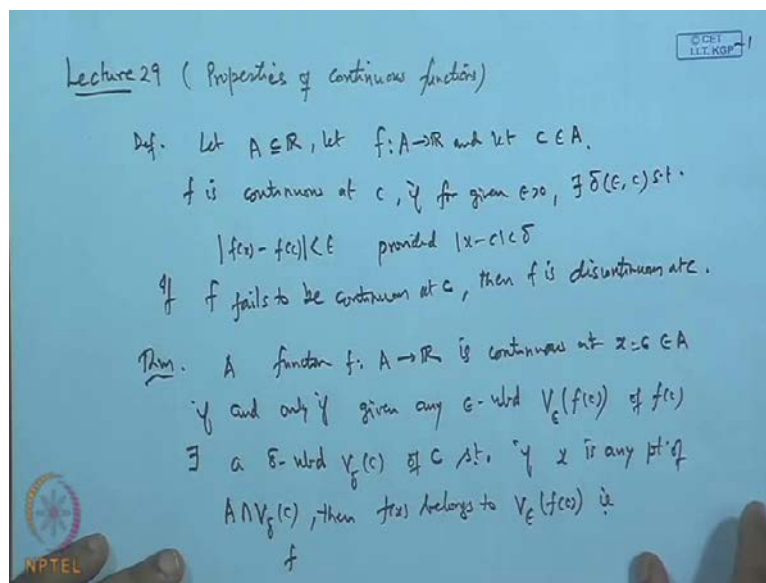


**A Basic Course in Real Analysis**  
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**Indian Institute of Technology, Kharagpur**

**Lecture - 29**  
**Properties of continuous Functions**

In the last lecture, we have defined the continuity of the function using the Cauchy and Hennis definition and both are, later on shown to be equivalent. And in that case, we have considered  $f$  to be continuous over an interval, but we can extend it, we can take an arbitrary set  $A$ , which is sub set of  $\mathbb{R}$ , where the similar way, the continuity of definition, of the continuity over the set  $A$ , which is sub set of  $\mathbb{R}$  or any point  $x, c$  belongs to the subset can be introduced.

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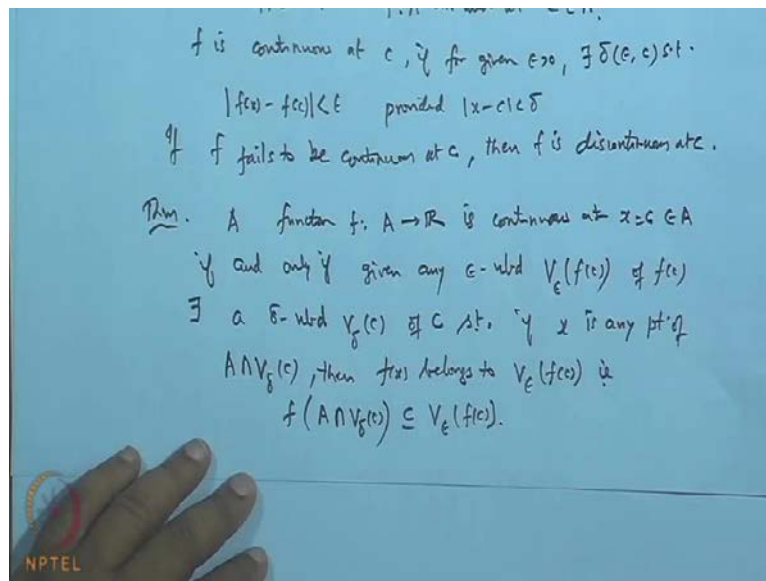


Let me just recall that, set we say let  $A$ , which is a sub set of  $\mathbb{R}$  non-empty sub set or a continuous sub sets of  $\mathbb{R}$ . Let  $f$  is a mapping from  $A$  to  $\mathbb{R}$ , and let  $C$  be a point in  $A$ ; then, we say,  $f$  is continuous at the point  $c$ , if for a given epsilon greater than 0, there exists a delta depends on epsilon and the point  $c$ , such that such that the mod of  $f(x)$  minus  $f(c)$  is less than epsilon, provided mod of  $x$  minus  $c$  is less than delta.

So, if  $x$  is any point in this side, certifying this then, the correspondingly we get this, if  $f$  fails to be,  $f$  fails to be continuous at  $c$ . Then, we say means, if this condition does not hold then,  $f$  fails to be continuous at  $c$  then  $f$  is said to be discontinuous function,  $f$  is

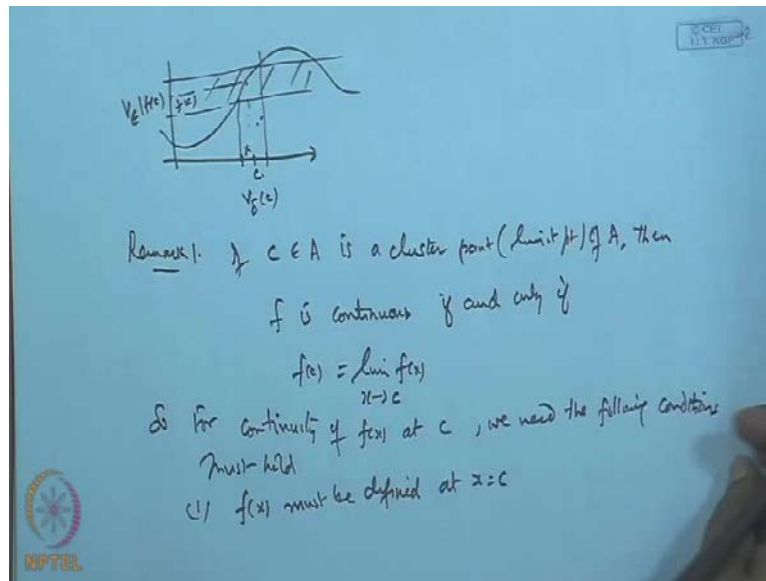
discontinuous at  $c$ . And in the neighborhood form we can say, in the neighborhood form is a function  $A$ ,  $f$  from  $A$  to  $\mathbb{R}$  is continuous at  $x$  equal to  $c$ , belongs to  $A$  if and only if, if and only if given any epsilon neighborhood epsilon neighborhood say,  $v$  epsilon depends on the of  $f(c)$ . Given, any epsilon neighborhood of  $f(c)$ , there exists, there exists a delta neighborhood say,  $v$  delta  $c$ , delta neighborhood of  $(c)$ ; such that, such that if  $x$  is any point any point  $x$  is any point of the  $A$  intersection  $V$  delta  $c$  of the common point of this neighborhood and  $A$ .

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Then means, it belongs to the neighborhood then  $f(x)$  belongs to belongs to  $v$  neighborhood of epsilon neighborhood of  $f(c)$  that is, the image of this image image of this interval, the delta neighborhood of  $(c)$  will be content in the epsilon neighborhood of  $f(c)$  that is what, is said.

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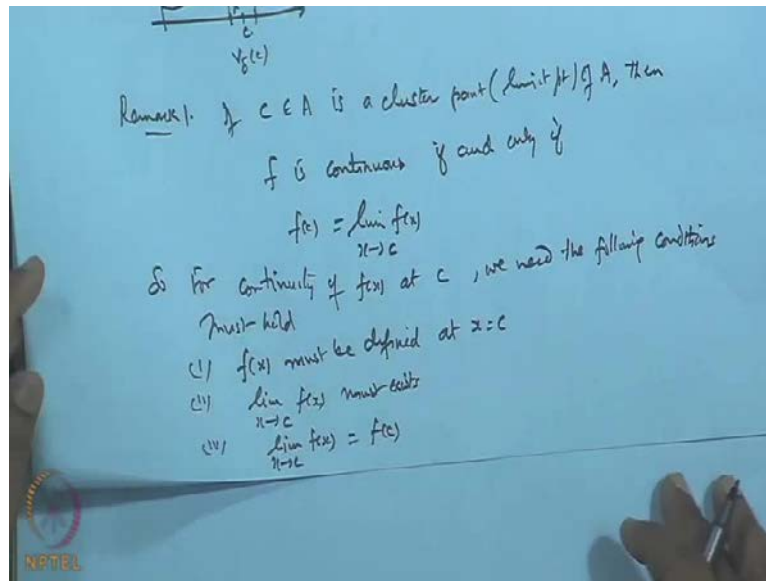


And we have seen that graphically also that, if a here say, this one is there so, this is our function and here, this is our say delta. So, this we get  $\delta$   $c$  here is our epsilon, this is  $\epsilon$   $f(c)$ ,  $c$  is point somewhere here, here is  $f(c)$ , this is  $f(c)$  where, the  $c$  is there. So, we get given any epsilon neighborhood of  $f(c)$ , there exists a delta neighborhood, this is the delta neighborhood  $\delta$   $c$  such that, image of any point in this will fall inside it then, it is a continuous function.

So, here  $c$  is a point in this neighborhood now it would may or may not be the limit point but, if  $c$  is a limit point, if  $c$  belongs to  $A$  is a limit point or is a cluster point. That is, the limit point and cluster point of  $A$  then, the continuity of definition can be written as then,  $f$  is continuous if and only if, the limit of this  $f(x)$ , when  $x$  approaches to  $c$  is  $f(c)$ .

And in fact, this is now, if you look at henni's theorem, henni's equivalent definition says, the result you can exchange in the neighborhood, which goes to  $c$  then, the corresponding  $f(x)$  will go to  $f(c)$ . So, basically it is the same as the hanes theorem so,  $f(c)$  it means, if a function we want to be continuity, what the condition required. So, for the continuous, for the continuity of the function  $f(x)$  at a point  $c$ , we need the following condition we need the following conditions must hold. The first condition is that, the limit of the function, the function  $f(x)$  must be defined at the point  $x$  equal to  $c$  where, the continuity is tested.

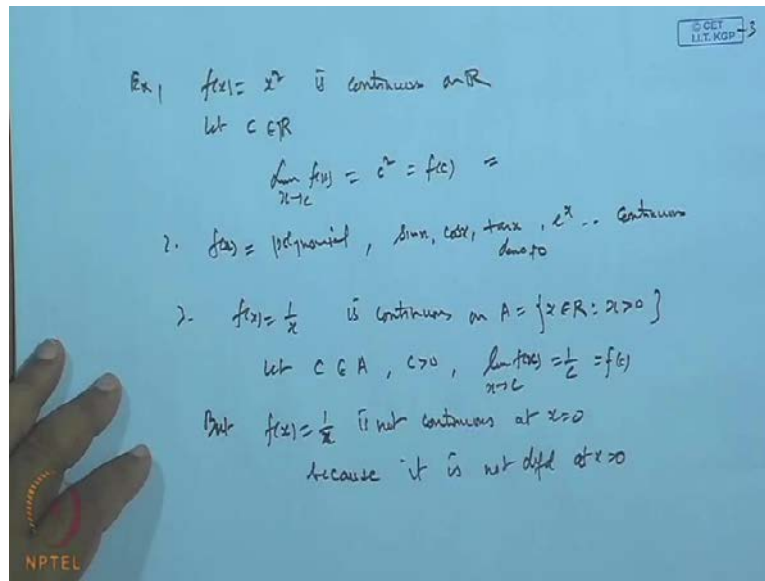
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Second part is the limit of this function  $f(x)$ , when  $x$  approaches to  $c$  must exist must exist and third condition is both should be identical, third is the limit of  $f(x)$ , when  $x$  tends to  $c$  must coincide with the value  $f(x)$ . If these three conditions are satisfied then, we say, the function  $f(x)$  is continuous at a point  $c$ . So, it using this definition, it is very easy to confirm, find out whether the function is continuous or not.

If suppose, in this case, any one of the conditions fails then, we say the function is discontinuous. So, again there are different types of discontinuities that, we will discuss later on that. If suppose, the function is defined, limit exist both may not be equal then, that case the discontinuity will be the removable discontinuity and so on so forth, so that we part, we will take up later.

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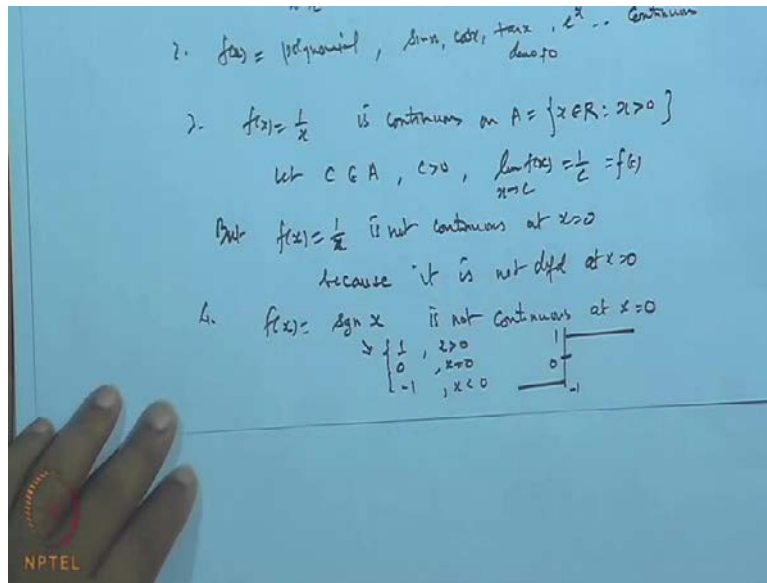


But, using this definition, we can show many examples suppose, I take  $f(x)$  equal to  $x$  square  $f(x)$  is equal to  $x$  square then, we claim is a continuous function on the entire real line  $\mathbb{R}$ . Because, let choose  $c$  as any point in  $\mathbb{R}$  then, what is the limit of  $f(x)$ , when  $x$  tends to  $c$ . As we have seen already this is  $c$  square, this is the same as  $f$  of  $c$  so, it is continuity follows similarly, the function  $f(x)$ , all the polynomial function, polynomials they are also continuous function, sin functions, cosine  $x$ ,  $\tan x$  where, the denominator is not 0, denominator is different from 0.

These are all function  $e$  to the power  $x$ , etcetera are continuous functions and follows from this definition easily. So, we are not touching, we are not going in detail the proof of this but, take the limit of this and so much, we have already discussed the limit of various functions and that, will help in identifying whether the given function is continuous or not.

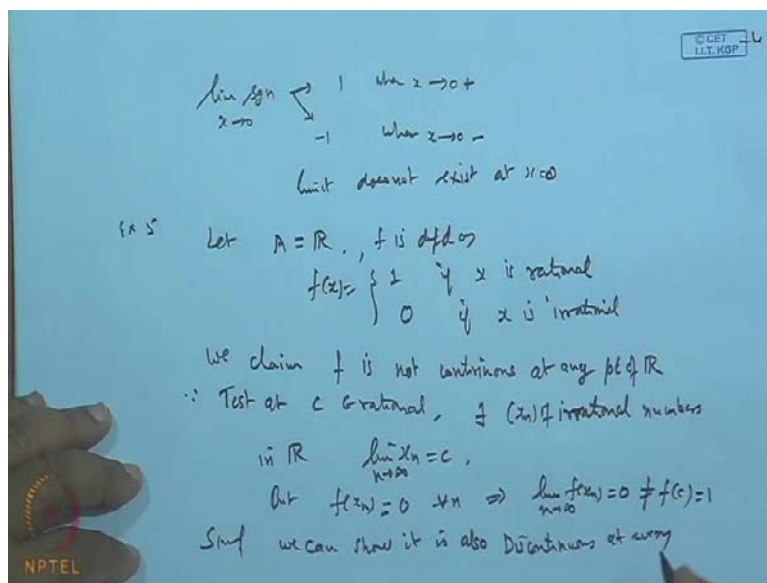
Now, let us take the function  $f(x)$  equal to  $1$  by  $x$  now,  $f(x)$  equal to  $1$  by  $x$ , this function when  $x$  is is continuous over the set, continuous on the set  $A$  of all real line, all real numbers, real is it positive, because if we take  $c$  any point in  $A$  so, this is  $c$  is greater than 0 so, limit of  $f(x)$ , when  $x$  tends to  $c$  is nothing but,  $1$  by  $c$ , which is the value of the function  $f(c)$  so, continuity follows but,  $f$  of  $x$ , which is  $1$  by  $x$  is not continuous is not continuous at  $x$  is equal to 0, because it is not defined at  $x$  equal to 0 so, there is no question of continuity lies here.

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But similarly, if we take the function  $\sin x$  sorry  $f(x)$  equal to signum of  $x$  signum of  $x$ , this function is not continuous at 0 at  $x$  equal to 0. The reason is because, the signum function is defined like this when  $x$  is negative, the signum value is minus 1, 0 it is 0 when  $x$  is positive, it is 1. So, signum function, this is the function, which is defined as 1 if  $x$  is positive 0, when  $x$  is 0 and equal to minus 1 when  $x$  is negative.

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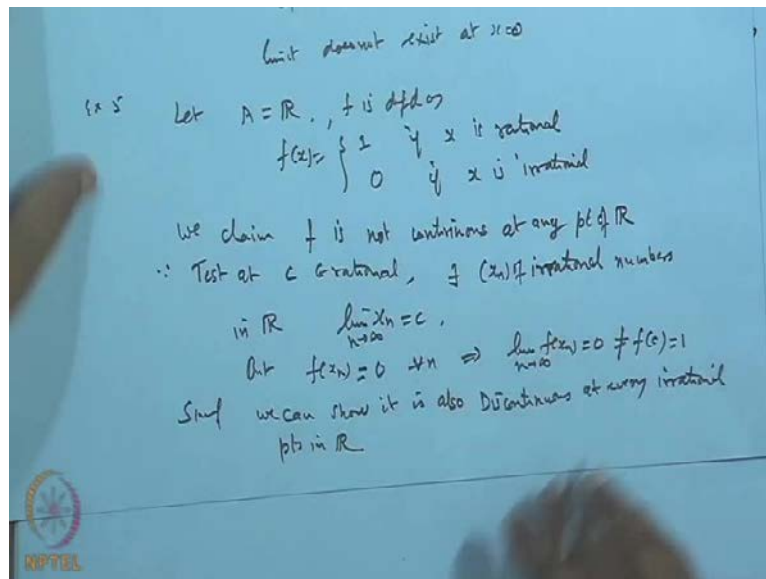


So, when you take the limit of this signum function, as  $x$  tends to 0, the limit will go to 1 and minus 1, when  $x$  is tending to 0 from positive side or when  $x$  is tending to 0 from the

negative side. And therefore, limit does not exist at  $x$  equal to 0 so, it is not a continuous function. Another example, which is also very interesting example let  $A$  be the entire real line and  $f$  define  $f$  is defined as,  $f$  is defined as the value is 1, if  $x$  is rational and 0 if  $x$  is irrational, irrational.

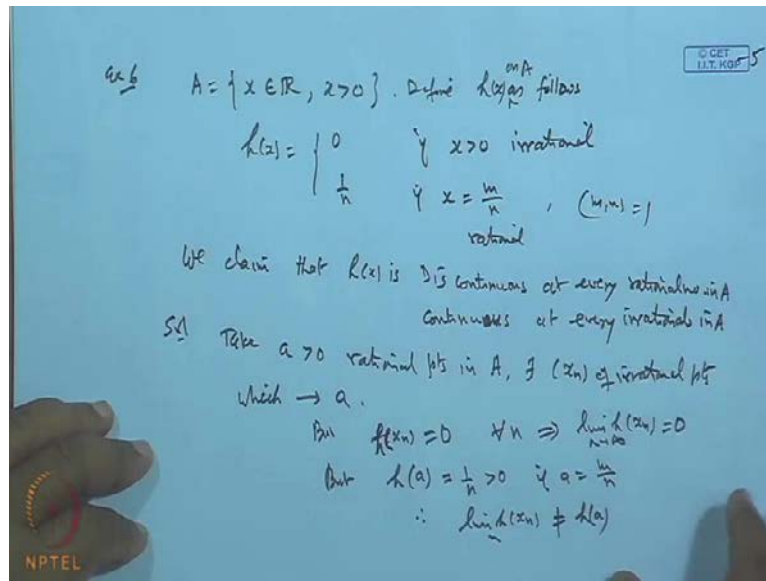
Now, we claim that, this function  $f$  is not continuous at any point of  $\mathbb{R}$  because the reason is simple because suppose, I take a point  $c$ , let us take, let if  $c$ , test it at  $c$ , which is rational number side. Now, every rational number can be approximated by means of the sequence of the irrational so, there exists a sequence  $x_n$  of irrational point, irrational numbers in  $\mathbb{R}$ , whose limit is this, is  $c$ . But what is the behavior of, what is the value of  $x_n$  but,  $x_n$  when  $x_n$  is rational, 0 always for each  $n$  therefore, the limit of this  $f(x_n)$  when  $n$  tends to infinity, will also be 0, which differs from the limit. But limit is  $c$  because,  $c$  is a rational number so, the value of this is 1, according to this so, it is not continuous at any rational point inside.

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Similarly, we can show that, it is also discontinuous at every irrational point every irrational points in the real line  $\mathbb{R}$ , in a similar way.

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A function however, is modified and made it rational at some point and that function is let us define, let  $A$  be the set of all real number, which are greater than 0, positive. Then, define the function, define  $h$ , function  $h$ , as follows, follows define  $h$  on  $A$  as follows so,  $h$  of  $x$  equal to say, 0, if  $x$  is greater than 0 are rational number is a irrational number. If this is irrational number, we are taking  $h$  of  $x$  to be 0 and equal to  $1/n$ , if  $x$  is a rational number  $m$  by  $n$ , which is in the lowest form  $m$  by  $n$ , which is in the lowest form natural number  $m$  and  $n$ .

And that lowest form is that means  $m$   $n$  is 1, the divisor, the greatest common divisor of this is 1 here. So, let us define the  $x$ , if it is rational number is of the form  $m$  by  $n$ , we will write  $1/n$  and if  $x$  is irrational then, we will write to be 0. Now, we claim that  $h$ , this function  $h$  is basically, is discontinuous at every at every rational number in  $A$ , in  $A$ . But it is continuous, continuous at every irrational number in  $A$  so, let us see the proof for it.

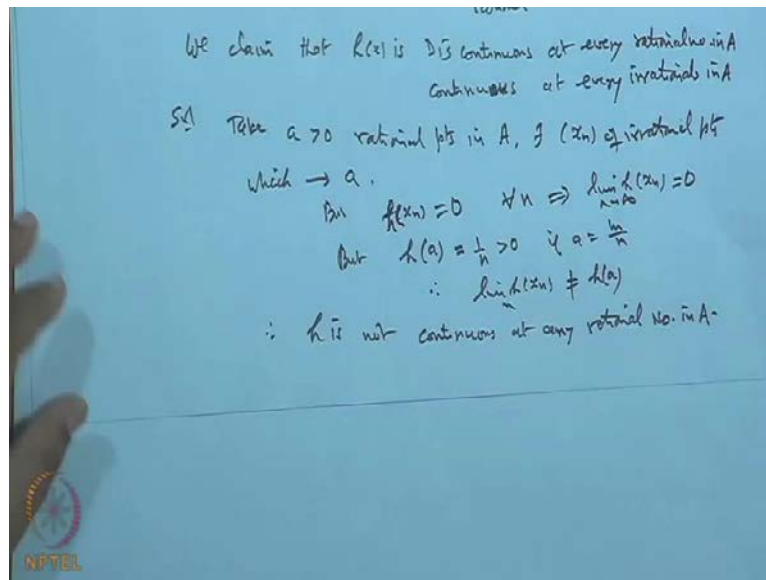
The solution is suppose, i take first irrational number let us take,  $a$  be  $a$ , take  $a$  which is greater than 1 0, as a rational points in  $a$ . So, we can find a sequence so, there is a sequence  $x_n$  of irrational numbers, irrational points, which, which, which converges to  $a$ . But, what is the value of  $f(x_n)$   $h(x_n)$   $h(x_n)$  is defined to be 0 for every rational points.

So, along the 0,  $x$  this sequence,  $h$  of  $x_n$  will be 0 but, what is the  $h$  of  $a$ ,  $h$  of  $a$  is given to be what, if it is rational number then, it will be some say  $1/n$  or whatever may be,



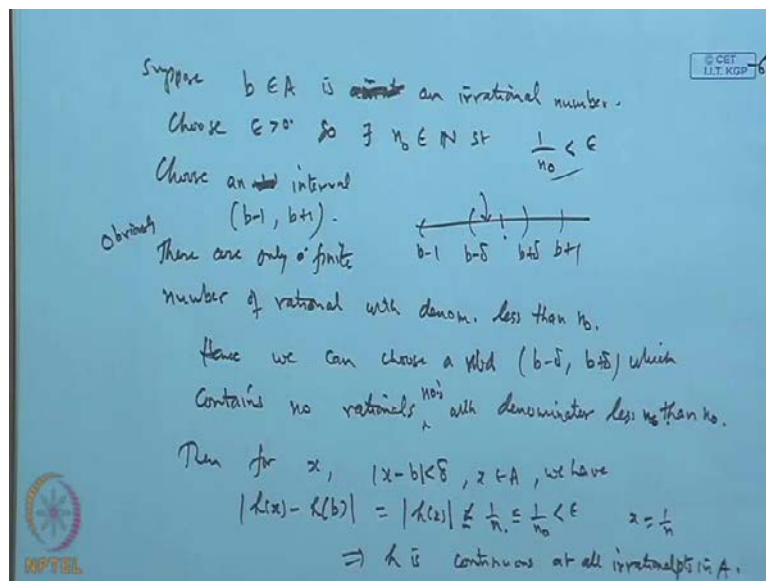
which is positive, which is positive. So, here the limit of  $h(x_n)$  as  $n$  tends to infinity, will come out to be 0 because, for all points, the values are 0 so, limit will be 0 and here the value is coming to be non zero. Therefore, this limit of  $h(x_n)$  over  $n$  is differs from the value of the function at a point  $a$  so, when  $a$  is rational it is not a continuous function.

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So,  $h$  is not continuous at any rational numbers in  $A$ .

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Let us see the irrational point so, suppose,  $b$  is a irrational point,  $b$  belongs to  $A$  is a irrational point, is an irrational number, is a irrational number. So, when  $b$  is irrational let

choose,  $\epsilon$  greater than 0 so, for this given  $\epsilon$  we can identify a integer  $n$ , a natural number. So, there exists a natural number  $n$  such that,  $1/n$  can be made less than  $\epsilon$ , which is possible by our arithmetician properties so, this one.

Now, let us consider an interval, the interval if I take the interval  $b - 1, b + 1$  now, in this interval, we can get the rational point, whose denominator is greater than  $n$  as well as less than  $n$ . So, what we claim is that, they natural denominators are less than  $n$ , there are only we choose a neighborhood or choose any interval, interval  $b - 1, b + 1$ .

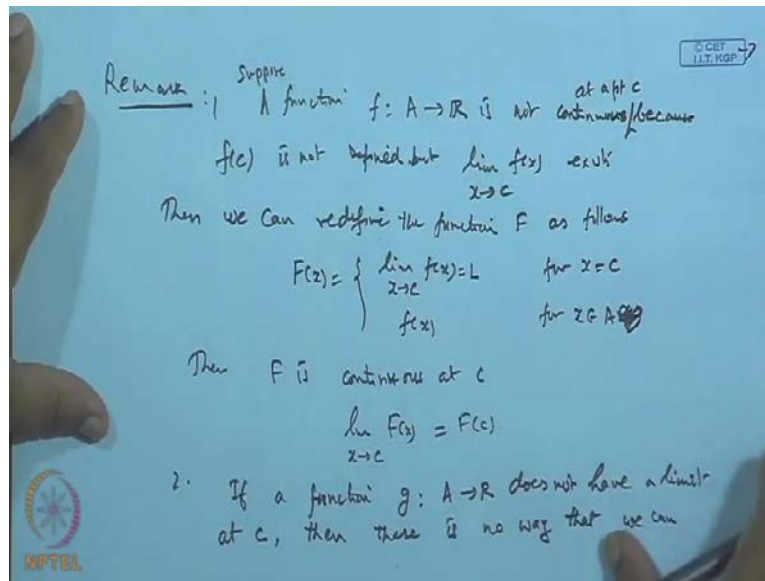
And choose a interval then, there are; obviously, there are only a finite only a finite numbers of rational finite only finite number of rationals finite number of rationals with denominator less than  $n$ . Because, they obviously, when this denominator less than  $n$ , it will be greater than  $n$ , this will follow, this will follow and  $b - 1$  here,  $b$  will get somewhere number.

So, it will approach to this number so,  $\delta$  is a any arbitrary number we can get hence, we can choose choose a neighborhood a neighborhood  $b - \delta, b + \delta$ , exquisite. So, we can get the neighborhood  $b - \delta, b + \delta$ , which contains which contains which contains no rational number with no rational numbers with numbers with denominator less than  $n$ .

Means, this neighborhood we can find in  $b - \delta, b + \delta$  where, all these point  $x$ , whatever the number, that denominators will be having, the denominator greater than  $n$ . So, it will satisfy this condition that is, what we wanted to have that neighborhood so, once you have this neighborhood then, then clearly then if  $x$  then, for any  $x$ , which satisfy this condition,  $|x - b| < \delta$  where, the  $x$  belongs to  $A$ , we have  $|h(x) - h(b)| < \epsilon$ .

But,  $h(b) = 0$  because,  $b$  is irrational and the function is defined irrational to be 0 so, this is  $h(x)$ ,  $h(x)$  means,  $x$  is of the form say,  $1/n$  so, this is  $1/n$  and this will be less than  $1/n$ ,  $1/n$  say,  $1/n$  or  $1/n$ ,  $h(x) = 1/n$  so, which is less than equal to  $1/n$  so, this will remain less than  $\epsilon$  help therefore, following  $\epsilon$  neighborhood of this, we can find that is  $\delta$  such that, for all  $x$  lying between these, the function  $h(x)$  will be less than  $\epsilon$ . So, this shows that,  $h$  is continuous at irrational point or irrational points in  $A$  and that is proved the result.

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Now, one more thing, which we have remark, we have seen the continuity of the function in terms of the limit, if suppose, the limit does not exist but, the function is well defined everywhere except the limiting point. Then, the two possibilities are there, either the limit will exist, function is defined everywhere except the point where, the continuity is required. So, the possibility is, when you take the limit of the function  $x$ , when  $x$  tends to that point, limit may or may not exist.

If it exists then, we can redefine the function in such a way so that, the new function so obtained becomes continuous so, that is the remark. Sometimes, suppose, suppose if function  $A$  suppose, a function  $f$  from  $A$  to  $\mathbb{R}$  is not continuous because, the value of the function at that point is not continuous at a point  $c$ . Because, the value of the function  $f(c)$  is not defined is not defined then, the two possibilities are there.

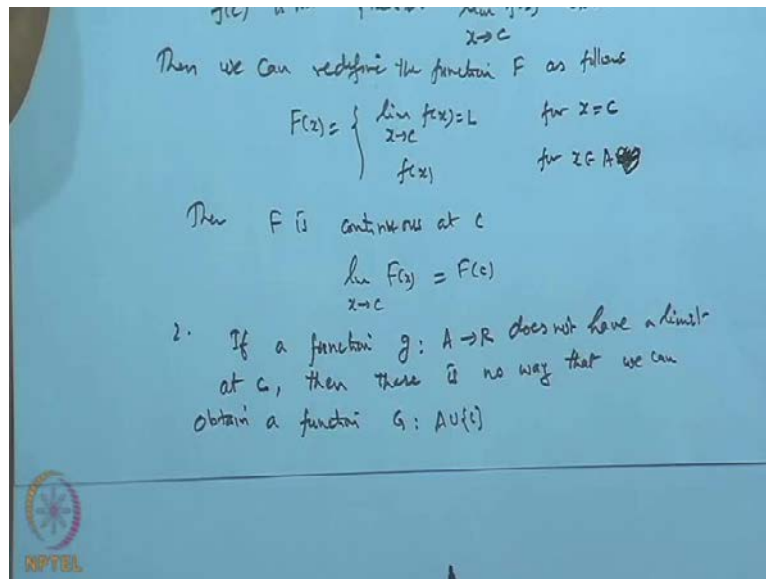
Then, but, the limit of this function limit of say,  $f(x)$  limit of  $f(x)$  when  $x$  tends to that  $c$  exist then, we can we can redefine the function, function capital  $f$  as follows, capital  $f(x)$  is equal to the limit of  $f(x)$  when  $x$  tends to  $c$ ,  $c$  for  $x$  equal to  $c$ . We are defining this and  $f(x)$ , for  $x$  belongs to  $A$ , which is not same, different from  $c$ , minus  $c$  minus  $c$  that point the function is defined in practice so, this.

So, that is equal to suppose, I now, we claim this  $f$  is continuous is continuous is continuous at  $c$  because,  $f$  is already given to be continuous function except at is continuous except at at the point  $c$ . So, when you take any point  $x$  sorry function  $f(x)$  is

not defined only at A and rest of the point is defined. So, when you take the limiting value of the function  $f(x)$ , this is comes out to the value of the function  $f$  at the point  $c$ . So, limit of this  $f(x)$ , when  $x$  tends to  $c$  is equal to  $f$  of  $c$  and that is, equal to limit  $f(x)$  exist.

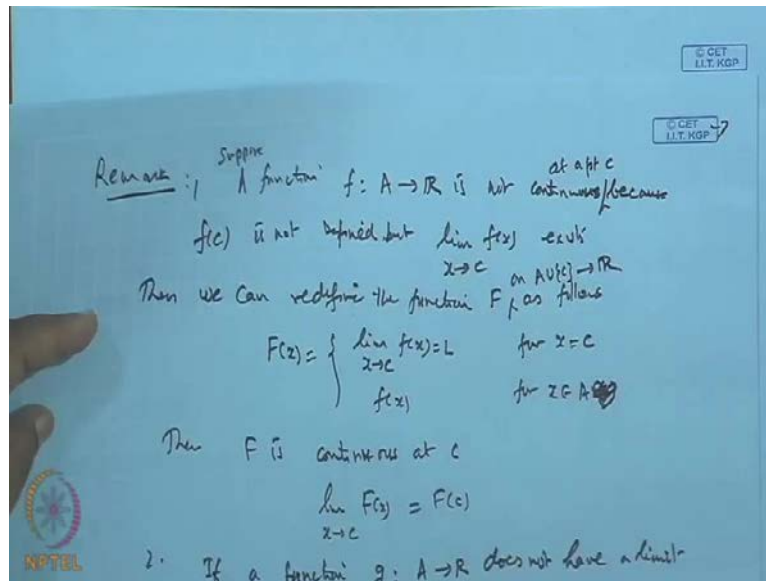
So, this function is continuous it means, the original function  $f$ , from the original function, we can remove the continuity, we can remove the point of discontinuity and made it to be continuous function. And this way, we call it that point to be a removable discontinuous point, this but if, if the limit of the function, if a function does not have a limit  $c$ , if a function if a function  $g$  from  $A$  to  $\mathbb{R}$  does not does not have a limit does not have a limit at  $c$  does not have the limit at  $c$ .

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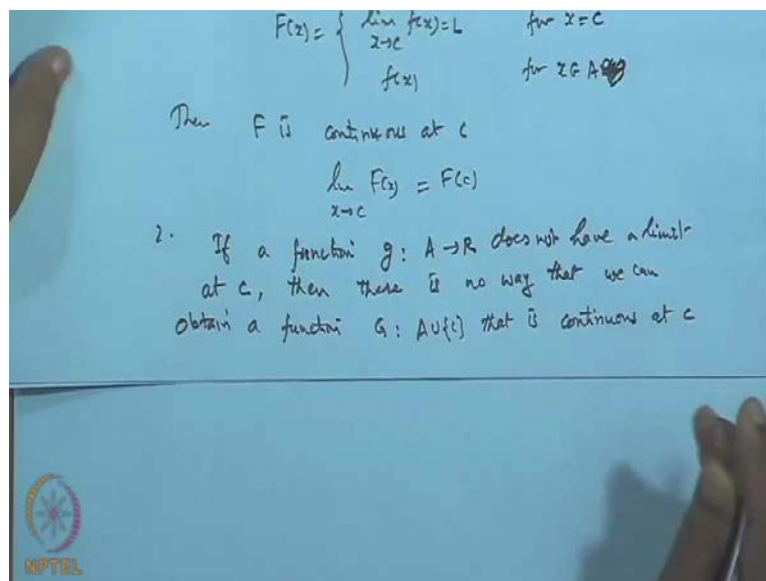
Then, there is no way there is no way that, we can obtain we can obtain there is no way that we can obtain a function  $G$   $A$  union  $c$ .

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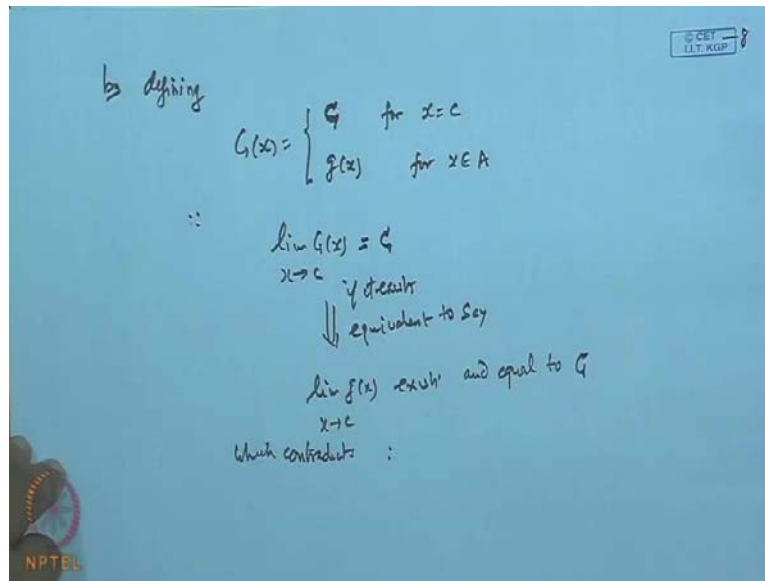
Here, we will write on  $A \cup \{c\}$ , this is one left to  $\mathbb{R}$  to  $\mathbb{R}$ , please check it to  $\mathbb{R}$  that is.

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So,  $A \cup \{c\}$  then we can that is continuous that is continuous at  $c$ .

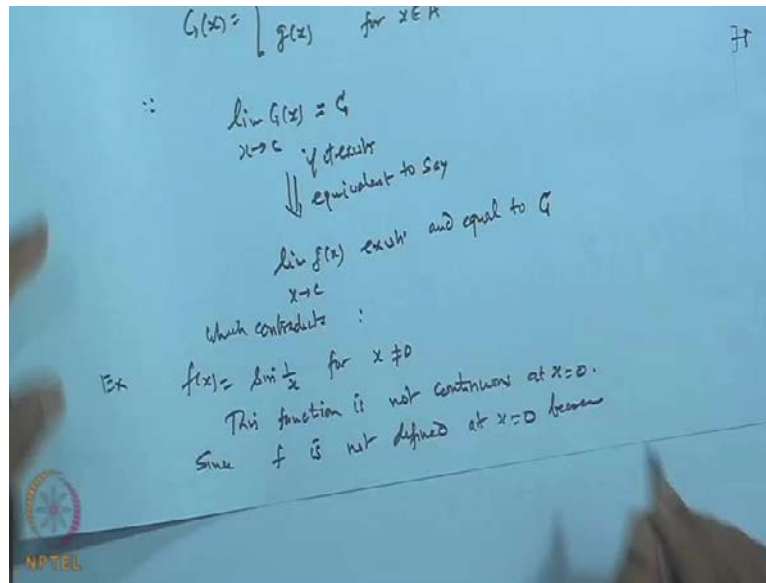
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By defining by defining by defining,  $G(x)$  as equal to  $C$ , for  $x$  is equal to  $c$ , for  $x$  is equal to say, capital  $C$  i am just trying capital  $C$ , for  $x$  equal to  $c$  and  $g(x)$ , for  $x$  belongs to  $A$   $x$  belongs to  $A$ . The reason is because, if we assume this way then, what happens because, the reason is this because, what is the limit of  $g(x)$ , when  $x$  tends to  $c$ . This is giving to be, if it exists if it exist then, it is giving to be the same but, this limit when  $x$  tends to  $c$ , it is equivalent to, which is equivalent to say, the limit of this  $g(x)$ , when  $x$  tends to  $c$  exist.

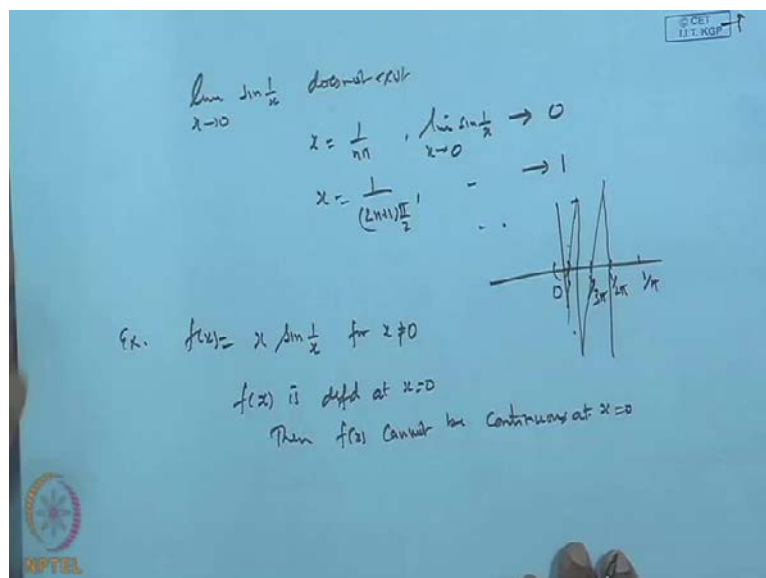
Because, if this limit exists it means, when  $x$  is different from  $c$ , the  $g(x)$  is defined like this small  $g$ . So, when you are choosing the limit of  $g(x)$ , when  $x$  suppose, you have you have for all  $x$ , we are differs from  $c$ , the  $g$  will be replaced by small  $g$ . So, limit of this  $i$  and if this limit adjust, this will adjust, which will contradicts, must also is and equal to  $c$  and equal to capital  $C$ . But this is not given, which is a contradiction which is which contradicts therefore, we cannot define redefine the function so that, it becomes continuous that is what, is used.

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Now, let us take an example in support of this suppose, I take a function  $g f(x)$  equals to  $\sin 1$  by  $x$ , for  $x$  different from  $0$  then, it does not have a limit at  $x = 0$ . Then this function is not continuous at  $x$  is equal to  $0$ , because we cannot define the value of the function at the point  $0$ . Because, at the point  $0$ , the function  $f$  is not defined since  $f$  is not defined at  $x$  is equals to  $0$   $f$  is not defined at  $x$  is equals to  $0$ , because it does not have a limit because, does not have a limit because it does not have a limit is not defined at  $0$ .

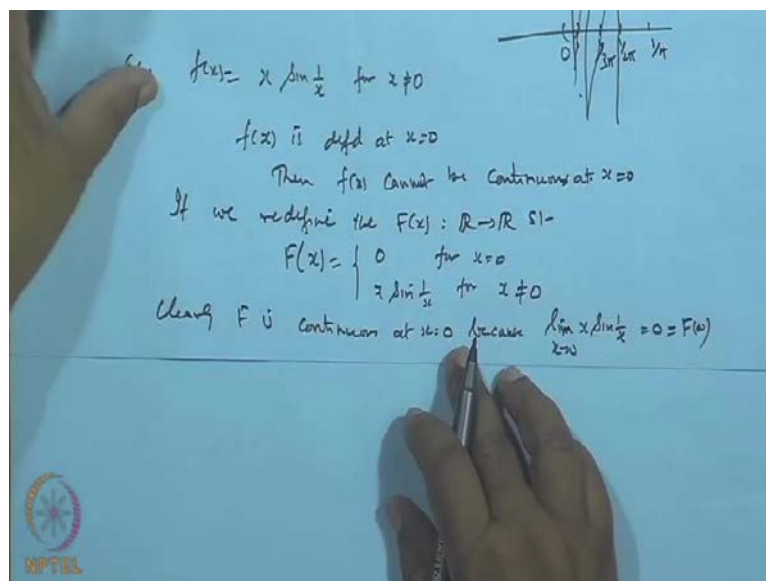
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Because the limit of this  $\sin 1/x$  as  $x$  tends to 0 does not exist and this we have seen already that, if we take  $x$  is equal to  $1/\pi$ , the limit of the function, limit of  $\sin 1/x$  as  $x$  tends to 0 will go to 0. When you take  $x$  is equal to all multiple of  $\pi/2$ , the limit of this function will go to 1 and like so on or there is a large of fluctuation. In fact, the if we go for the limit then, this is  $1/\pi$ ,  $1/2\pi$ ,  $1/3\pi$  and like this.

So, there is a large of fluctuations is there like this, something like goes so, it has a positive negative value in the neighborhood of zero. So, the fluctuation exceed by epsilon therefore, it cannot be continuous also and again limit also, we can say, does not existing. The second example, if we look the function  $f(x)$ , which is  $x \sin 1/x$  suppose, I take for  $x$  is not equal to 0. The function in that case,  $f(x)$  is not defined at  $x$  is equal to 0 at  $x$  equal to 0 then, the function cannot be continuous at this point then,  $f(x)$  cannot be continuous at  $x$  equal to 0 but, however, if I define the redefine the function as follows.

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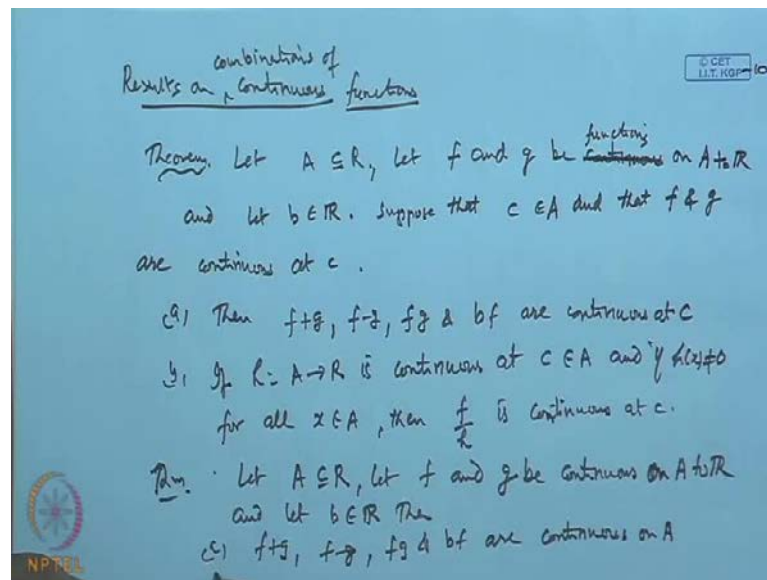


So, if it defines, we if we redefine the function capital  $F(x)$  from all to all means, all to all such that,  $F(x)$ , the value is 0 if  $x$  is 0 and equal to  $x \sin 1/x$ , for  $x$  different from 0. If I redefine this function then, what happens then, clearly  $F$  is continuous at  $x$  equal to 0 why because the limit of this  $x \sin 1/x$  as  $x$  tends to 0 is 0. Because, it is dominated by the  $x$  so, limit will come out to be 0, which is the same as the  $F$  of 0.

So, this gives continuity follows therefore, this function for this function, 0 is the removable continuity.



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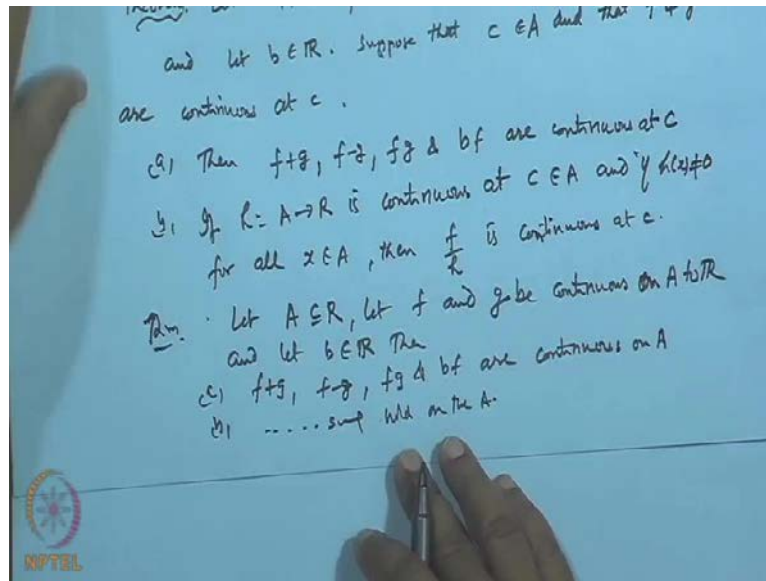


Now, there are certain properties of the continuous functions, which we will just there, some results or results on continuous functions, on combinations of continuous functions combinations combinations of continuous functions. So, i was just listed the results and the result is, for any arbitrary point it is true so, i will just give the general results. Let  $A$ , which is a subset of  $\mathbb{R}$  and let  $f$  and  $g$  be continuous on  $A$  on  $A$  or at a point let us take this  $f$  and  $g$  be continuous on  $A$ ,  $A \rightarrow \mathbb{R}$  be a continuous function, functions  $A \rightarrow \mathbb{R}$  no be a functions be a functions on  $A \rightarrow \mathbb{R}$ .

And let and let  $b$  belongs to  $\mathbb{R}$  suppose that suppose that,  $c$  is an element in  $A$  and and that  $f$  and  $g$  are continuous are continuous at  $c$  then, the following result shows. Then,  $f$  plus  $g$ ,  $f$  minus  $g$ ,  $f g$ ,  $b f$  and they are all continuous at  $c$  and if, if  $h$  is a mapping from  $A$  to  $\mathbb{R}$  is continuous is continuous at  $c$  belongs to  $A$ . And if  $h$  of  $x$  is not equal to  $0$ , for all  $x$  belongs to  $A$  then, the coefficient  $f$  by  $h$  is continuous is continuous at  $c$ .

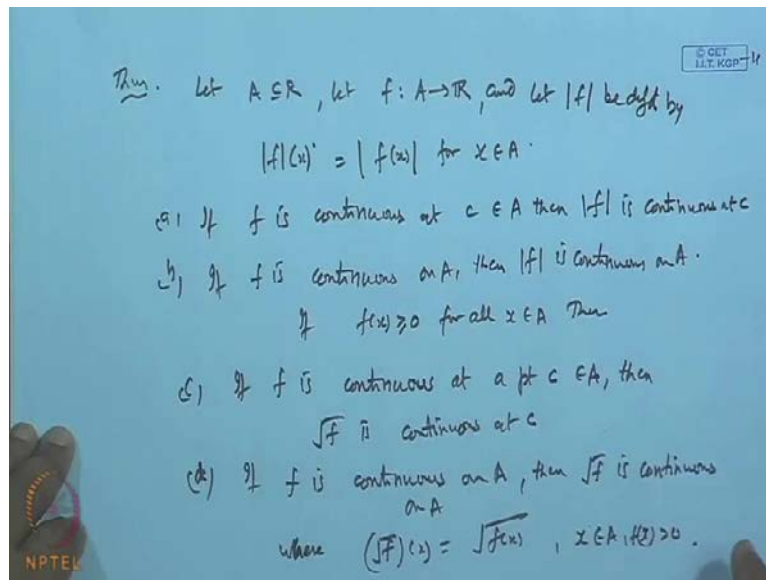
So, the results follows now, this can be extended to a one, as corollary or theorem, the same results holds if  $f$  and  $g$ , let  $A$  which is subset of  $\mathbb{R}$  and let  $f$  and  $g$  be continuous be continuous on  $A \rightarrow \mathbb{R}$  and let  $b$  belongs to  $\mathbb{R}$  then,  $f$  plus  $g$ ,  $f$  minus  $g$   $f$  minus  $g$ , product  $f g$  and  $b f$  are continuous on  $A$ .

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And similarly, b holds similarly, similarly b holds hold as above on the set A so I am not writing that is, what true. This one, the proof is just based on the earlier cases, we have seen the function f is limit function is there, limit of f (x) plus g (x) is limit of plus f (x) g (x) so, similarly, based on this, we can write down the proof for it.

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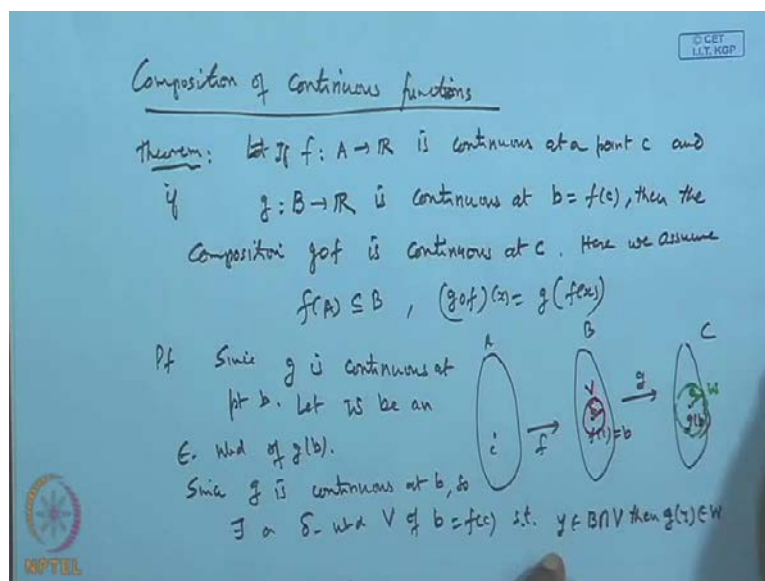


Now, another result are again without proof, let A which is a subset of R and let f is a mapping from A to R and let mod f, mod f means, absolute value of this f (x). We define we define by mod of f (x) means equal to the modulus of f (x), for all x belongs to A.

Then, if  $f$  is continuous at a point  $c$  at a point  $c$  belongs to  $A$  then,  $\text{mod } f$  is also continuous then  $\text{mod } f$  is also continuous at  $c$ . Then second part is, if  $f$  is continuous  $f$  is continuous on  $A$  then,  $\text{mod } f$  is throughout continuous on  $A$  is continuous on  $A$ .

Similarly, if our domain, if  $A$  is continuous, if  $f$  is a if  $f(x)$  is greater than equal to 0, for all  $x$  belongs to  $A$  for all then, then the following result show then,  $f$  is continuous continuous at a point  $c$  belongs to  $A$  at a point  $c$  belongs to  $A$ . Then,  $f$  then under root of  $f$  is also continuous at  $c$  and if, if  $f$  is continuous on  $A$  on  $A$  then, under root of  $f$  is also continuous on  $A$  where, where under root  $f(x)$  is defined as the under root of  $f(x)$ , for  $x$  belongs to  $A$   $x$  belongs to  $A$  and  $f$  of  $x$  is greater than 0, that is what. So, similarly, result holds and we are not proving giving that because, it is very easy to show the proof of all these results so, we are just skipping the proof for it.

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Now, next result is, which is in testing, composition of the continuous function composition of continuous functions suppose, we have the function  $f$  and  $g$  both are say, one is continuous at some point and other is also continuous say, at point where, the range set is containing this then, we get the compound. So, let us so let  $f$  is a mapping from  $A$  to  $\mathbb{R}$ ,  $b$  is a continuous at a point  $c$ , let this is continuous at a point  $c$ .

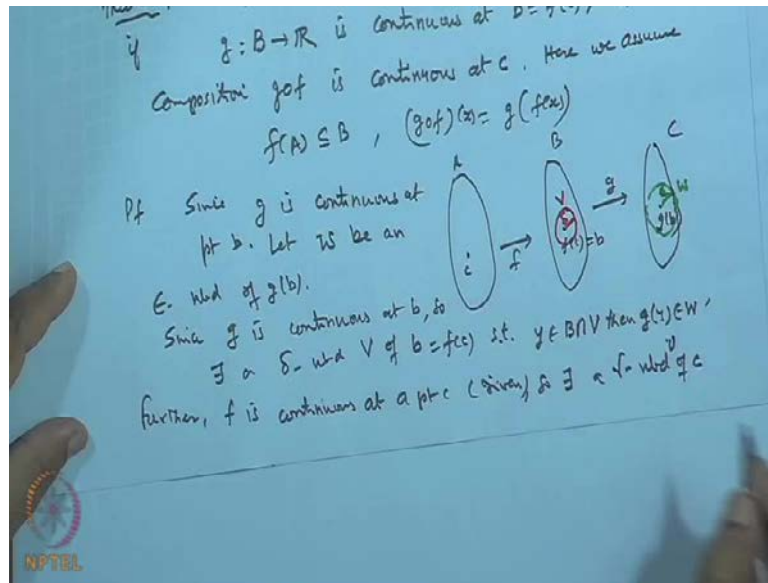
If  $f$  which is continuous at a point  $c$  and if and if  $g$  which is a mapping from  $B$  to  $\mathbb{R}$  is continuous is continuous at a point  $b$ , which is the image of  $c$  under  $f$ . Then, the composition function, composition  $g$ , composition  $f$  is continuous is continuous at a  $c$ ,

function at a  $c$ . Here, we assume here we assume that, image of  $A$  under  $f$  is containing  $B$  so, the function is well defined dominating and  $g \circ f(x)$  means  $g$  of  $f(x)$ , this is the meaning.

So, let us see the proof of this, this is the result, what he says is suppose, i take this one suppose, this is the set  $A$ , this one is the set say  $B$ , this one is the set say  $C$ .  $f$  is a mapping from  $A$  to  $B$ ,  $g$  is a mapping from  $B$  to  $C$  and this now, what is given is,  $f$  is continuous at a point  $c$ . So, here, we are getting  $f$  of  $c$  that is, equal to  $b$ ,  $g$  is continuous at a point  $b$  so, here you are getting  $g$  of  $b$ .

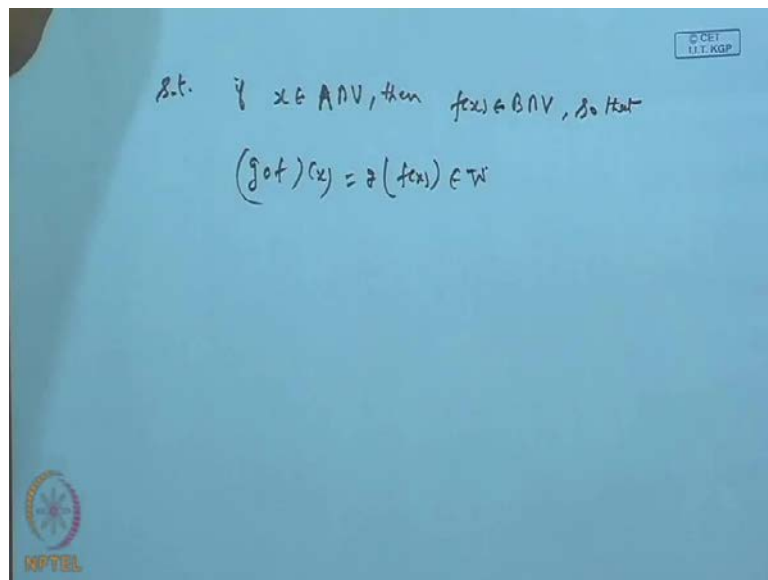
So, since  $g$  is continuous at the point  $b$ ,  $g$  is continuous at a point  $b$  so, what we get so, there is an let so, let us take a epsilon neighborhood of  $g(b)$ . So, let  $W$  be an epsilon neighborhood epsilon neighborhood of  $g(b)$ , let us consider this epsilon neighborhood of  $g(b)$  say, here this is, in this inside this is epsilon neighborhood of so, this is  $W$ . Take the epsilon neighborhood of  $W$  as an epsilon neighborhood of  $g(b)$  now,  $g$  is continuous. Since  $g$  is continuous at  $B$  so, there is a, since  $g$  is continuous at  $B$  so, there exist are delta neighborhood delta neighborhood  $V$  of a small  $b$ , which is the value of the function at  $c$ . Such that, whenever the  $y$  belongs to the  $B$  intersection of  $V$  then,  $g$  be image of this  $y$  under  $V$ , belongs to  $W$ , this is by definition. So, what is saying is, if I picked up a for corresponding to this epsilon neighborhood, we can find delta neighborhood of this that is,  $V$ . Such that, whenever the any point lies in the intersection of  $V$  in this, the image will fall within this line so, by definition of the continuity of  $g$  at a point  $B$ .

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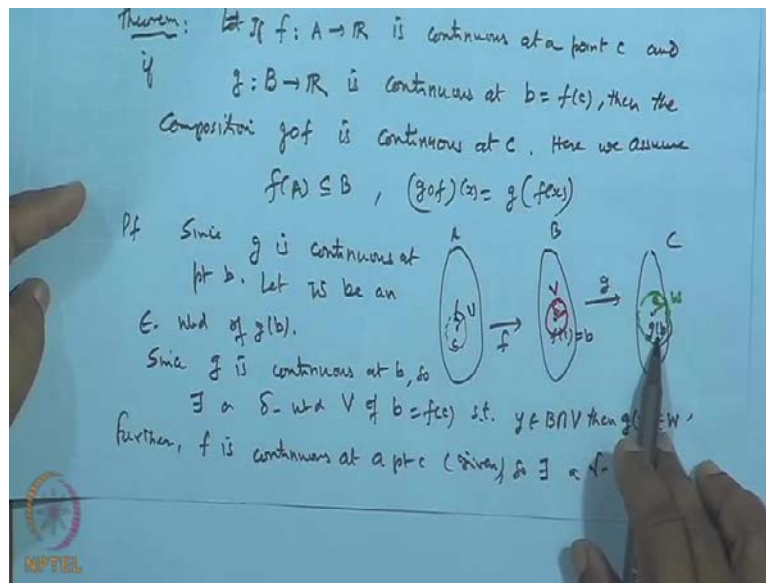
Now further, further g is giving f is giving to be continuous further, f is continuous continuous at a point c at a point c this is giving. So, there exist say, gamma neighborhood of c gamma neighborhood say, U of c.

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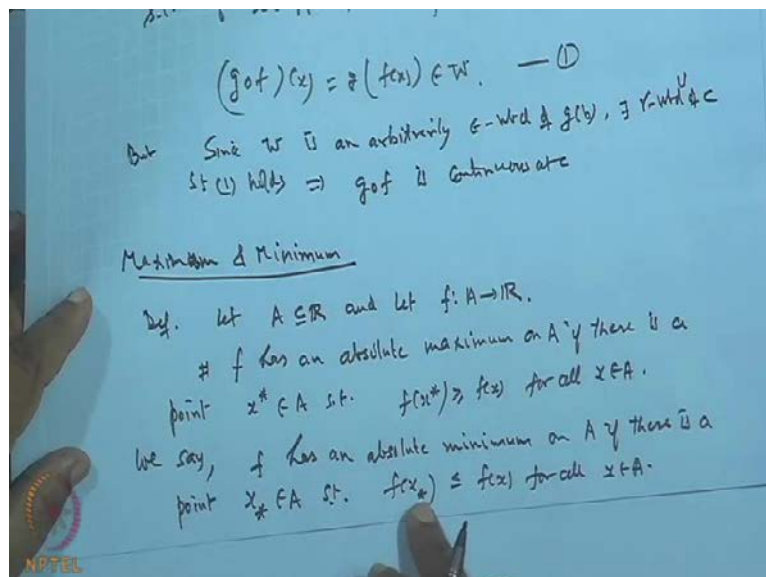
Such that such that, if x belongs to A intersection U A intersection U then, the image will go f (x) will go to B intersection V B intersection V. So that, so, what we get, the composition mapping g of f (x), this will be equal to g of f x and that, belongs to W.

(Refer Slide Time: 49:23)



So, what he says is that, if since  $f$  is continuous at  $c$  so, for given this delta neighborhood of this, we can find a gamma neighborhood of  $c$ . Such that, the any point, which lies inside this gamma neighborhood, gamma intersection  $A$  then, image will go to  $V$  and every point, which is in  $V$ , the image will go to  $W$ .

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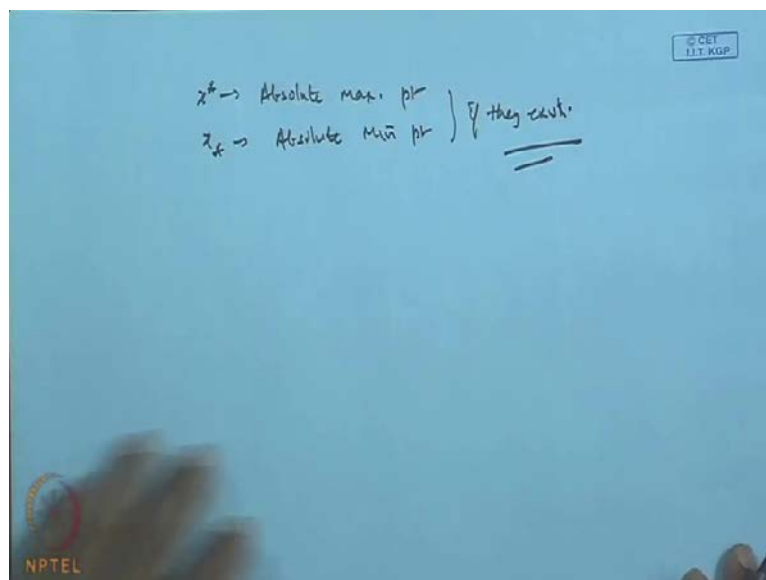
So,  $g$  composition  $f(x)$  will go to  $w$  that is, what is said but, since  $w$  is but since since  $w$  is an arbitrary epsilon neighborhood,  $W$  is an arbitrary is small epsilon neighborhood of  $g(b)$ . So, this shows that, this implies that,  $g$  composition  $f$  is continuous at  $c$  because, for

arbitrary is small epsilon here, there exist neighborhood say, gamma neighborhood of say, U of point c. Such that, image of any point one holds such that, one holds so, this implies, this is continuous function for this so, we get this.

Now, we wanted to do some more maximum minimum also for that, let me just define the maximum and minimum theorem. First the definition, maxima maximum and minimum, we have already stated, the local maxima local minima. Here, just it will be required definition, let A be a subset of R and let f f which is a mapping from A to R. We say, f has an absolute value, has an absolute maxima maximum on the set A, if there is a point is a point say, x star in A.

Such that, the value of this x star under f, will always be greater than the value of f (x), for all x belongs to A then, we say the function f has a absolute maxima on the set A. Then, we say we say f has an absolute absolute minima minimum on A, if there if there is a point there is a point say, x star. If there is a point x lowering star i am saying, belongs to A such that, the f of lower image of this lower x star is less than equal to f (x), for all x belongs to A then, we say, x star is an absolutely the lower absolute minimum for it.

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The x star is called the absolute maximum point, x lowering star is called the absolute minimum point if there exist, x star is called the absolute maximum point and x lowerly star is called the absolute minimum point if they exist and let us see. So, we will continue next time this one. Thank you very much.