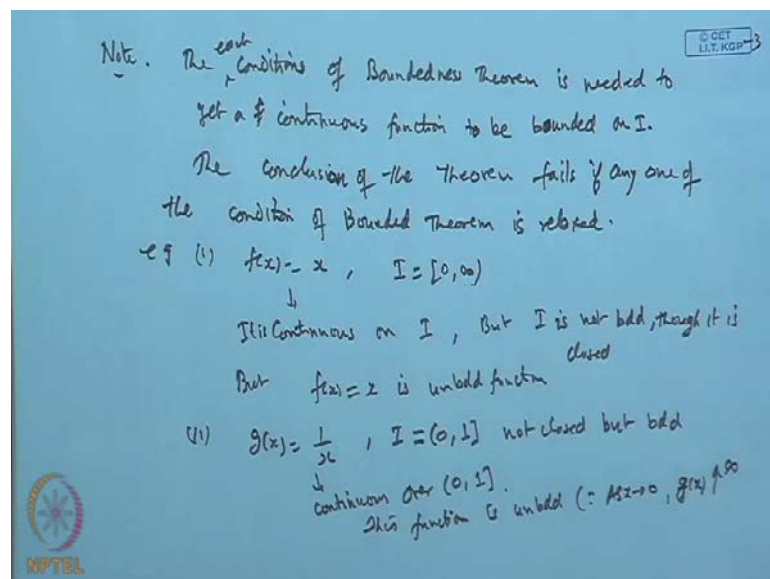
Now, f is given to be continuous. Since f is given a continuous function over the interval I and x is one of the point inside the I . So, this implies f is continuous at x . So, by theorem, therefore, this implies limit of x_{n_k} , when k tends to infinity is x , will give f of x_{n_k} – limit of this – k tends to infinity is nothing but what? f of x , because by the converges part. So, what he shows? This implies the sequence $f x_{n_k}$. This sequence is a convergent sequence and the convergent sequence is always bounded. So, it is bounded. But that gives a contradiction to the result, because this sequence f of x_n is greater than 1 from here from 1 . But from 1 , what we get? We get mod of $f x_{n_k}$ is greater than n of k , which is greater than k . And this for all k belongs to \mathbb{N} . So, this continuous function is not bounded on the closed bounded interval. So, this shows that, function f over this sequence...

Suppose if the function is not bounded, is a contradiction. So, this gives a contradiction of over second part. But from we get this, which contradicts 2, because here this shows this is unbounded. But here we have already shown it is bounded. It gives a contraction. And contraction is because our assumption that the function f is not bounded on I . So, this shows that, function f is bounded. This implies f is bounded on I , that is, closed

interval. So, this proves the result, which is known as the boundedness theorem. Now, in the boundedness theorem, we have assumed these conditions? What are the conditions in the boundedness theorem is, first condition is, the interval on which the function is defined must be closed and bounded. So, this is one of the conditions, which we have taken.

Second condition, which have the function must be a continuous function on this. So, interval is closed bounded and function is continuous. Then only, we can say, f is a bounded function. If any one of the condition is relaxed, that is, if we take I to be a simply bounded interval, not closed or simply closed, not bounded or function is not continuous, then our conclusion that, f is bounded on I will cannot be drawn. In fact, we get a contradiction; we will get the example, where we get this function is unbounded when we relax any one of the condition.

(Refer Slide Time: 12:56)



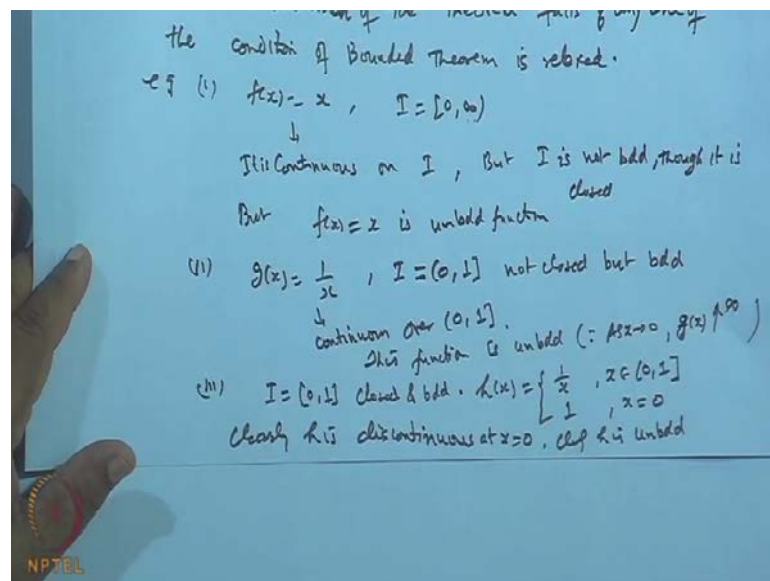
So, let us see the examples, the condition. Note: The conditions of the boundedness theorem – the hypothesis of boundedness theorem is, each condition is needed to justify or to get the function f to get a continuous function to be bounded on I . If we relax any one the conditions, then we get the conclusion of the theorem fails if any one of the hypothesis or if any one of the conditions of boundedness theorem is relaxed.

For example, suppose I take say the function $f(x)$ is suppose x ; the interval I – I am choosing I as the interval 0 to infinity. Now, this function is continuous function. It is

continuous function on this interval I . But I is not... that is, the function is closed, but is not bounded. It is unbounded one though it is closed, because all the limit points are between 0, 1 are inside it. So, I is a closed interval, but it is not bounded. Then what happens? The function is continuous throughout, but the bound for this function, function $f(x) = x$ is an unbounded function, because as x increases, the value of the $f(x)$ keeps on increasing and interval is up to infinity. So, it is without bound. So, it is unbounded. Therefore, the conclusion fails.

Second – if suppose I take the same function $g(x)$ say is $1/x$. This function and the interval I to be the interval say $(0, 1]$. Now, this interval is not closed, but bounded. Function $f(x)$ is continuous over the interval $(0, 1]$, because 0 is not included. But we have seen, this function is unbounded, because as x tends to 0, the function $g(x)$ will go to infinity – unbounded.

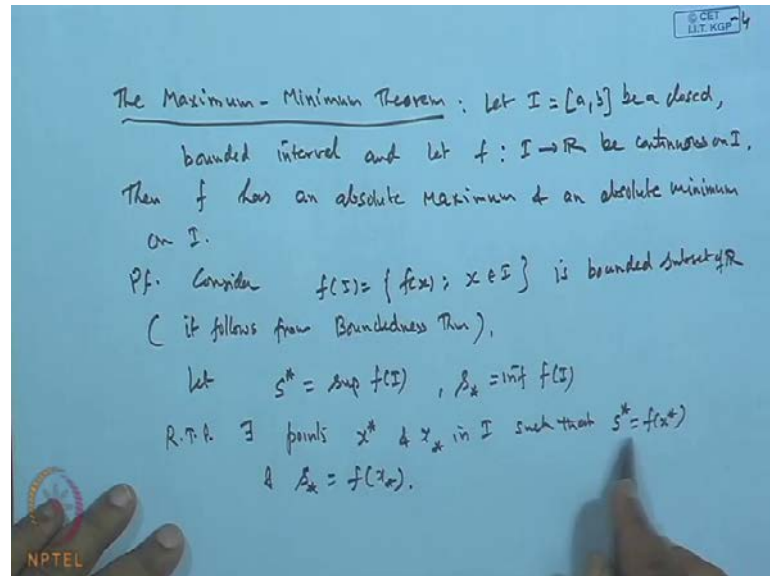
(Refer Slide Time: 16:53)



So, again, relaxing the condition is again not going to help; that conclusion fails. Then if I take the interval I as a closed interval, which is a closed and bounded; but now, I am taking the function $h(x)$ as $1/x$ if x belongs to the interval say $(0, 1]$ and 1 if x is 0. I define this function. The function is continuous over the $(0, 1]$ interval. But it is discontinuous at the point 0. Clearly, h is discontinuous at x equal to 0. So, again, the condition is not satisfied and obviously, clearly, h is unbounded when x approaches to 0. This is not a bounded function; $1/x$ for this $(0, 1]$; and, 0 when it is continuous and

unbounded on $()$. So, this will be there. So, we can see that, in boundedness theorem will only be applicable when all the three conditions are taken in consideration.

(Refer Slide Time: 18:21)



Next result, which we $()$ is maximum-minimum theorem. The maximum-minimum theorem – the theorem says let I be a closed, bounded interval and let f , which is mapping from I to \mathbb{R} be continuous on I . Then f has an absolute maximum and an absolute minimum on I . This results...

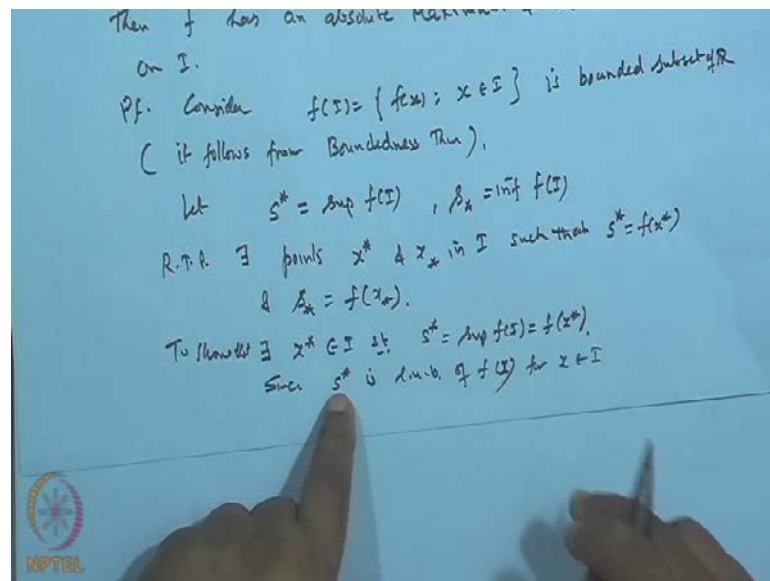
Now, we have already discussed – the absolute maxima and absolute minima in the last lecture. So, proof we go; what is the absolute maxima we mean that, if suppose a function is from A to \mathbb{R} , then f has absolute maxima at a point... There exist some point x star such that $f x$ star is greater than $f x$ for all x ; and, minima when $f x$ lower star is less than equal to $f x$. So, that is the way we have introduced the maxima-minima of this function. We will discuss after this proof. What we want is, suppose a be a closed, bounded interval and function is given to be continuous; then it will attains its maximum value as well as minimum value over the interval I ; that is, there exists some point, where the absolute maxima will be attained, absolute minima will attained. So, let us see the proof.

Consider the set f of I . The set of those value is $f x$, where x belongs to I ; means consider the range set of function f , which is defined over I . Now, this range set the values of f for I is clearly is bounded subset of \mathbb{R} . And it follows from the previous boundedness

theorem, because boundedness theorem we have seen – if I is a closed, bounded interval; f is a continuous function; then f is bounded on I ; that is, the range set will be a bounded set on I . So, it is a bounded subset of \mathbb{R} , this much. So, once it is bounded subset, we can talk about the upper bound and lower bound and supremum of least upper bound and greatest lower bound.

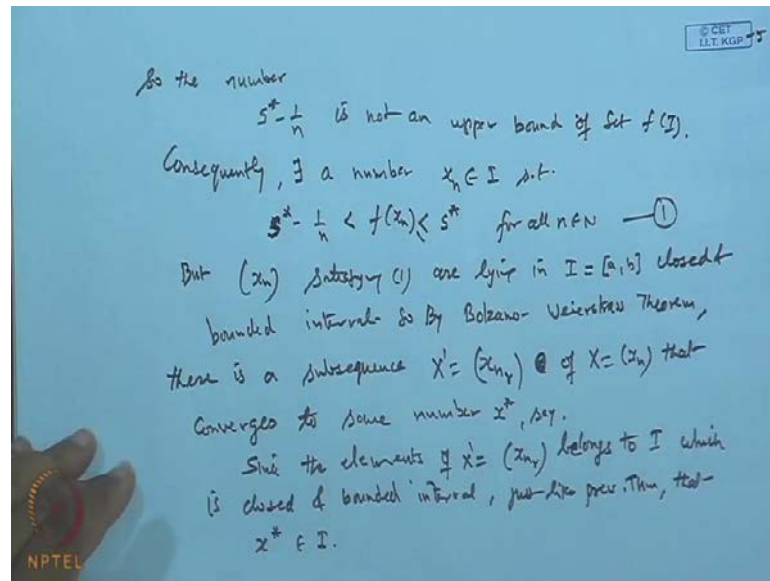
Let us suppose s^* be the upper bound supremum value of the function f on I ; and, s lower star be the infimum value of the function f on I ; that is, the least upper bound of the function f over the interval is suppose s^* and... Now, what we want to do is, this s^* and this small s exist. It means there exist some functions; what we require to prove is that, there exist points x^* and x lower star in I such that the s^* will coincide with the f of x^* and s lower star will coincide with f of x lower star. This we wanted show. So, first, we will prove for this side and other will follow in a similar way. So, let us assume.

(Refer Slide Time: 23:13)



First, to show that, there exist an x^* in I such that s^* , which is the supremum value of f on I exist and equal to the value at this point. This we wanted to show. Now, since s^* is the least upper bound of the function $f(x)$ for x belongs to I . When x belongs to I , least upper bound of this function is s^* . So, if I choose a number slightly lower than this, it cannot behave as a least upper bound.

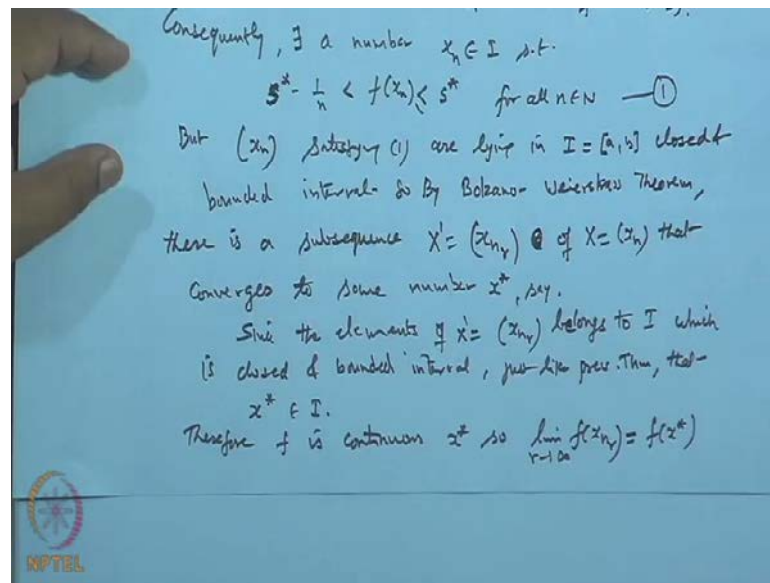
(Refer Slide Time: 24:05)



So, the number $s^* - 1/n$, this number is slightly lower than... So, it is not an upper bound of the set $f(I)$. Therefore, there exist... So, we can say, consequently, there exist a number x_n in I such that $s^* - 1/n < f(x_n) \leq s^*$, which is less than or equal to s^* for all n belongs to capital N , because this is the upper bound. So, when you take a number slightly lower than this, then we can find some number x_n in I , so that the functional value of x_n will exceed by this number and obviously it will remain less than or equal to this, because this is a least upper bound for this.

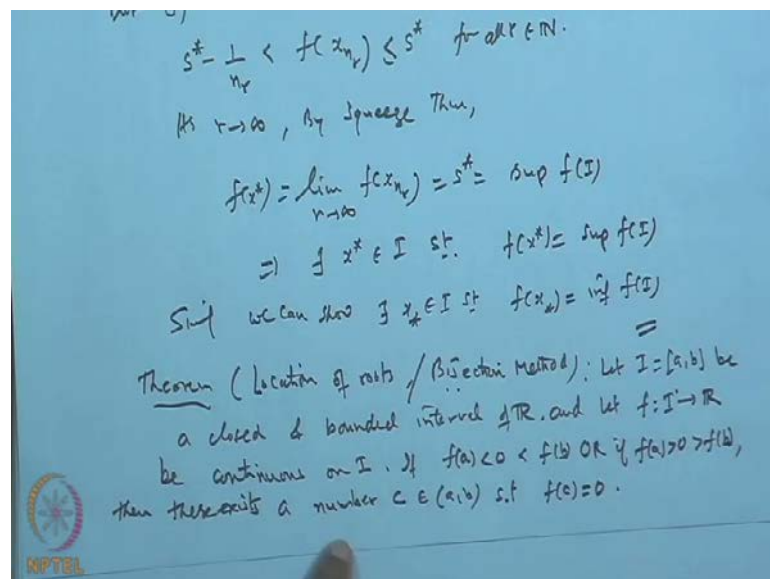
Now, this sequence number x_n , but this sequence is x_n satisfying (1) are lying in the interval I say a, b , which is a closed and bounded interval. So, by Bolzano-Weierstrass theorem, there is a subsequence say X' ; elements are x_{n_r} belonging to of X ; X is a sequence x_n of x that converges to some number x^* say. In fact, this number which we have got it, must be a point of a, b . This we wanted show first. So, how to show is, since all the elements of this are again the elements of I . So, since the elements of X' , that is, x_{n_r} , they belong to I , which is closed and bounded interval. So, just like previous theorem, we have seen that, if the sequence of the point belongs to closed and bounded interval, then limit point will also belongs to it. And since it is close to limit point, it follows from this that, the limit point x^* is also a point in I . Therefore, f is continuous at I .

(Refer Slide Time: 28:13)



Therefore, f is continuous at this point also, because it is continuous throughout over the I . So, once it is continuous, apply the definition Heine definition. Heine theorem says, if a sequence converges to x^* , then f of x_n will also converges to $f(x^*)$. So, by theorem, limit of $f(x_n)$, when n tends to infinity coincide with f of x^* , because it is continuity. But by the first one... Use the first one.

(Refer Slide Time: 28:52)



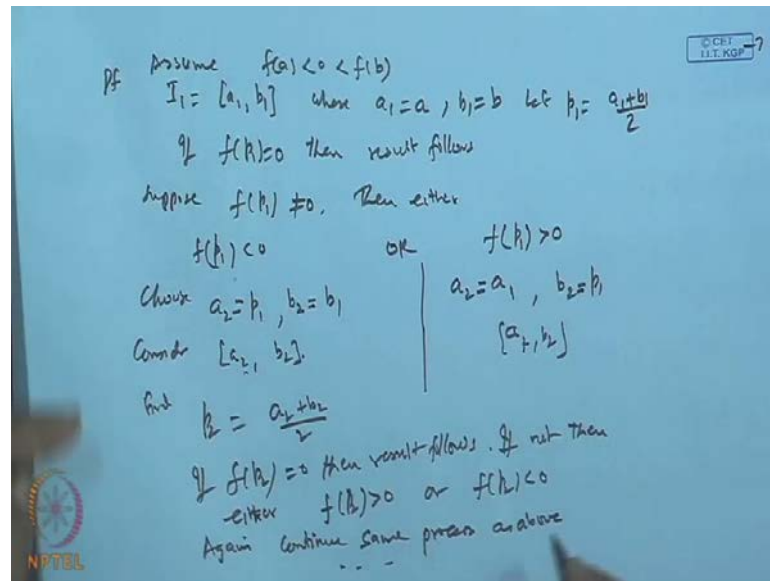
From first, what we get is $s^* - \frac{1}{n}$ is less than $f(x_n)$. This is less than or equal to s^* . So, this is true for all n belongs to \mathbb{N} . Now, let n tends to infinity. So, this

limit is s^* ; this is s^* . So, by Squeeze theorem, the limit of this function $f(x_n)$ as n tends to infinity will be equal to s^* . But this limit is nothing but what? $f(x^*)$. So, this implies there exist an x^* belonging to I such that the supremum – because this is the supremum value of the function f on I ; such that this $f(x^*)$ is the supremum of f on I . And that proves existence of it. Similarly, we can show for that, there exist an x_* in I such that $f(x_*)$ is the infimum of f on I . And that completes the proof of this. So, this is another result, which (()).

Now, next result is also interesting. That shows, the location of the roots. In fact, this is also known as the bisection method. It will use in the bisection method for this known as the... (()) is known as the bisection method. So, we are not touching the location... What this result says is, let I be a closed and bounded interval of \mathbb{R} . And let f is a mapping from I to \mathbb{R} be a continuous on I .

Now, if at the point a is suppose negative; at the point b , it is positive; or, if at the point of a , it is positive; and, at the point of b , it is negative; that is, at the corner point, if the function attains the different sign, then there exist a number c belongs to the interval a, b such that the value of the function at the point c will be 0. So, this shows that, we can identify the root of the function if a function is defined over a closed interval a, b , which has a alternate sign; that is, at the point a is negative, at the point b is positive. So, there will be some point c . And the function is continuous. So, obviously when the function is continuous, there is a continuous graph. So, when the function is negative, it means the part of the graph is below the x -axis and part of the graph is above the x -axis. So, obviously because of the continuity of the curve, the curve definitely crosses the x -axis. So, that point where it crosses will be the point c ; where $f(c)$ will be 0. And that is the the location.

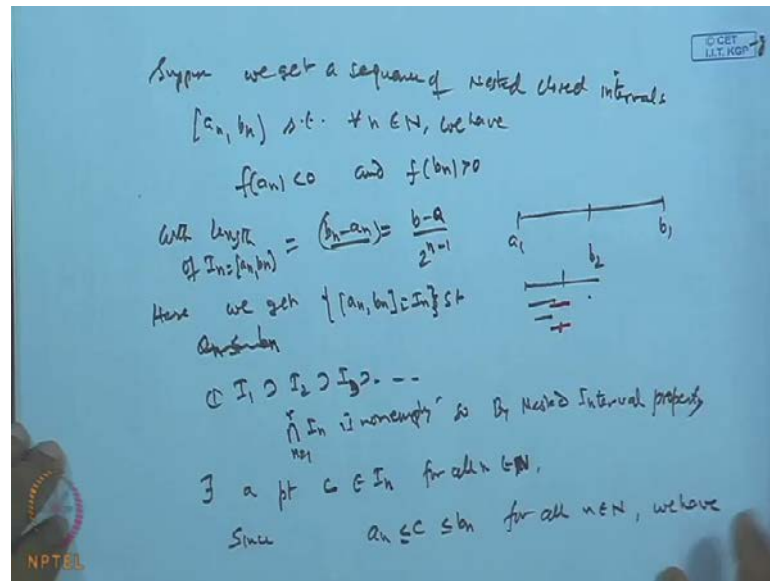
(Refer Slide Time: 33:00)



Now, proof of this... In fact, we will generate the sequence of successive by section just like (()). So, let us suppose I_1 is an interval say a_1, b_1 . And assume... Let us first assume that, f of a_1 – negative and f of b_1 is positive. Now, I_1 is the interval, where a_1 is suppose a and b_1 is suppose b and let p_1 is the middle point of a_1 plus b_1 by 2. Now, if f of p_1 is 0, then result follows. Suppose f of p_1 is not equal to 0. Then either f of p_1 will be negative or f of p_1 will be positive. If f of p_1 is negative, then in that case, we take the a_2 as p_1 ; b_2 as b_1 . And in case, if this is positive, then take a_2 as a_1 , while the b_2 is p_1 . And consider the interval a_2, b_2 like this. And then one of the case will be open, so one of the interval a_2, b_2 .

Then find the point p_2 , which is again the interval half of this; that is, basically, a_2 plus b_2 by 2. So, basically, this length when you are taking this a_2 by 2, then test the functional value as f_2 . If it is 0, then result follows. If not, then again, either f of p_2 will be positive or f of p_2 will be negative. So, again, continue the same process as above. Suppose we are getting after at the n eth stage, what we get?

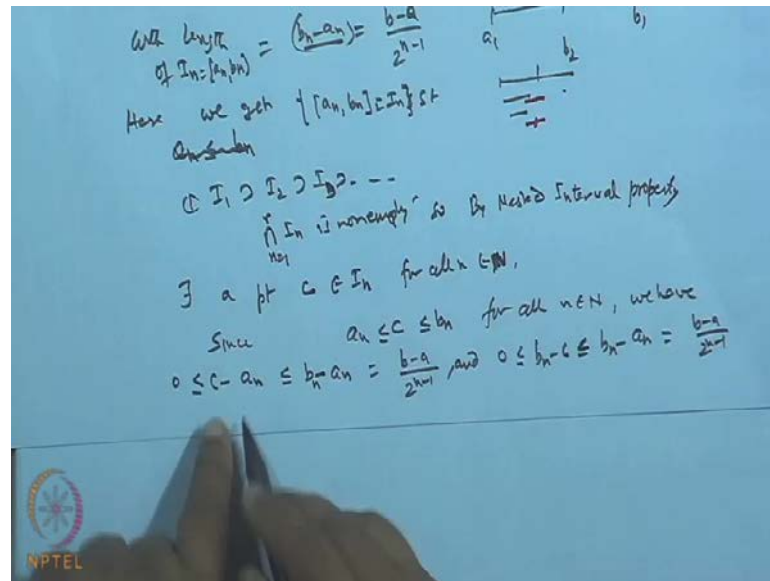
(Refer Slide Time: 35:57)



Suppose we get the sequence of nested closed intervals a_n, b_n with length such that for every n belongs to capital \mathbb{N} , we have the value of f of a_n is negative and the value of the function at the point b_n is positive. So, this is the sequence of nested intervals say a_1, b_1 . Then maybe once you divide, here is a_2, b_2 . Like this further divide; and, like this. So, we get this nested sequence of the nested intervals we are getting; or, maybe sometimes here or there – that also, possibility may be like this also that instead of, we get this or maybe this and so on like this. So, we get a sequence of the nested intervals, which is contained totally in the previous one. And length of this – with the length, will be b_n minus a_n . And that is equal to b minus a over 2 to the power n minus 1 . This will be the length of the interval a_n, b_n – length of I_n , which is a_n, b_n interval – this one.

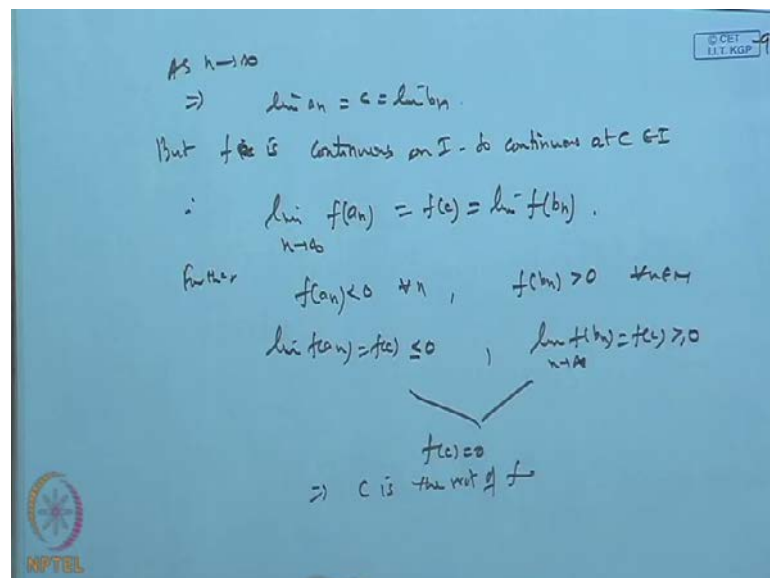
Now, let us see, here we get the sequence a_n, b_n , which is less than equal to b_n – nested interval. So, what we get it here is... So, here we get a sequence of the nested interval a_n, b_n say I_n such that I_1 covers I_2 covers I_3 and so on. And the finite intersection of I_n when n is equal to 1 to R is non-empty. So, by the result, which we have nested interval property – by nested interval property, there exist a point c that belonging to I_n for all n . This is nested interval property for all I_n .

(Refer Slide Time: 39:32)



Now, since c lies between a_n and b_n for all n belongs to \mathbb{N} , we have that 0 less than equal to c minus a_n , which is less than equal to b_n minus a_n , which is equal to b minus a over 2^n minus 1 and 0 less than equal to b_n minus c , which is less than equal to b_n minus a_n and which is equal to b minus a over 2^n minus 1 . This is true. So, when n tends to infinity, this is tending to 0 , this is tending to 0 . So, this shows limit of a_n is c , limit of b_n is c .

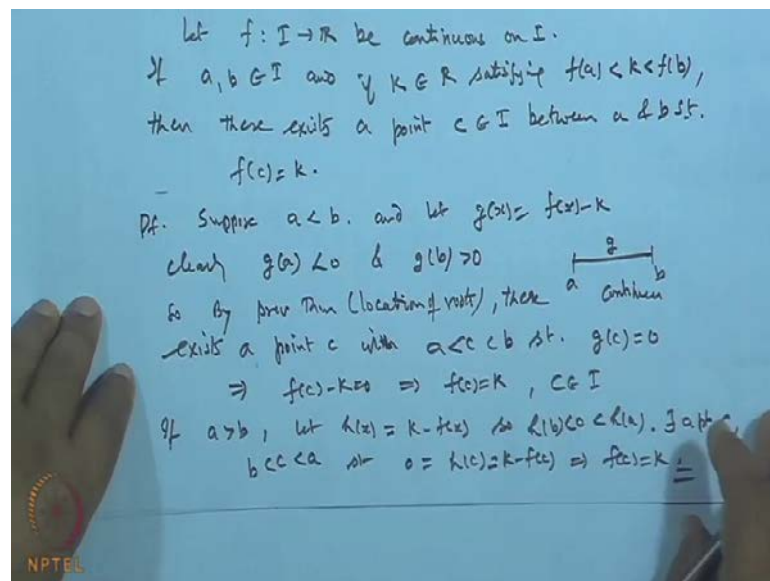
(Refer Slide Time: 40:18)



So, this implies that, as n tends to infinity, limit of a_n is c , which is the same as the limit of b_n . But f is given to be continuous – f is continuous. So, continuous at this point; continuous on I . So, continuous at c also, which is in I . Therefore, limit of the sequence f of a_n as n tends to infinity is nothing but the value of the function at the point c , which is limit of b_n . So, this shows...

Now, further, f of a_n will always be negative for each n . And f of b_n will always be positive for each n . Therefore, the limit of this sequence f_n , which is equal to f_c will be less than or equal to 0. And from here, the limit of $f b_n$ when n tends to infinity, which is also f_c , will be greater than or equal to 0. So, when you take these two together, we get f of c is equal to 0. And that proves the root. This shows, c is the root of f ; that is, there exists a c , where the function will be 0. So, if alternate positive, negative, then we get this thing. So, that is very (()), it is basically used by numerical methods in numerical to get the approximate root for the function $f x$ equal to 0.

(Refer Slide Time: 41:59)



Next result, which we have, the Bolzano's intermediate theorem – what this theorem says, let I be an interval and let f , which is a mapping from I to \mathbb{R} be continuous on I ; f be continuous on I . Now, if a, b belongs to I and if k is any real number, satisfies the $f a$ is less than k , which is less than f of b means in between f and a, b , we are choosing a real number k . Then there exists a point c in I between a and b such that the value of the function at the point c is k . This is known as the intermediate theorem; means if f is a

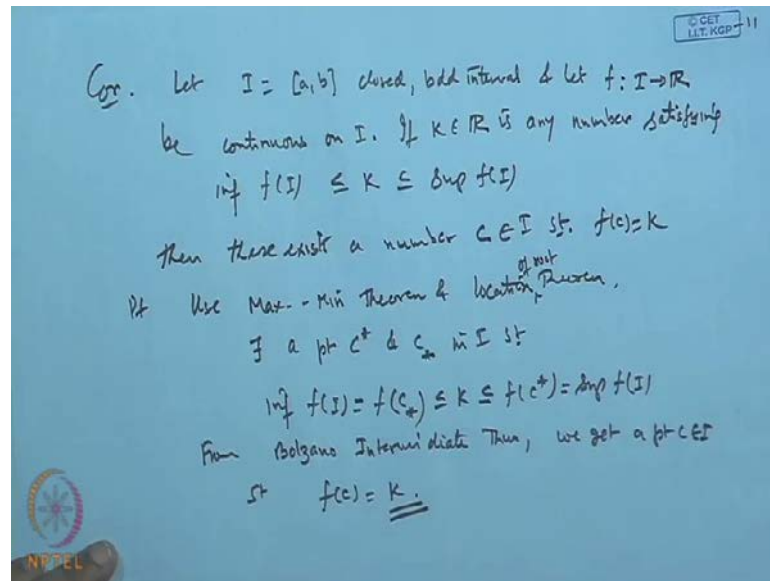
continuous function, then it will attain all its values in between the maximum and minimum value. So, in fact, here we are not discussing about the maxima-minima. What we are saying, we are taking two particular values of the function $f(a)$ and $f(b)$; and, they are distinct. So, if we picked up any number in between $f(a)$ and $f(b)$, and f is continuous, there will be at some point c available, where this number will be attained by the function at some point. So, that is known as the intermediate theorem.

Proof of this is like that. Suppose a is less than b ; let us take this one first. And let g is defined as $g(x)$ is chosen as $f(x) - k$. So, if we look this function, then clearly, at the point a , $g(a)$ is $f(a) - k$; $f(a) - k$ is negative. And $g(b)$ is positive. So, this function g is a value negative at the point a ; positive at the point b . g is continuous function, because f is continuous; k is constant.

So, addition and subtraction of the continuous function is continuous. So, g is continuous over the interval a, b . Therefore, by the intermediate theorem, by the location of the root, which we have proved earlier, there will be some point in between a, b , where the function g will be 0. So, by previous theorem, that is, location of roots, there exists a point c such that with c lying between a and b , such that the value of this $g(c)$ is 0. But what is $g(c)$? But $g(c)$ is nothing but the $f(c) - k$ is 0. So, this implies $f(c)$ is equal to k . So, there will be a point c available in I , where the function will be attained. And the k will be attained by this function at this point.

Similarly, if we take suppose a is greater than b ; second – if a is greater than b , then what happens? We consider the function $h(x)$; instead of this, we say $k - x$, so that $h(b)$ will be negative and $h(a)$ will be positive. So, again, there exists a point against c lying between b and a such that the value of h will be 0 at this point, that is, $k - f(c)$. So, this implies $f(c)$ is equal to k . And that proves the... Now, we can extend this result, because this is for any value lying between the two unequal values of f . If we replace this f and $f(a)$ and $f(b)$ by its minimum value or the maximum value, that is, infimum of this and supremum of this; we choose a k in between infimum and supremum; then also, we can get some point value.

(Refer Slide Time: 47:30)



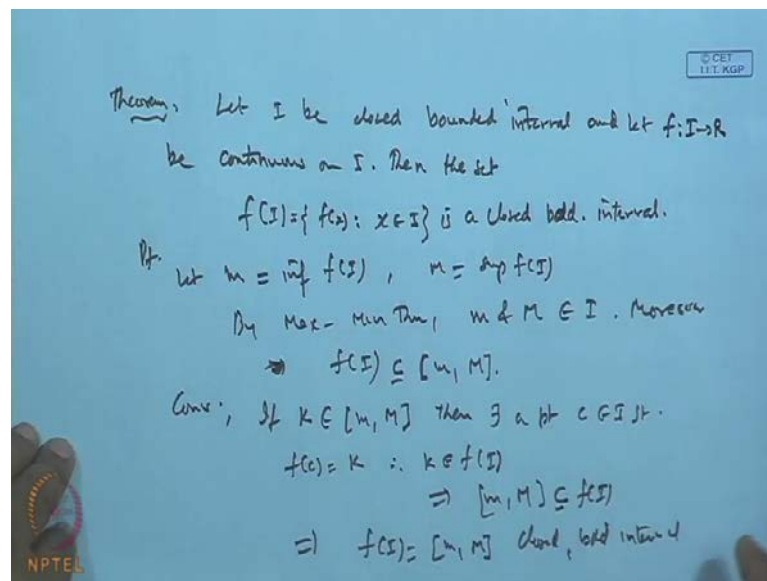
As the corollary to this, we can say, let I be a closed, bounded interval and let f is a mapping from I to \mathbb{R} be continuous on I . Now, if K is in \mathbb{R} , is any number satisfying the infimum of f of I , that is, infimum of I is less than equal to k , which is less than equal to supremum of I . That is the minimum and maximum value. So, k lies between minimum and maximum value. Then there exists a number c in I such that the value of the function at the point c will be k ; means this k will be attained by the function at some point.

Proof follows just from the maxima-minima theorem and above this previous Bolzano's intermediate theorem. So, it follows from use maxima-minima theorem and location of roots theorem. So, what we see here, there exists a point say c^* - c upper star and $c_{\#}$ lower star in I such that the infimum of this will be f of $c_{\#}$; infimum will attain, because I is a closed and bound interval, f is continuous function. So, infimum will be attained. And there exists a point $c_{\#}$, where the f of $c_{\#}$ is the infimum value. And then this is less than equal to k , which is less than equal to f of c^* , which is the same as the supremum of f of I . Now, conclusion follows from Bolzano's (()). So, from Bolzano's intermediate theorem, we get a point c belongs to I such that the value of the function at the point c is k follows. And that proves the result, which is (()).

Now, we get one more result, which is also true in case of the continuous function. The transfer of an interval - if f is a continuous function, then it will transform the close

interval to the closed interval. But if interval is not closed, then the image of the interval other than the closed interval, that is open interval or semi-close interval, need not be remain to open or semi-closed; that is, the nature of the closed interval is only retained by a continuous function. But if the interval is not open and closed, then its nature may change depending on the function.

(Refer Slide Time: 51:15)

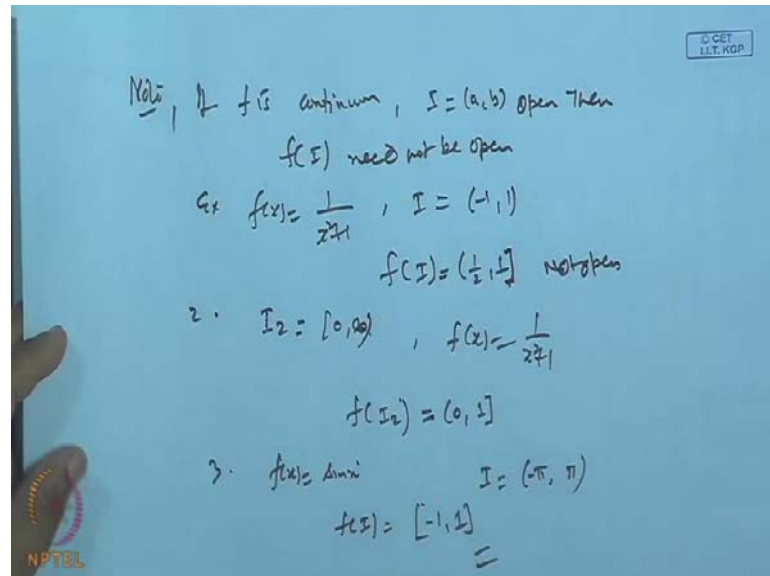


So, we get this result. First, for this, let I be a closed, bounded interval. And let f is a mapping from I to \mathbb{R} be continuous on I . Then the set $f(I)$, which is the set of $f(x)$ such that x belongs to I , is a closed, bounded interval. So, proof is just like, suppose m is the infimum value of $f(I)$ and capital M be the supremum value of $f(I)$. Suppose then we know that, from maximum-minimum theorem, this m and capital M belongs to I , because there exists and there will be a point, where it belongs to I . Therefore, the every value of $f(I)$, moreover moreover the functional value will lie between the interval m and M , because its maximum value and minimum value only will be there. So, we can get the maximum value and minimum value. All the values lie in between this.

Conversely, if k is any point belonging to this interval – k is any value in between m and capital M , then there exists a point c in I such that the value of the function at the c is equal to coinciding with k . It means k is an element of $f(I)$. So, we conclude that, any value in between $f(I)$, m , is also contained in this. Therefore, we can say, m , capital M – this closed interval is contained in $f(I)$. So, combine these two. We get $f(I)$ is nothing

but m , capital M , is a closed, bounded interval; that is, the image of the closed, bounded interval under the continuous function is closed and bounded.

(Refer Slide Time: 54:10)



Let us see the note. The image of... If f is continuous and I is an open interval, then image of $f I$ need not be open. For example, if we take say, function $f x$ is 1 over x square plus 1 and I is minus 1 to 1, then f of I – you can see just half, 1. This is not open; closed at point, not open. Similarly, if we get... If f is say semi-closed interval, then also, suppose I_2 is our interval say 0, infinity – a semi-closed interval and f of x is the same as x square plus 1, we see f of I_2 ; that f of I_2 comes out to be a semi-closed interval, but open at this point. So, what we say, I_2 , which is not a closed interval. So, if it is not, then we can... So, this shows that.

Similarly, third, if we take $f x$ is equal to $\sin x$ and if we choose the interval say minus π , 2π , then image of this interval $f I$ will be the closed interval minus 1 to 1. So, this shows that, only the closed intervals under the continuous function remains closed image; otherwise, if the interval is not closed, the image of that interval is a continuous function, need not be the same nature.

Thank you very much.