A Basic Course in Real Analysis
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Lecture - 3
Continuum and Exercises

So the last lecture we have discussed the cuts related to the real numbers, and in fact we have also given the idea of the Dedekind's theorem. And that theorem says, if you remember that if and Dedekind if you just go through the recapitulate about things. What is the Dedekind's theorem is that if the system of the real number is given, then we can or divide this system in to the two classes; lower class and the upper class, such that each class will contain at least one number, and second part is every number belonging to one class or the other.

So, these classes will be non empty, and the real number will belongs to either one class or the other class, and third point is the every member; every number is lower class is less than the every member of the upper class. Then this cut this section we denoted by alpha and we say alpha represents the real number. The alpha may or may not belongs to the curve. If it a rational number, it will belongs one of the class either L or R and if it is a irrational number, then it may not belongs to that it will not belongs to the class L or R.

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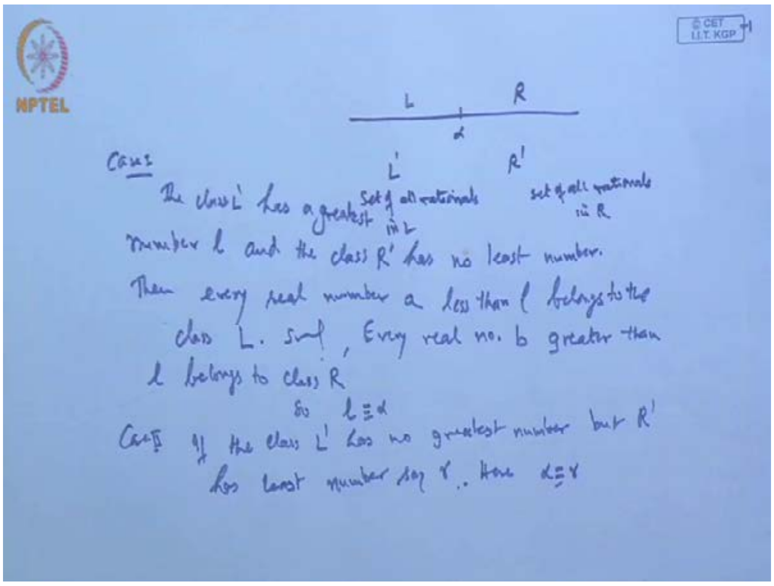

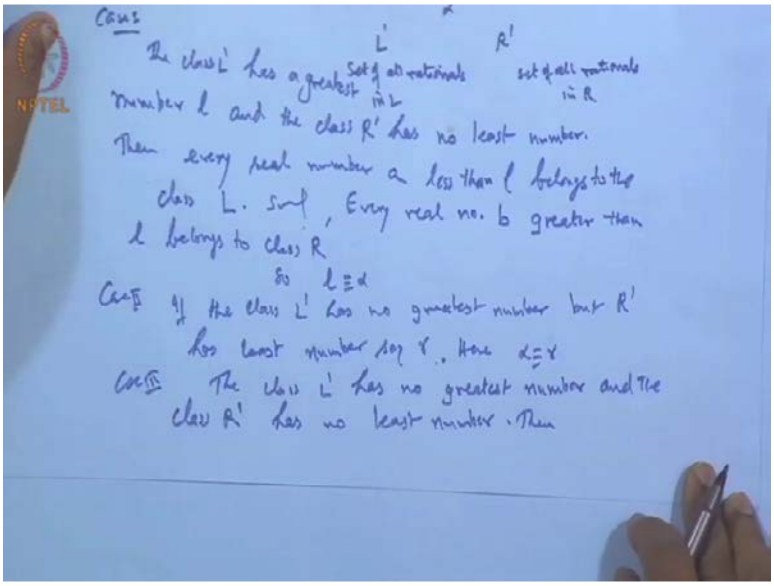

This can also be justified this theorem this results can also justified as follows. Suppose we have they cut alpha is a real number which bifurcate the entire or the real numbers into two classes; lower class and the upper class. Now in this case, suppose we construct the L dash and R dash as the set of rational numbers, set of all rational numbers rationales, set of all rationales which is in the class L found it L and set of all rational which are in the upper class R, rational in L set of all rational in the upper class R, and let us not these (( )) R. So, they will find the two classes; every rational number will be one of the class either L dash or R dash and number will be (( )). Now, there are three possibility. One is the case one; if this low, because L dash and R dash these are collection of the rational point. So, L dash may have the greatest number greater number R dash may not have a least number. Second case when L dash does not have a greatest number, R dash has a least number and third case when none of them is (( )). So, first case if the class L dash the class L dash has a greatest number, say L number L and the class R dash has no least number no least number, then every real number a then every real number a which is less than alpha L which is less than L is well is less than L less than L belongs to the class L belongs to the L why, because this L dash has a greatest number L. So, any rational numbers which are less than L must be the point in less and between any two rational number there are the further rational number. Similarly, between two real number we will show that there are infinite numbers of real numbers.

So, if any number a any real number a weather it is rational or is a rational if it is less than L; it means, it must be the class in L dash if it is rational is a not. Otherwise, it will be the class in L, because L is the lower class; it contains all the rational and irrational numbers. So, basically any number a which is less than L must be the point in L. So, that is one point and similarly, any every real number b similarly, Every real number b greater than L every number b greater than L belongs to class r. Again the same thing any number be real number if it is a rational point than it is greater than the must be in the R dash so; obviously, it will be in R if it is a not rational than we can choose the all the points, because it is the lowest be all the number which are greater than will be come over here. So, that will be final in the in this class and every real number be greater than this where this is. So, thus L is a number alpha for this. So, what is? So, here, so L correspond to above alpha is a it not in this case the alpha real number basically is nothing, but L clear similarly, in the second case if the class L dash has no greatest number, but R dash has least number say a small R then the same repetition case will be there and here in the same case same alpha becomes r.

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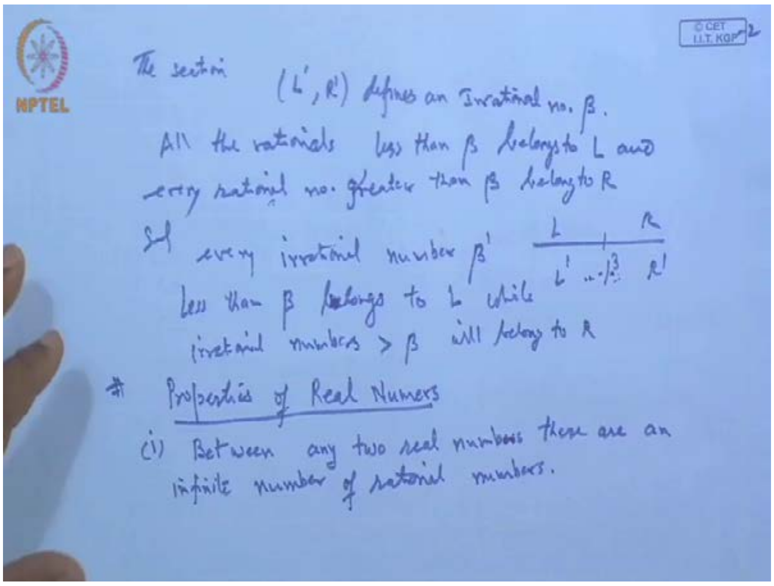

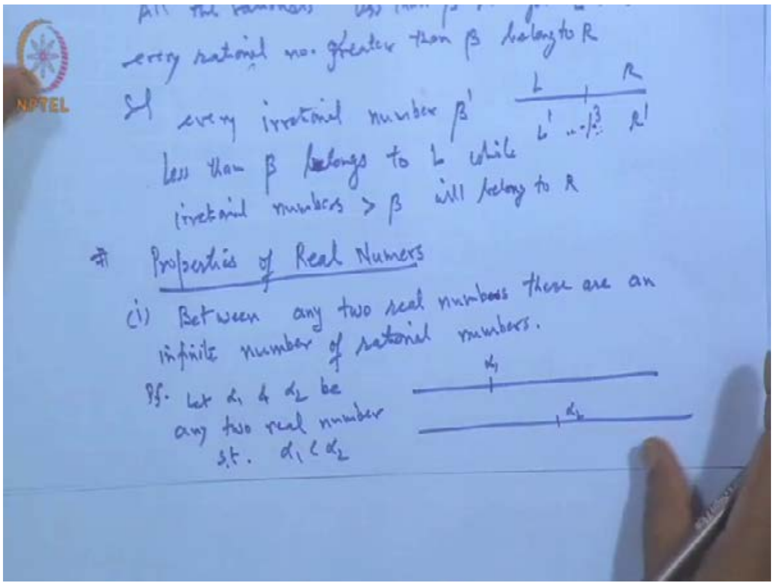

In the third case in the third case the class L dash has no greatest number and the class R dash has no least number then basically this section will define a rational point than the section is it then this section the section L dash R dash this defines an irrational number beta. Now, our aim is that this all the real number will be either in L or in R. So, all the rational number all the rational which are less than beta all the rational number less than beta belongs to L and or an d every rational number greater than beta belongs to R why.

So, because this is our beta here this L this is R I am choosing L dash and R dash and beta I am taking such a thing where L dash is neither having the greatest number R dash does not have the least number. So, it will the section represented by beta now we claim this beta is basically the number real number corresponding to our cut L R, then if we chose any number less than beta then that number if it is rational it will be the point in L dash. So, it will be the point in L. So, all the rational which are less than beta must be the n L suppose I take a rational point then what happen he points, which are say any point beta there which is less then beta in between beta and beta dash there are infinitely many irrational points. So, those points all in the L dash hence it is in L. So, beta dash has to be in L. So, therefore, all such number rational number greater than less than beta must be in L or rational numbers which are greater than beta must be in this. So, this is one can similarly, we can prove for irrational case similarly, if every rational number every irrational number sorry irrational number beta dash less than beta belongs to less than beta.

Belongs to L while the irrational number greater than beta will belongs to R and the reason is I just justified the reason, because there are rational number in between beta dash and beta which are in L dash. So, it is in L therefore, this much similarly, the other. So, this way we can say that any if it take a aggregate of the real number or the set of real number we can always divide into two classes we are both the class will be nonempty and elements either lower class will have the largest elements upper class will not have a least elements and vise versa. And every element of the lower class is less than the every member of this and alpha corresponds to this section. So, if it is rational belongs to one of the class if it is irrational then if not belongs to any other class.

So, this is the way the Dedekind's I introduced the concepts here in this process be a though we have not justify, but what we have assumed it between any two real number there are infinite numbers of real's also is a not that is the way justify between any two rational number we can justify it as a rational number what about the irrational number if there are two irrational number can you say again there are the infinite number of the t of rational irrational number in between it the justification is follow. So, we can go through the some properties the properties of real number.

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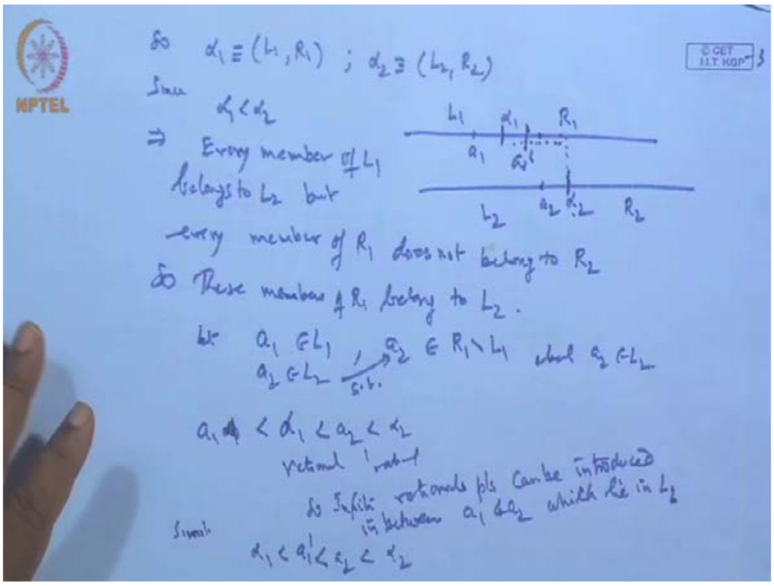

The first property is between any two real numbers between any 2 real numbers there are an infinite number of rational numbers between any two real number there are an infinite number of number of rational numbers let us see how suppose we have the two numbers

1 is alpha 1 another one is alpha 2. So, let alpha 1 and alpha 2 we any 2 real number such that alpha 1 is less than alpha 2. So, alpha 1 will correspond to the section L 1 R 1. So, alpha 1 will correspond to. So, alpha 1 will correspond to the section L 1 R 1 alpha 2 will correspond to the section L 2 R 2 is a not clear now the position is like this is alpha 1 here is alpha 2 this.

Correspond to section L 1 R 1 this correspond to section L 2 R 2. So, every element of a L 1 is basically element of L 2 every number which belongs to L 1 is also there in L 2 is it not because alpha 1 is less than alpha 2, but all the numbers of R 1 is not in R 2 it means some of the elements of R 1 must be in L 2 is a not. So, because of this. So, we can say that every member since alpha 1 is less than alpha 2. So, this implies that every member of L 1 every member of L 1 belongs to L 2 every member of L 1 belongs to, but every member of R 1 R 1 does not belong does not belong to R 2, because these are the points which are left out is it not these are the points which does not belongs to R 2 than, but they are in R L 2, but they are in L 2 this belonging (( )). So, these members of R 1 belong to L 2 is it .

So, now if you pick up any two elements from L 1 and L 2, so let a 1 belongs to L 1 and a 2 is an R 1, but not in L 1 not in L 1 it means I m taking here somewhere is a not this point it is in here this is the point a two sorry yes this is the point a two which is in which is clearly a 2 is in L 2 or a 2 belongs to the L 2 such that let us take this a 1 is in alpha 1 the here is the a 1 a 2 I am taking L 2 which is in R 1, but not in R 2, so those points. So, a 2 minus a 1 or. So, clearly a 1 is less than a 2. This a 1 is less than a 2, but what is the a 1 a 1 in a 2 are the point in the same class and class L 2 all the element which are less than alpha 2 must be the point in L 2 and there are infinitely many point if I chose between a 1 and a 2 there are many points which can introduce between a 1 and a 2 which are in less than alpha 2.

So, it is again in this. So, what we can between a 1 and a 2 if a 1 and a 2 are rationales' are rational than we can introduce many infinite number of. So, infinite rational point rational points can be introduced in between a 1 and a 2 which lies in which lie in L 2 is it or not it lies in L 2 yes yeah like this way. So, we are in fine it means between any 2 real number, but a 1 a 2 is satisfying this condition the alpha 1 less than this alpha 1 is less than a 1 the sorry I am sorry yes no a 1 a 1 is less than alpha 1 a 1 is less than alpha 1. So, it is alpha 1 a 1 is less than alpha 1 a 2 less than alpha 2 now if we take a 1 here say here I take a 1 dash than similarly, if I take alpha 1 less than a 1 dash which is less than a 2 which is less than alpha 2.

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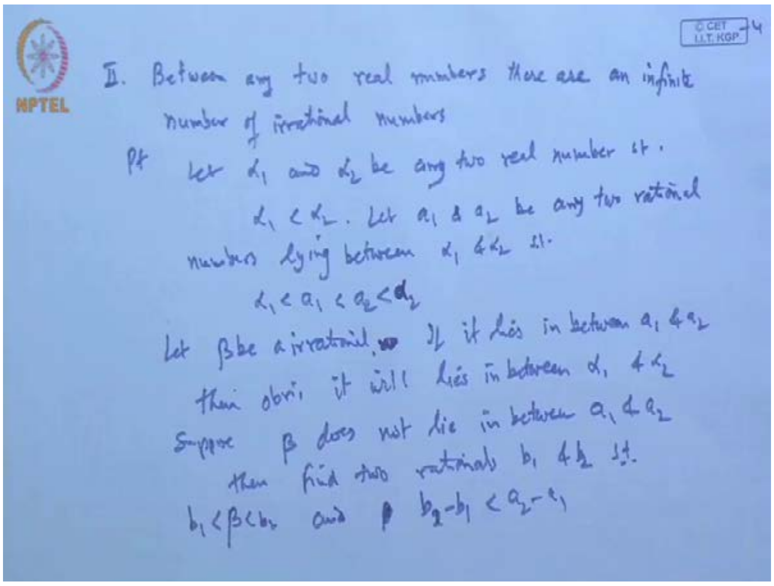

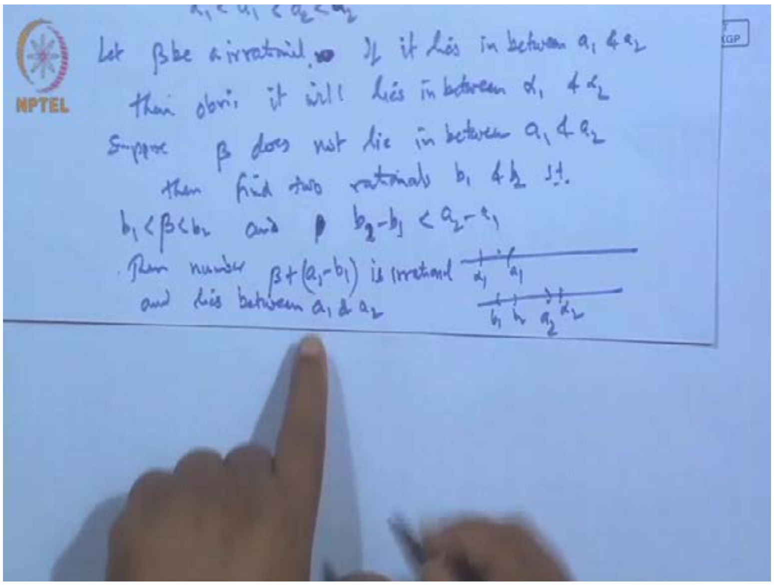

So, in between alpha 1 alpha 2 there are infinite many rational number can be introduced and that is what he says that between any 2 real numbers there are infinite number of rational points. So, again what I did suppose you chose the 2 real number alpha 1 alpha 2 then ordering relation is possible can be define. So, alpha 1 is suppose less than alpha 2 it means you are having cut of this type this is alpha 1 here is alpha 2 what we want in between alpha 1 alpha 2 there are infinite number of rational numbers. So, what I am taking is I am picking up 2 rational number 1 is a 1 dash another 1 a 2 a 1 dash is a point in R 1, but not in L 1 a 2 dash is a point in L 2 so; obviously. So, a 1 dash L 2 we can again order them let a 1 dash is less than a 2 a 1 dash and a 2 are rational number.

So, in between these 2 rational number we can introduce in infinite number of rational point again therefore, in between alpha 1 and alpha 2 we can introduce in infinite number of this rational points that is what he says. So, this is second case is second property is between any 2 real number between any 2 real number there are an infinite there are an infinite number of irrational numbers. Now, proof is suppose let alpha 1 and alpha 2 we any 2 real number real numbers such that alpha 1 is less than alpha 2 now in the previous property between any 2 real number we can introduce rational points. So, let a 1 and a 2 with the 2 rational number.

Lying between alpha 1 and alpha 2 such that alpha 1 is less than a 1 is less than a 2 less than alpha 3 alpha 2 sorry less than alpha 2, because alpha 1 alpha 2 reacts in between we can introduce the rational now what we want it show that in between these rational number are there. So, let beta be a rational number let beta be a rational number let beta be a rational number rational irrational sorry irrational number beta be a irrational number lying let beta be a rational number. So, if it lies in between. If it lies in between a 1 and a 2 then our problem is solved then; obviously, it will lie between it will lie in between alpha 1 and alpha 2 suppose it is not suppose beta does not lie or you can write their exist a rational number beta suppose beta does not lie in between a 1 and a 2 suppose beta do not lie between then we can choose a 1 and a 2 then find the 2 rational number b 1 and b 2 such that beta lies between b 1 and b 2 and b 1 minus b 2 or b 2 minus b 1 is less than a 2 minus a 1 let us see how what we did suppose we have this alpha 1 here we have alpha 2 in between alpha 1 alpha 2 I am taking the point a 1 and here is say b a 2 there are infinite many points alpha a 1 a 2 now in between a 1 a 2 there are the rational number irrational number seemly. So, suppose beta is a point irrational number lying between this then ; obviously, it will lie between this alpha if it does not lie then it will lie outside of it something.

So, suppose beta corresponding to the beta we can identify the rational number b 1 b 2 such that beta lies with b 1 b 2, but the difference between beta b 2 minus b 1 less than this is this difference a 2 minus a 1 is less than b b 2 minus b 1 here is some thing b 1 b 2 like this whose difference is less than this 2 then the consider then the number beta plus a 1 minus b 1 this number is a irrational number lying between and will lie and lies between a 1 and between a 1 and a 2 why. Suppose beta is does not lie between a 1 and a 2 then I have I can choose the 2 rational number which can enclose the beta 1 beta 2 sorry beta b 1 b 2. So, construct a number beta plus a 1 minus b 1 what I claim this is a irrational number irrational is; obviously, to because beta is irrational.

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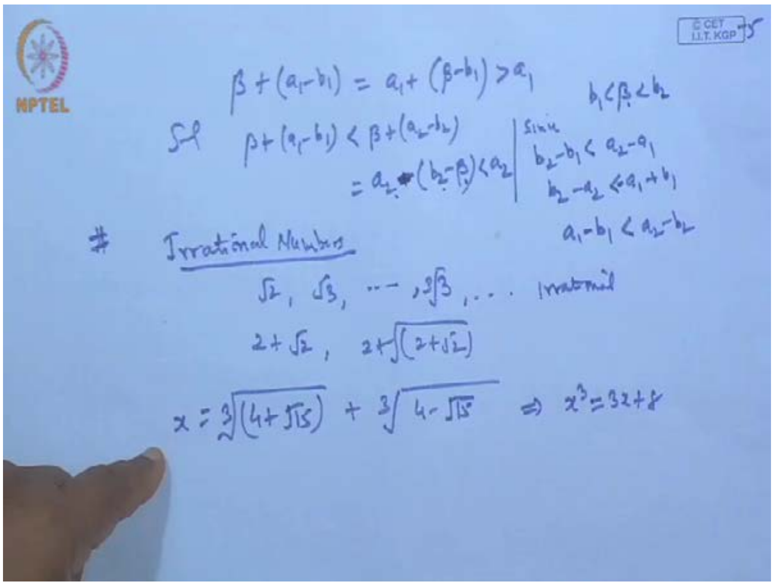

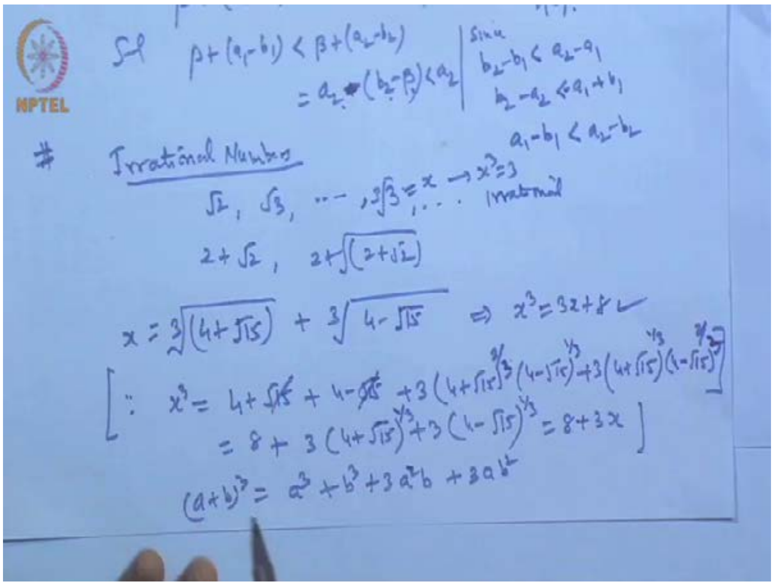

So, any number say under root 2 is a irrational then 2 plus root 2 will as be a irrational. So, this a irrational then no problem, but what he says is lie between a 1 and a 2 why because the beta plus a 1 minus b 1 this number when you write when you write beta plus a 1 minus b 1 this can be written as a 1 plus beta minus b 1, but beta I am assuming lying between b 1 and b 2. So, beta minus b 1 will be positive. So, this entire thing will be greater than a 1 by this is positive say any number at positive number at this quantity will be bigger than this. So, we are getting this number will lie will be greater than a 1 similarly, beta plus a 1 minus b 1 now a 1 minus b 1 from here we can write this since we have assumed b 2 minus b 1 is less than a 2 minus a 1 this we have assumed. So, what happen is when you take this side b 2 minus a 2 is less than a 1 minus b 2 minus a 2 is less than minus a 1 plus b 1 is it not or b 1 minus a 1. So, this a 1 minus b 1 is less than a 1 minus b 1 if you transferred here is less than a 2 minus b 2. So, if I take this is strictly less than beta plus a 2 minus b 2 and again this is a 2 plus minus times b 2 minus beta, but beta lies between what beta lies between b 1 and b 2. So, b 2 is greater than be beta. So, b 2 minus beta is positive your subtracting positive quantity from a 2. So, will it not be less than a 2. So, be this number lies between a 2 and a 1.

So, we have constructed a rational irrational number does not lie between a 1 a 2 then we can rewrite this number in such a way. So, that this new number irrational number will lie between a 1 and a 2 it means between any 2 real number we can introduce the a irrational number and there are infinite in numbers. So, this these are the 2 property which we enjoyed by this now we have seen that apart from the rational there are irrational number like under root 2 under root 3 etcetera, but weather this are the only irrational rational number the question arise can we say there are some other irrational number irrational number other than the surds these are called the surds say irrational numbers under root 2 under root 3 and so on.

So, far or may be 3 to the power cube root of 3 like this these are all irrational numbers. And not only this if I take 2 plus root 2 it will be irrational number or 2 plus 2 plus root 2 and then root 2 this is also irrational number and like this way we can go ahead further. So, what it says is there are. So, many irrational numbers now if we look these numbers suppose I take root 3 or the number like this suppose 4 plus root 15 cube root of 3 plus cube root of 4 minus root 15 suppose I take this number. So, if I take the number we can write it suppose this is x then if I simplify this number then we can say that this comes out to be this is equivalent to x cube equal to 3 x plus 8. x cube is 3 x plus why because if you take the x cube then what happen is, because x cube means a plus b whole cube is a cube plus b cube plus 3 times a square into b plus 3 times a into b square this is this is the expression for this x cube now this gets cancel. So, what we get 8 plus 3 now this will be equal to what 4 plus root 15 and then a plus b into a minus b a square minus b square. So, 16 minus 15 is 1.

So, nothing then here is plus 3 4 minus root 15 is it just I am combing this a plus b a minus a square minus. So, you are getting this, but again if you take 3 outside then what you get is it not the same as x cube plus 3 x this is cube root yes. So, what you are getting is a cube plus here something mistake I did what is this expression a plus b cube is a cube 3 will be out b cube 3 will be out then square of this. So, 2 by 3 is it not into one-third then again one-third into 2 by 3 is it correct a square means what this is the cube root power 1 by 3. So, square means. 2 by 3 multiply this 1 by 3. So, so when you take this outside this becomes the what is one-third if you take outside one-third. So, finally, what you are getting is square of this one-third and one-third. So, one-third is outside and that now this is x. So, it becomes the 8 plus 3 x is it. So, we get this 1 no it is not clear by this is the formula is a plus b cube is a cube plus b cube plus 3 a square b plus 3 a b square this is the formula, so using this formula this. So, you can get yes a plus b is x yeah that is also be clear.

So, you get it no a b here will not because power 1 by 3 is there n a. So, how can you it is 1 you just open it and take here what I am doing is 4 plus 1 by power 2 by 3 is there I can write 1 by 3 into 1 by 3 4 plus root 5 a power 1 by 3 multiply by 4 plus root 5 power 1 by 3 and then 1 by 3 combine 4 minus that becomes one. So, only this term is there similarly this. So, that is what. So, what it says is this one similarly, if you take this number this is say x 1 can easily write it this x cube equal to 3 now let see the converse part suppose an expression is given x cube plus 3 x plus 8 and it is ask to find the x if it is. X equation algebra equation of degree 1 we can write it algebra equation to we can also be find x explicitly even x equal to 3 power is 3 still we can find the expression, but it is complete, but when the power when the algebraic equation is having the degree more than 3. In fact, more than 3, 4, 5, etcetera it is very difficult to write the x in the form of this surds thou theoretically it must set come, because it is a solution of the algebra equation clear. So, what we conclude is that when we have a general algebraic equation with the integral coefficients than this solution of this algebraic equation will gain either a rational number or may be a irrational number. So, always you find the roots of this algebraic equation will be rational or irrational.

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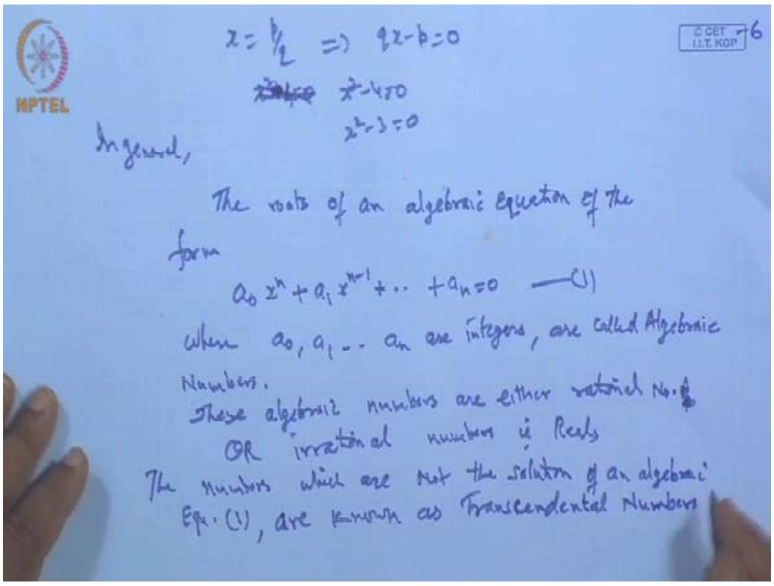

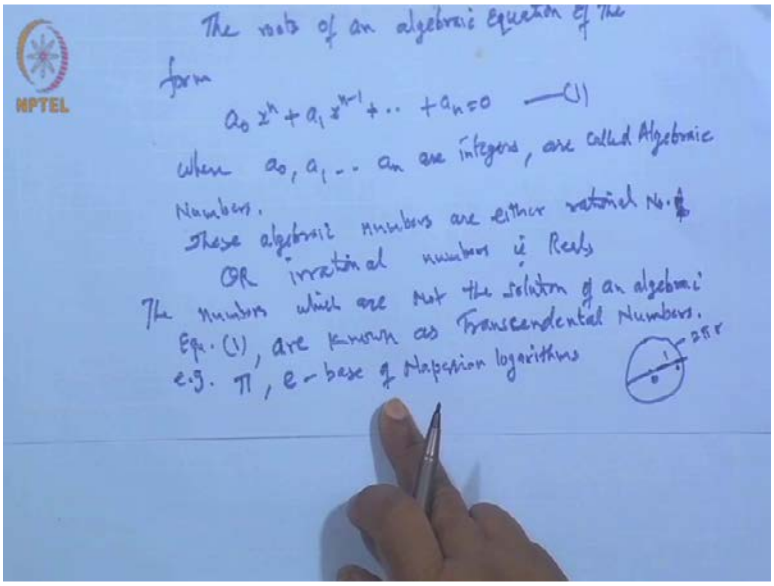

The sometimes it is possible to find the explicit form of the irrational number depending on the degree of the polynomial is it not polynomial in x if a degree is comfortable we can identify x as insert if a degree is higher we cannot write, but theoretically it should come. So, what we concluded is that the solution of all algebraic equation will always give a real number that is irrational number or rational number both is it clear. Why rational number suppose I take just for x equal to p by q this is rational point is it not a equation q x minus b equal to 0 is not a algebraic equation of degree 1. So, it means the solution of algebraic equation may be rational also, but if we take the equation like x square plus 1 equal to 0 or. X square plus 4 equal to 0 something like that some where then we are getting some irrational points or minus 4 you can say, because otherwise will take the lea dip complex. So, if I get this 1 then you are getting x equal to plus minus 2, but if I get x square minus 3 equal to 0 you get a irrational point. So, in general we can say that the roots of n the roots of n algebraic equation algebraic equation of the form a not x to the power n a 1 x n minus 1 plus a n equal to 0 where a naught a 1 a 2 n are integers are called algebraic number algebraic numbers are called the algebraic numbers and these algebraic numbers are these algebraic numbers.

Are either rational numbers or rational numbers or irrational numbers. So, basically there are the here that is real's the solution of the algebraic equation will be give a real number is it clear, but just by taking the solution of the algebraic equation we are getting a irrational numbers also will it exhaust the entire irrational points means this is these are the only irrational number which can be obtained the answer is no is not only the solution of the algebraic equation only can come give the irrational point and the irrational set of irrational number is not true there are some other rational irrational number which are not the solution of algebraic equation. So, others number the numbers which are not the solution which are not the solution of an algebraic number of an algebraic equation 1 of an algebraic equation.

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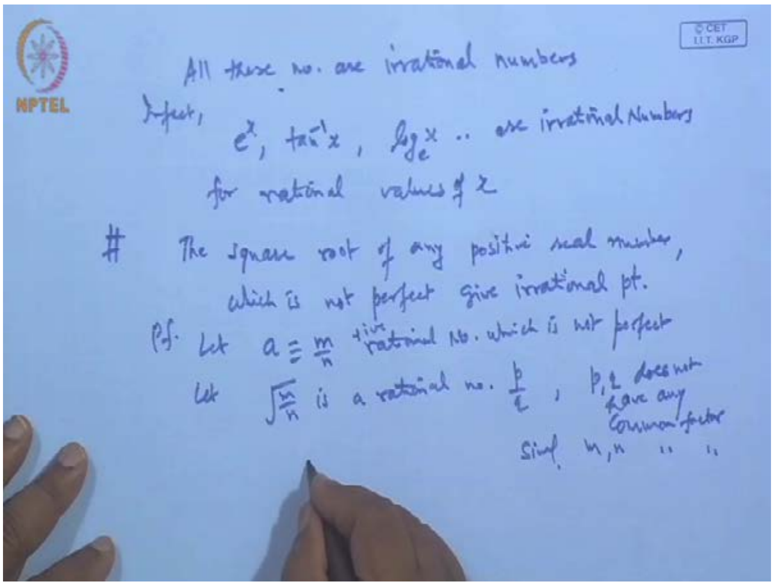

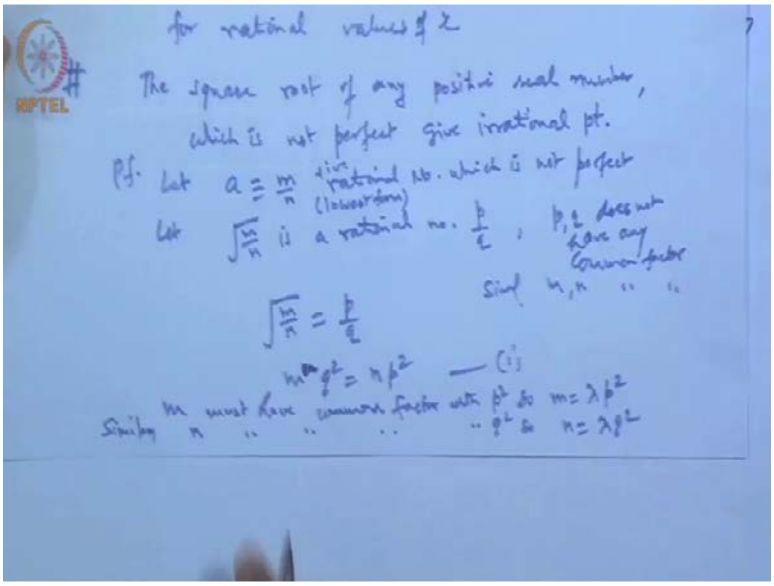

One are known as transcendental function transcendental number t R a n s transcendental numbers for example, our pie is a transcendental number it is not a solution of a any algebraic equation what is pie is basically the circumferences of a unit circle if you take a unit circle centered at 0 with the radius a 1 then what is the circumference 2 pie R is it not the if R becomes 1 R becomes 1 one then the diameter of this as diameter is 2 pie 2 pie diameter. So, that diameter 2 pie will give the pie. So, basically the pie is a transcendental number then and another 1 e that also a number what is the e is a dash of dash of naperian logarithm base of the log natural log with base e. So, this e is a transcendental number is not a then these numbers similarly e to the power x all these numbers are irrational number. Are irrational numbers. In fact, e to the power x ten inverse x into ten inverse x log x to the base e etcetera these are all irrational number irrational numbers for rational values of x ten inverse x is irrational number when x is rational e to the power x irrational when x is rational log of x is a irrational like that. So, the set of collection of the irrational number is a very weak set real set and here we can clear. So, this one now there is a result the result says is the square root of any a square root of any positive real number any positive real number which is not perfect gives irrational point give irrational point if a is a positive real number, but it is not a perfect means that a cannot be a square.

Of a is not n say a cannot express a square of some integers is not perfect the square root does not come out to be integer. So, if it is not a perfect than the square root must be a irrational point the proof is very simple in case of root 2 we have already shown suppose I take general 1 let a real number a let a equal to m by n which is rational point rational number positive rational number which is not perfect it is not perfect, but we want to show that the square root of this will not be a will be a irrational point. So, let us let square root of this is a rational number suppose p by q here p and q there not have any common factor in fact it is the least similarly, m n we assume m and n does not have any common factor this is also in least form. So, both are in least form. Now, let us see under root m by n is p by q. So, we get from here is m is square into q sorry m into q square equal to n and p square let it be 1.

Now, when we take this 1 q is square m into q square is n into p square. So, we can say that m divides p square, because m cannot divide n because m n is already in the lowest form m and n its given in the lowest form because m by n is in the lowest form do not have any common factor. So, only the possibility m must have a common factor with p square is it not. So, from here m must have must have common factor with p square similarly. So, m can be written as lambda times of p square is it not once it is a common factor is means the similarly, an has a an must have a common factor with q square. So, n must be equal to lambda in to q square is it not, but this lambda m and n are in the lowest form. So, lambda must be 1 because m and n does not have any common factor it is in the lowest form m by n we are assuming in the lowest form this is the lowest form a rational number.

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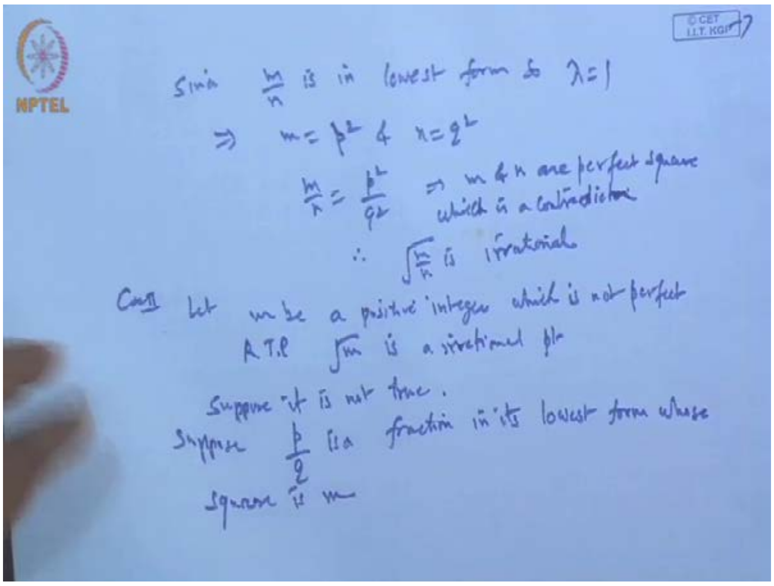

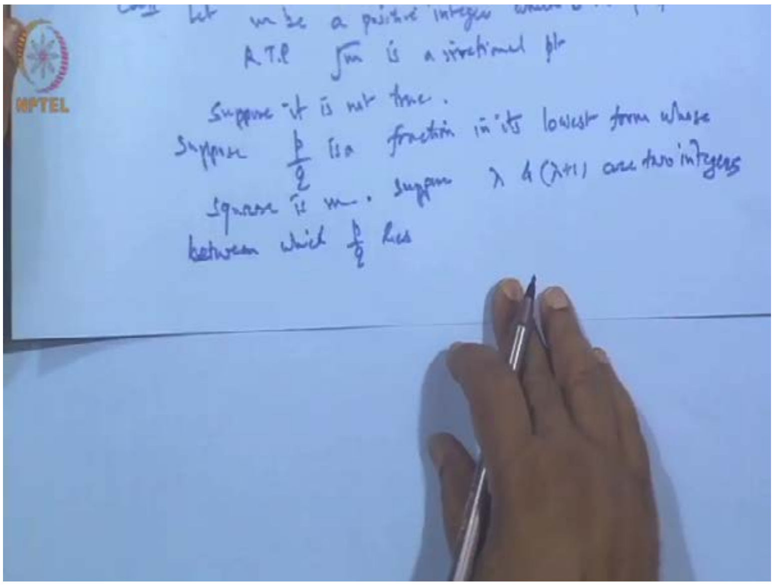

When you write a rational number 1 by 2 or say 3 by 4 it is in the lowest form suppose it is 6 by 8 we do not take it as 6 by 8 what we do we put it in the lowest form is it not. So, that there no common factor in between. So, m by n as a only common factor as 1 that is all. So, at the most lambda will be 1. So, when lambda is 1 since m by n is in lowest form. So, lambda must be 1 therefore, what does imply therefore, m is equal to p square and n is equal to q square what you mean by this what you mean by this it means m by n is nothing, but the p square q square. So, if I take a square root it comes out to be p by q it means m must be is square of number m must be a perfect square n must be a perfect square clear. So, it counteraction. So, a contraction.

So, this shows there implies m and n are perfect squares. Which lead which is a contraction is it or not. So, therefore, m by n is square root m by n is irrational now suppose we have the any number of the real number rational number perfects positive integer second case is let m be a positive integer which is not perfect we wanted to show that this is it is equal to prove is that square root of m is a irrational point suppose it is not true suppose it is not true. So, let p by q be the number. So, suppose p by q is a fraction in its lowest form lowest form whose square is m we are assuming this is a rational number and p by q and is square is this. Now, since p by q is a rational number. So, there are the 2 point lambda and lambda plus 1 which can in circle this p by q p by q lie between 2 integers if any rational number you choose you can always identify the 2 conjugative integer in between the rational number lies.

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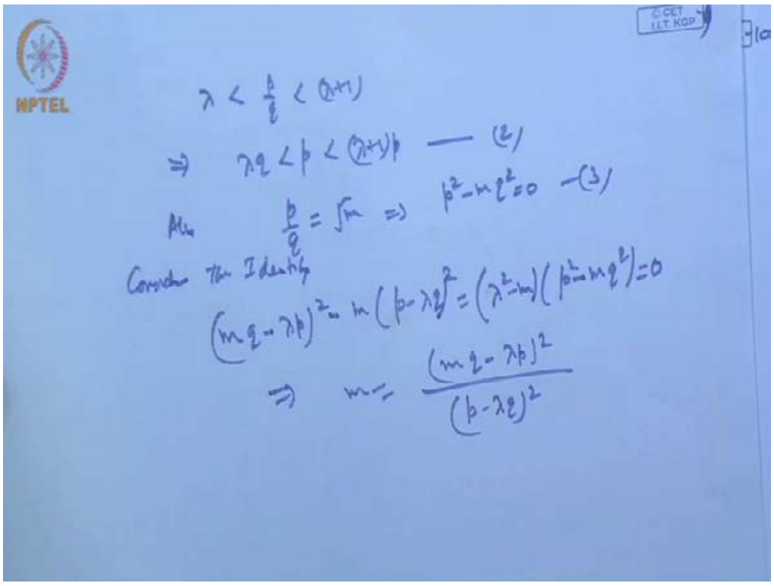

So, is it not? So, we get there exist. So, suppose that. So, suppose lambda and lambda plus 1 are the 2 integers are the 2 integers between which p by q lies p by q lies it means lambda. So, what we get is p by q lie. So, lambda plus 1. So, we get from here is. So, we get lambda is less than p by q less than lambda plus 1 this implies that q into lambda into q is less than p less than lambda plus 1 let it be 2. Also p by q is equal to what under root m. So, this shows that p square minus m q square is 0 write 3 with the help of this if we write this equation identity consider the identity this just you verify m q minus lambda p square minus m p minus lambda q square you will see the value of this will come out to be the lambda square minus m or we can rewrite this expression into this form which is 0 this expression m q minus lambda p whole square minus m times p minus lambda q whole square I just open it and open it arrange in this form then you are getting this expression. But, lambda square is m lambda square is m, because this is p square sorry p square minus m q square is 0. So, this part is 0 means total thing is 0. So, total thing is 0 means this is also 0. So, from here we can write m h m as m q minus lambda p whole square divided by p minus lambda q whole square is it not p minus lambda q whole square m. So, what you are getting is the m this is also number, but what is the denominator yes I think its m yeah the denominator is p minus lambda q while the m earlier was this p by q I think here something lambda p m q minus lambda q .

Is another fraction square. So, denominator of this fraction is less than q why less than q this will be yes. So, when you write here p minus lambda is what p minus this will come from here here you take it this 1 p minus lambda q is positive p minus lambda q is positive it means this whole thing is less than p this is whole thing is less than p because p minus lambda q is less than p similarly, here we can say. So, we are getting a m into another form rational where the denominator changes denominator changes to a lower form, but m is in the lowest form. So, discontradiction. So, this leads the I think this I will continue next time.

Thank you. Is this clear?

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