A Basic Course in Real Analysis
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Lecture - 28
Continuity of Functions

We will discuss today the continuity of functions. We have already seen how to define the functions. It is basically a mapping a rule from a set A to set B such that each element of A, we can have one unique element in B. Then function is well-defined. If there are more than one values of A, then we say it is not well-defined function. However, when corresponding to each value x, we have one value, then it is a single-valued function. And for each x, if there is more than one value imagine, then we say it is a multi-valued function. Say for example, if f x is the root x; there for each real x – positive real x, we have two values. So, it is a multi-valued function. So, we will include all sorts of the functions whether it is a single-valued function or multi-valued function.

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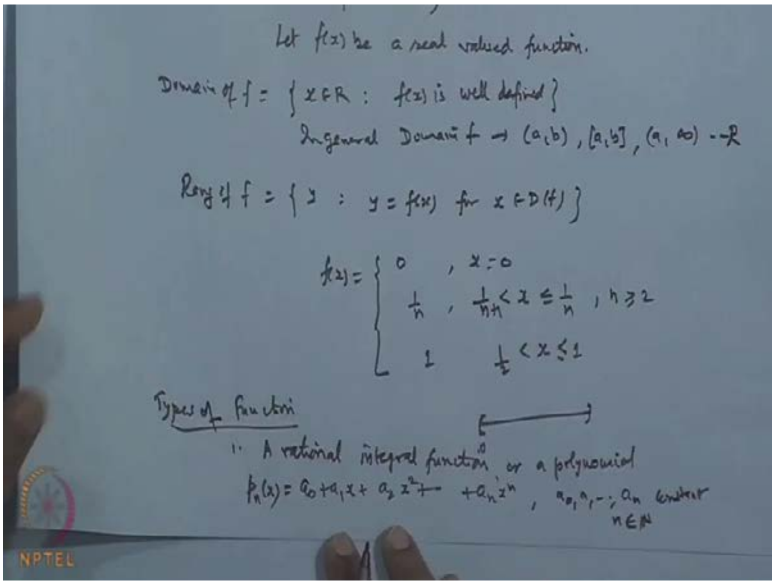

And, when we say f x is a real valued function, it means that values of the f x lies in the real. Let f x be a real valued function. That by the domain of f, we mean the set of those real numbers x belongs to R such that the f x is well-defined. Well-defined at this means f x should not be infinity or minus infinity or the functional at x it must be a finite value well-defined. Normally the domain of f, in general the domain of f normally, we consider either an open interval or maybe a close interval or maybe a semi-close interval say like this or in the entire real line R. So, it depends on the function, where it is defined. And the set of those values of a is the range of f; range of f means those y's such that there exists some x such that f x equal to y for x belongs to the domain of f. So, all the images set is the range of the f. So, this much we have.

Now, the domain of f either function when we say the function is defined over the domain a, then it is not necessary that function; the domain will be continuously defined like a, b. It may be in the breakup form. Maybe the same function may not be defined over the entire domain; it may be a several functions can be a take up over the domain of the f. For example, if we say f x is say 0 when x is 0; say f x is equal to line between say 1 by n plus 1 less than x less than equal to 1 by n; where, n is an integer greater than equal to 2 say.

Then the value of this is defined as the functional; f x is defined by 1 by n. So, in the interval, when it is lying between 1 into 1 half less than x less than equal to 1, the function is defined. So, this is also a way of defining the function; maybe take up of this type of also. The domain of definition is from 0 to 1; but is not the same function. By means of the same function, we are not defining over the interval. What we have here is over the several parts of this interval 0, 1. This is 0, 1 interval. We have a different form of the f at different points. We have all in different subintervals like this. So, this will be (( )).

Now, there are different types of the functions which we come across like the rational types of real value function or types of functions, which we will come across the first (( )) a rational integral function; we can also say or polynomial, which is of the form like a naught plus a 1 x plus a 2 x square plus a n x to the power n; where, the coefficients a naught, a 1, a 2, a n – these are all real constants. And n is a positive integer belongs to n. So, this is also a functional. And we denote by p n x – a polynomial of degree n.

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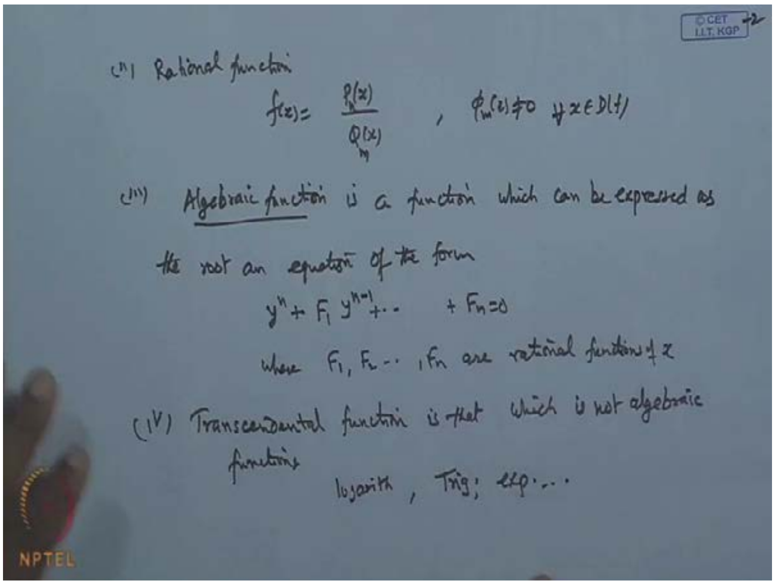

Another type of the function, which we come across – about the rational functions. Rational functions – a function f x, which is of the form p x by q x; where, p x is also a polynomial say of degree n; q x is a polynomial of degree say n. But, phi of m x is not 0 for any x belongs to the domain of f; for every x belongs to f, phi m x is not 0. Then such a function we call it as a rational function. And then a third type of the function, which we can approach as function – algebraic function is a function, which can be which can be expressed as the root of an equation of the form say, y to the power n plus F 1 y to the power n minus 1 plus F 2 y to the power n minus 2 plus F n equal to 0; where, these F 1, F 2, F n are rational functions of x. So, the roots of this equation will be a function, because these are all rational functions; coefficients are rational functions. So, the roots will be a function; that function we call it as algebraic function.

Then, transcendental function are those functions, which are not algebraic – is that function, which is not algebraic function. So, transcendental functions. Now, in this case, we have seen so many examples like logarithmic functions, and then trigonometric functions, and then exponential functions. All these functions we will come across (( )). So, we will discuss in general first, the continuity of these type of functions; and then we will see that, these functions, which are smooth, continuous and also some uniform continuity will also be tested, will be more useful to develop further the theory. So, let us come to… Now, before going for this definition for continuity, we will also look… We require the bounds.

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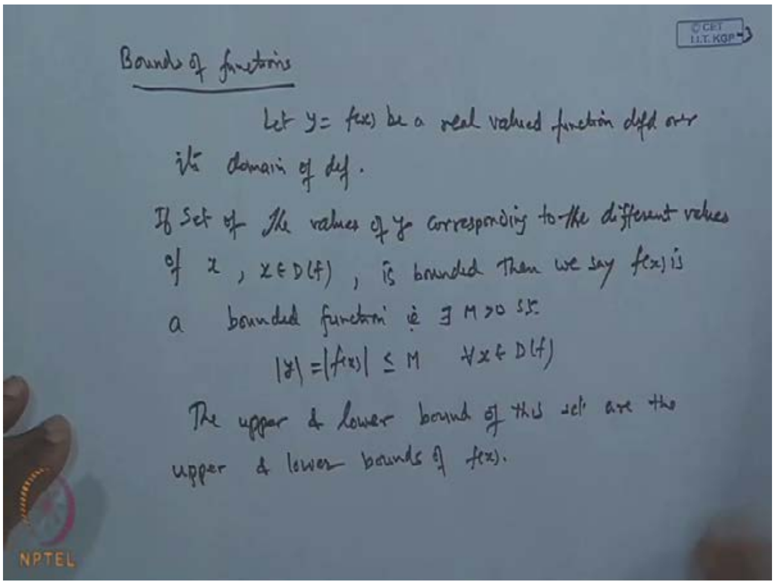

Bounds of functions; in fact, this is… When we say the function f x is bounded, when do we say? Let y is equal to f x be a real valued function defined over its domain of definition, whatever – whether in interval or may be a union of the disjointing interval and like this – intervals. If the set of values of y corresponding to the different values of x; where, the x belongs to domain of f is bounded, then we say the function f x is a bounded function. It means, that is, mode of f x is less than equal to some constant; that is, there is exist some M greater than 0 such that this is true for x belongs to the domain of f. What is f x? f x is y basically. The value of f at the point x denoted by y. So, set of values of f x, if this set is a bounded set, then we say, the function f is a bounded function.

And, in the similar way, the upper and lower bound of this set are the upper and lower bounds of the function f x; that is one – the upper bound of f x; then find out all images and then among these images, find the upper bound. So, that upper bound will be the upper bound for f x; similarly, the lower bound of f x like this. If this upper bound is finite, fine; otherwise, this may be unbounded set if that upper bound we are not able to get M to be finite.

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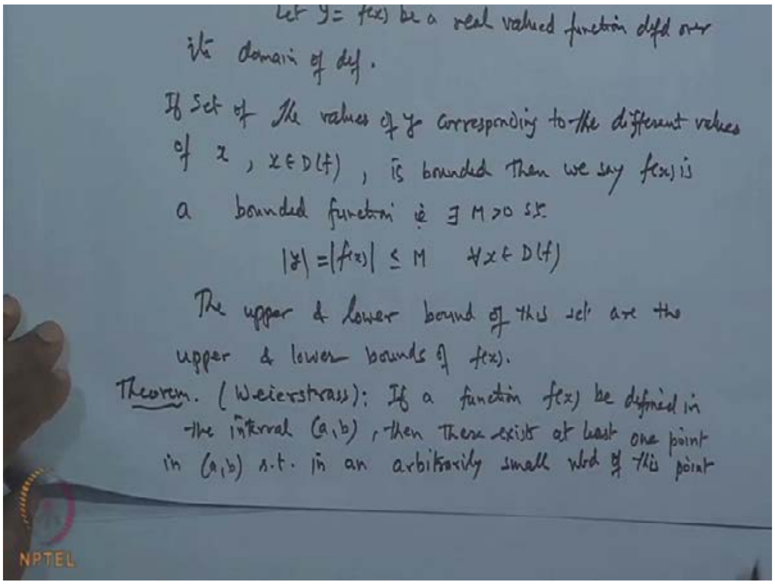

Now one more result, which we require here; that is, a theorem given by Weierstrass. The theorem says if a function f x be defined in the interval a, b open interval a, b, then there exists at least one point in the interval a, b such that in an arbitrarily small neighborhood of this point, the least upper bound of f x is the same as in a, b.

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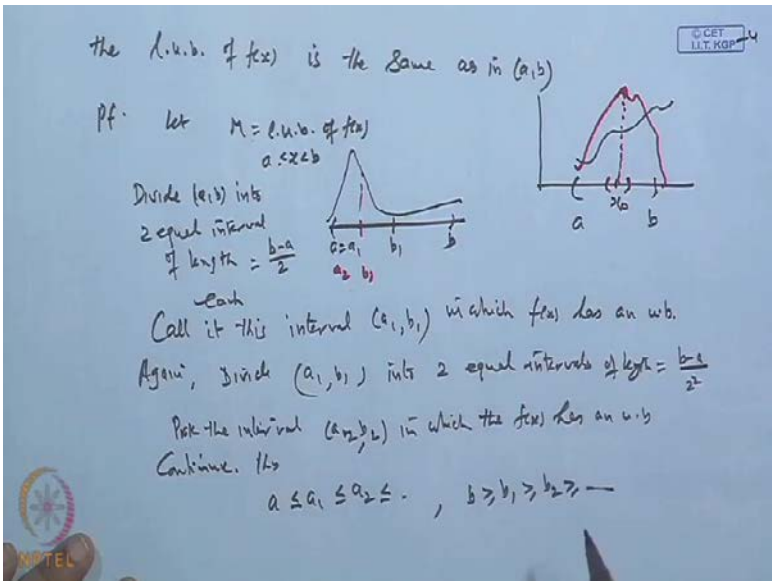

So, what he is says is, if suppose the function we define… Suppose this is the interval and the function is defined everywhere; it is the function, is well-defined; then there exists a point say x naught in the interval a, b such that an arbitrarily small neighborhood if I choose, then this function will have the same upper bound as it is over the entire thing. Say if I choose this function suppose; this is only function. Suppose I take this function. Now, you see the upper bound maximum value is attained at this point. Now, we can find a point x naught, so that in an arbitrarily small neighborhood, the upper bound of the function f x is the same as the upper bound of the function over the entire range a, b. That is what this result says. The proof is… Obviously, the function is throughout well-defined function.

The proof is very simple. Suppose the upper bound of this is suppose M; then what we do? Let us find that bisection of this interval such that we get this… Let M be the upper bound of this. So, let M be the upper bound of f x over the interval x belongs to a, b. This is an upper bound for this; say least upper bound we can say is this. If it is closed interval, then we can say least upper bound of this. On the closed interval, let it be… Now, this interval is given. We divide this interval into two parts of equal length. Now, we pick up that interval in which that least upper bound occurs. Suppose we have this – say this curve suppose like this; suppose this curve.

So, what we do is, we divide this in two equal intervals and then we pick up the interval a 1, b 1; which is divide a, b interval into two equal intervals of length b minus a by 2 each of this length. So, let us pick that interval in which the least upper bound is there. So, obviously, here a 1, b 1 – in this interval, this upper bound lies. So, call it this call interval a 1, b 1 in which the function f x has an upper bound, call it this way. Now, this a may coincide with a or b 1 may coincide with b depending on the function.

Now, further again, divide a 1, b 1 interval into two equal intervals of length equal to b minus a by 2 square. And then pick up the interval say a 2, b 2 in which the function f x has an upper bound. So, you are further dividing it into 2. So, now, I am picking up this. This we call is a 2; this is b 2, because in this only, the upper bound lies. Continue this process. So, if I continue this process. what happens? In this process, when you continue, you are getting the sequence of the points a 1, a 2, a n’s such that a is less than equal to a 1 less than equal to a 2 and so on while b is greater than equal to b 1 greater than equal to b 2 and so on. So, a is a monotonic increasing sequence; b is a monotonic decreasing sequence. But, the upper bound of a cannot exceed by b 1. And the lower bound of this when you take the b 1, b 2, b n – all these things, the lower bound of this will be…

Again, when you take b, b 1, b 2; b 1 is an upper bound here. So, this is bounded above; this is bound it below; and, below will be at the most b.

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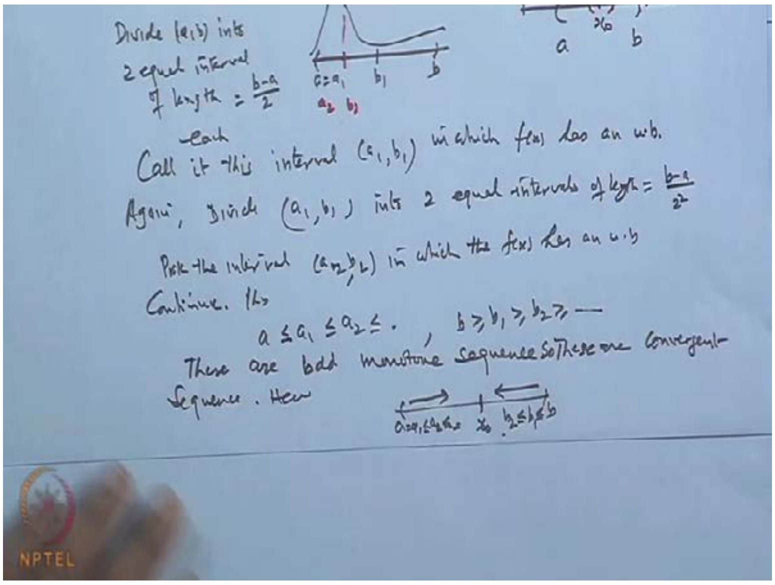

So, this will be that both are… So, these are the monotonic sequences. These are bounded monotone sequences. And we know, the bounded monotone sequence of real numbers or rational numbers, they are all convergent. So, they converges. So, these are convergent sequences. Hence, a n will converge to a point; b n will converge from there. So, we get from here a b. So, a less than a 1 less than equal to a 2 less than equal to a 3 and b greater than b 1 greater than b 2 and so on. So, a n will go this side; b n goes this side. So, a point x naught can be (( )) where the limit of this will coincide.

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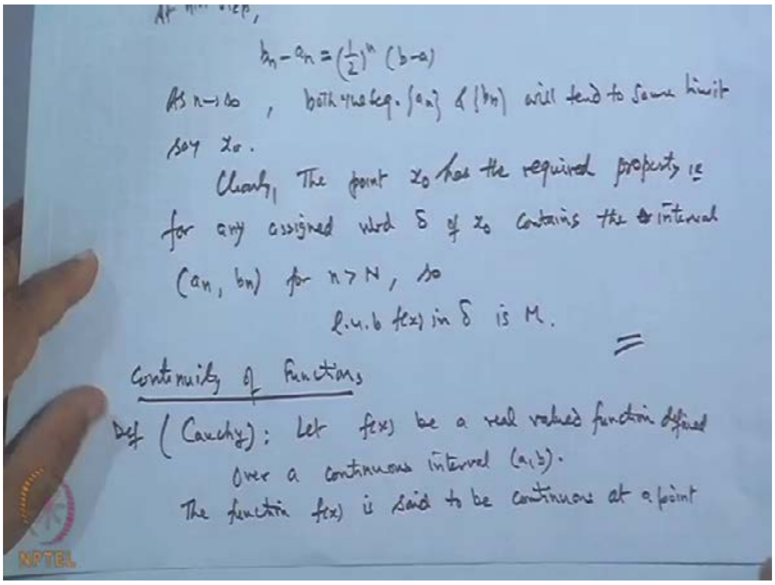

Why? because at the n-th stage step, what we get? The difference between b n minus a n will be half to the power n b minus a. So, as n tends to infinity, this difference is very very small. Therefore, as n tends to infinity, both the sequence a n and b n will tend to the same limit. And that limit point is our a; say x naught; it means there exist an x naught and a neighborhood around the point x naught will be there in which the upper bound lies. So, clearly, the point x naught has the required property; that is, for any assigned neighborhood, delta of x naught delta contains the interval a n comma b n for large n. So, the least upper bound of the function f x in delta is M. That is what it means. So, this is basically is proved assuming the function is well-defined at each point of this. So, the continuity is obviously, is taken in consideration. So, not looking of there; now, we come formerly the definition of continuity.

There are two ways of defining the continuity: one is given by the Cauchy; another one is given by Heine. So, Cauchy has taken in the form using the epsilon delta definition; that in terms of the intervals; while, Heine has defined the continuity of the function in terms of the limit of the sequences. But, both these concepts, both definitions are basically equivalent definitions. So, we will see first, what are these definitions and then we will justify that, these two definitions are basically an equivalent definition.

Let us see the continuity of functions. First, the definition by Cauchy – Cauchy's definition; what is the Cauchy's definition? Let the domain of (( )). Let f x be a real valued function defined over a continuous interval a, b. Of course, here we are taking continuous interval, but it maybe, partly, a function will also be defined with a different way. So, over a continuous function and interval a, b, let y equal to f x be the given values of the function at this interval.

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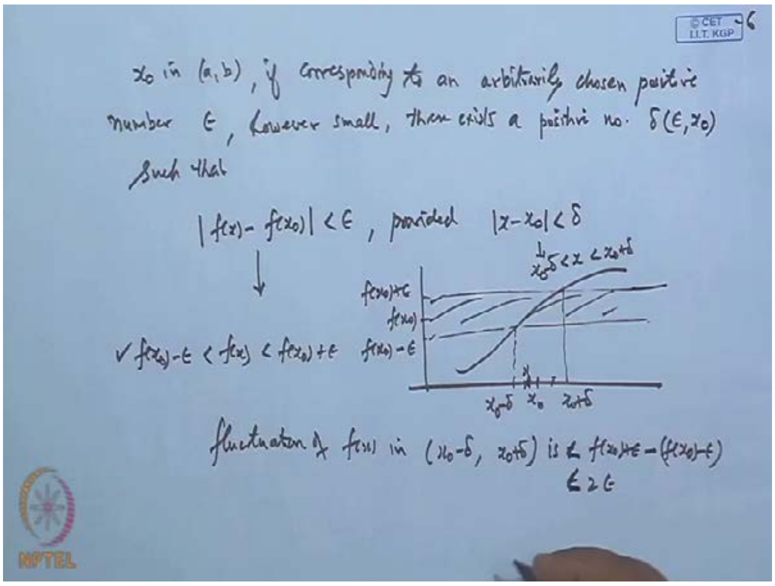

Then, the function f x is said to be a continuous at a point x naught in the interval a, b, if corresponding to an arbitrarily chosen positive number epsilon, however small. Corresponding to this, there exists a positive number delta, which depends on epsilon as well as on the point x naught such that the modulus of f x minus f x naught is less than epsilon provided mod of x minus x naught is less than delta. It means what? The meaning is this – a function f is continuous at a point x naught if for a corresponding to arbitrarily chosen positive number epsilon, there exists a delta, depends on epsilon such that this condition holds. It means that if we take epsilon is given, then we can find out a neighborhood of x naught say x naught minus delta and x naught plus delta.

A neighborhood can be obtained with respect to this given epsilon, with respect to the bond, which is given here such that, the image of any point x will always fall within this range; that is, here you say f x naught. Then the range fluctuation of this will be f x naught minus epsilon f x naught plus epsilon, because mode of f x minus f x naught less than this implies the f x lies between f x naught minus epsilon and f x naught plus epsilon. And this we mean that, x lies between x naught minus delta and x naught plus delta. It means…

So, if f is continuous, then a neighborhood of the x naught exist such that image of any point x in this neighborhood will always satisfy this condition; that is, will always fall within this two epsilon bits of that step whatever the point it choose. If so then we say it is a function. It means the function of the function f x in the interval x naught minus delta to x naught plus delta is what? Is basically, the difference between these two upper and the lower bound. So, x naught plus epsilon minus f x naught minus epsilon; and, that is equal to 2 epsilon. So, if this fluctuation is less than this 2 epsilon – very very small number, then we say the function is continuous. So, this is also considered as a necessary condition for the (( )).

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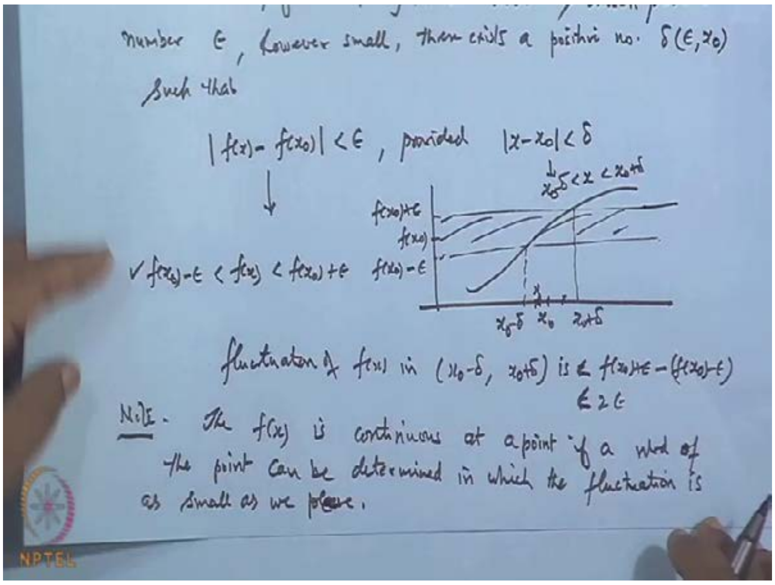

So, an important note we can say the function f x is continuous at a point if a neighborhood of the point can be determined in which the fluctuation is as small as we please. So, this is considered as a necessary condition, because if this does not satisfy, the function cannot be continuous function.

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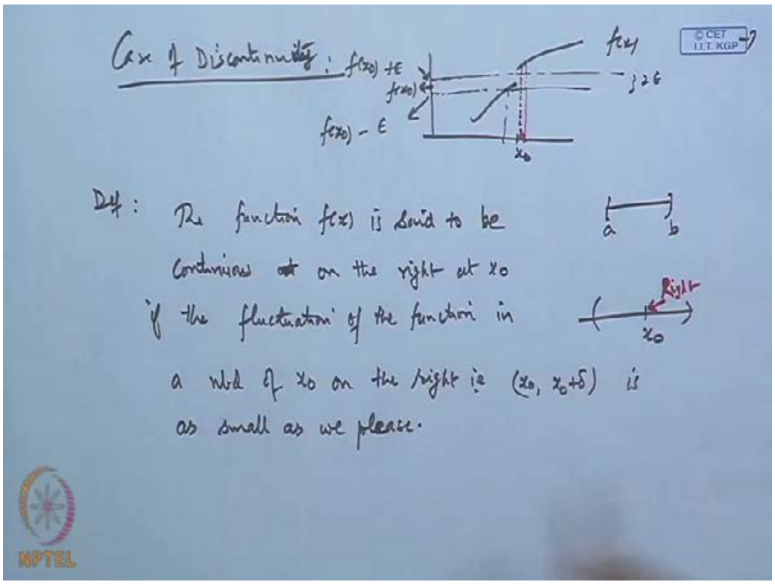

For example, suppose I take this graph; say this is our graph; point x naught is here; function f is this. Now, howsoever small say with this given epsilon, I take this point is say f x naught. So, this point is f x naught minus epsilon. This point will be f x naught plus epsilon. And here this is f x naught. So, for this epsilon… This is the width – 2 epsilon. For given a epsilon, if we get, there must exist an interval neighborhood around the point x naught such that the fluctuations should not exceed by epsilon. But, what happens?

If suppose I take this neighborhood; this is the neighborhood like this. Here as soon you take this point, the corresponding image of this will go here. It means the fluctuation may exceed by 2 epsilon. So, it cannot be made as small as we please. And this is because there is no continuity of the curve at this point x naught. So, such a function will not be considered as a continuous function. So, that is important. That is why we say, this is a case of discontinuity, because of this thing. So, that is why it is necessary. Now, as a corollary of this.

We further extend this. Suppose a function is given over a closed interval; the domain of definition is closed interval. And when we say the function is continuous over the closed interval, then what we do, we take the open interval first and then at the point a and b also we test it. But, before a, the function may not be defined; after b, the function may not be defined. So, we have to modify the definition of the continuity at the end point of the interval. So, we say, the function f x is said to be continuous on the right at x naught – suppose this is x naught; on the right – it means this side; this side from the right-hand side; this is right – on the right at x naught if the fluctuation of the function in a neighborhood of x naught on the right, that is, x naught, x naught plus delta – this is neighborhood on the right of this. The fluctuation of the function in the neighborhood of x naught on the right is as small as we please. Then we say, the function is continuous at a point x naught from the right-hand side.

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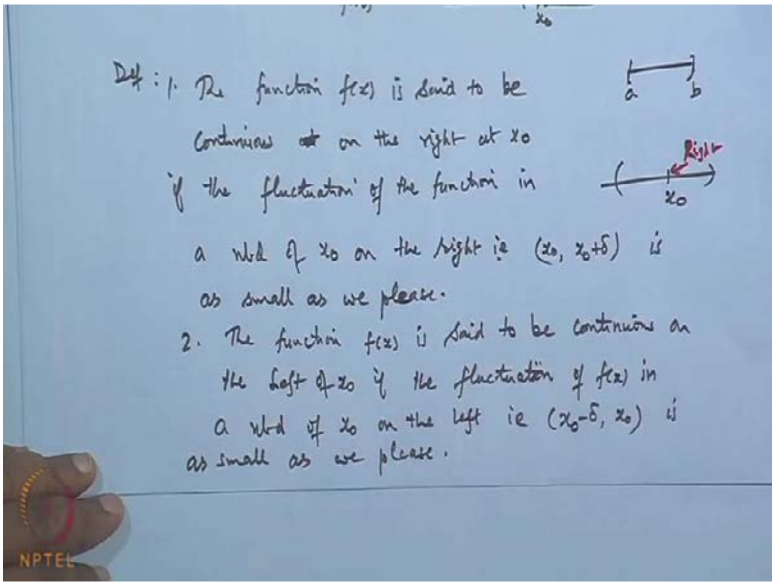

Similarly, the function f x is said to be continuous on the left of x naught if the fluctuation of the function in a neighborhood of x naught on the left – on the left means that is x naught minus delta, x naught on the left – is as small as we please. So, that is the (( )). So, now, we can define the function over the continuity in the interval.

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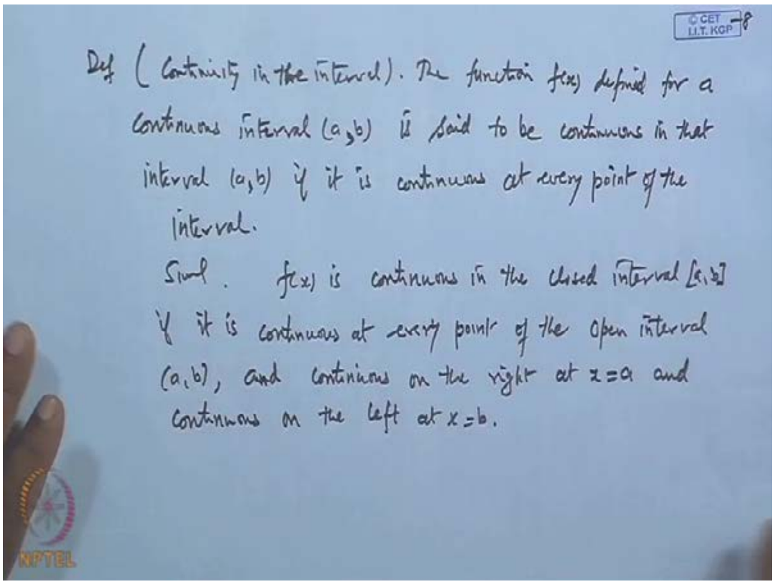

We define as continuity in the interval. The function f x defined for a continuous interval a, b is said to be continuous in that interval a, b, that is, at each point, if it is continuous at every point of the interval, if it is closed. Similarly, we say f x is continuous over the closed interval a, b if it is continuous at every point of the open interval a, b, and continuous on the right, at x equal to a, and continuous on the left at x equal to b. Then we say the function is continuous.

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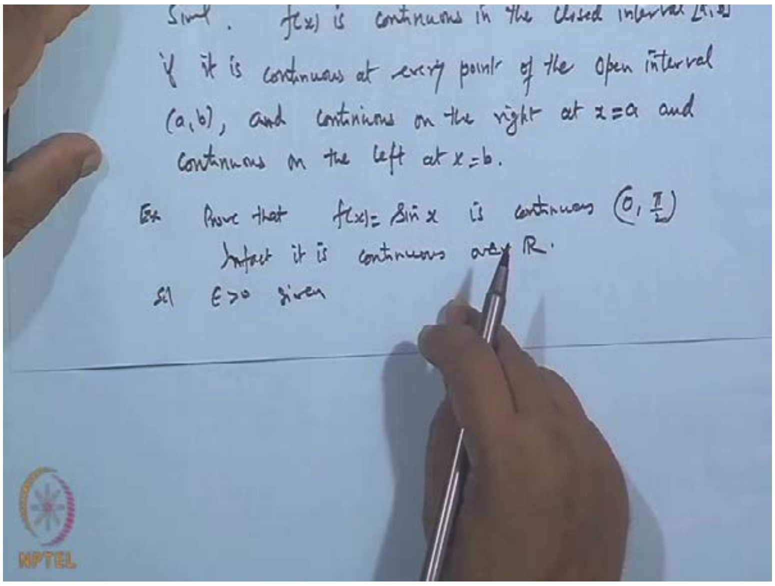

Now, let us take an example using the epsilon delta definition. Suppose prove that f x equal to sin x is continuous function, is continuous say over the internal 0 to pi by 2. In fact, it is continuous everywhere, because it is a periodic function. So, we can say it say 0 2 to pi by 2, is continuous. In fact, it is continuous over the entire real line R. So, the proof is simple. Suppose epsilon is greater than 0 is given. We have to identify delta; depends on epsilon, so that the mod of f x minus f x naught less than epsilon is less than epsilon provided x minus x naught less than delta.

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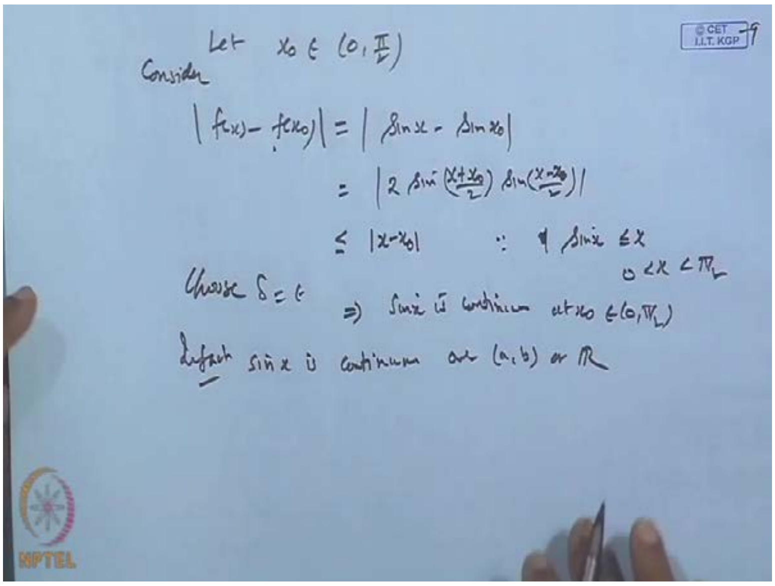

So, consider mode of f x minus f x naught; where let x naught be a point in this internal suppose. Then what is this? This is sin x minus sin x naught. But, this is equal to 2 times sin x plus x naught by 2 into sin x minus x naught by 2 – mode of this; which is less than equal to… Now, sin of this thing is a bounded function bounded by 1. So, this is less than equal to 1. And this sin will be less than equal to mod x minus x naught. Why? Because the sin x is less than equal to x for x lying between 0 and pi by 2; sin 0 is 0; sin 90 is pi by 2; or even 0 to pi it will help. So, any internal, you can just say this. In fact, this can be extended for any x real x. So, this is true.

Now, if I take the delta. So, choose delta equal to epsilon. So, if this is less than delta; obviously, this will be less than epsilon. So, this shows that, sin x is continuous at x naught. But, x naught is arbitrary. So, it is continuous everywhere. And even if we take interval sin x is basically is continuous over any closed, any open internal a b or in the real line. In fact, this one can prove easily. So, this is one of the example, which I have chosen.

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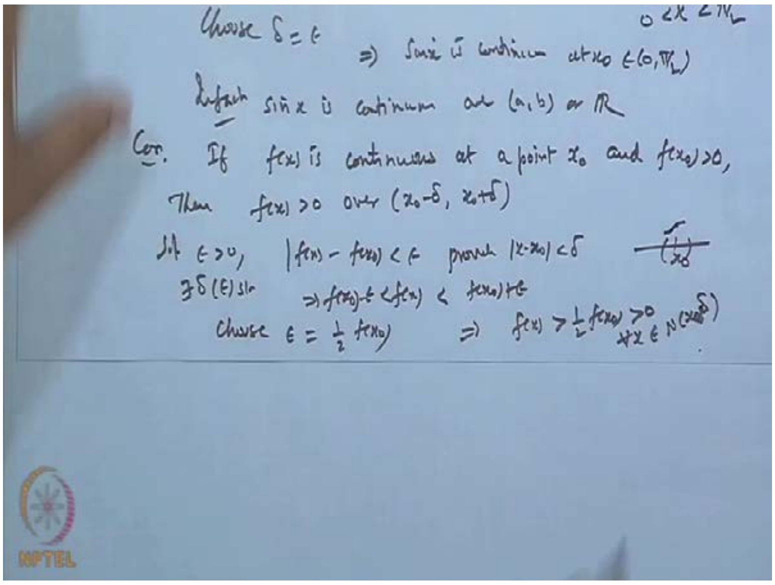

Now, the corollary… There is one small result, which we call the corollary. If f x is continuous at a point x naught and f x naught is positive, then f x is positive over the entire interval x naught minus delta, x naught plus delta; means if the function is continuous at point and at the point x naught, it is a positive function – graph is up; then we can find a neighborhood, where the function will remain positive.

The reason is, because it is continuous, this condition is satisfied. For a given a epsilon greater than 0, there exist a delta depending on epsilon such that this is true provided mod of x minus x naught less than delta. So, when the x lies between this, then what happen? This shows, the f x is lying between f x naught plus epsilon greater than f x naught minus epsilon. So, if I choose epsilon to be half of f x naught, then obviously, f x will be basically half of f x naught because of the left-hand side, which is positive. So, this is true for all x belongs to the neighborhood of x naught with the radius delta. So, this is true, this (( )).

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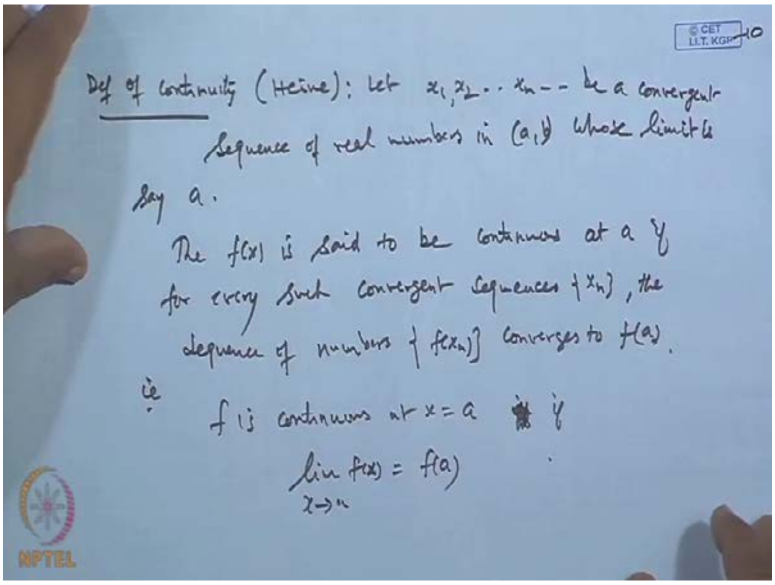

Now, Heine definition for n. Definition of continuity given by Heine – this is given in terms of the sequence. So, what he said, let x 1, x 2, x n be a convergent sequence say of real numbers or points on the real line in the given interval a, b, whose limit is say a. By definition, the function f x is said to be continuous at a point a if for every such convergent sequences x n, the sequence of their functional value – sequence of numbers; functional value means it is numbers now f of x n – this is real numbers, converges to f of a. That is, the meaning is, f is continuous at a point say x equal to a if the limit of the function f x when x tends to a, exists and coincides with the functional value at a. This definition is if an only if. In fact, this is the definition given by this condition. In fact, if an only if is also functional. So, what he says is that, if we take a sequence x n which converges to a, then f of x n will go to f a. That is the definition given by Heine.

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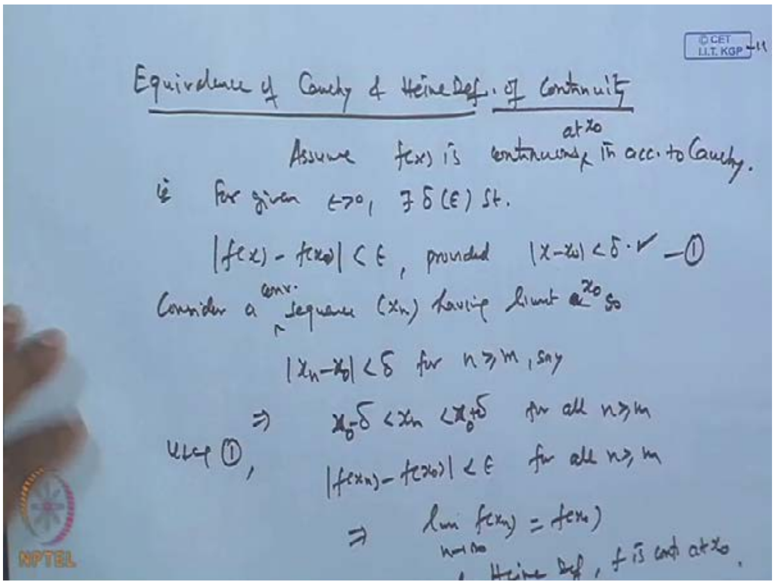

Now these two definitions are basically equivalent definition. The equivalence of Cauchy and Heine’s definition of continuity – let us assume the function is continuous. Assume f x is continuous according to Cauchy. It means, that is, for a given epsilon greater than 0, there exists a delta depending on epsilon such that mod of f x minus f x naught continuous at x naught. f x naught is less than epsilon provided mod of x minus x naught less than delta. Suppose this is true.

Now, let us consider a convergence sequence x n having the limit a; means x n sequence converges to a. So, by definition, modulus of x n minus a is less than say delta for all n greater than equal to say small m say, because limit of x n is a. So, x n minus a will remain less than after a certain stage. But, if x n lies it this implies that x n lies between a minus delta and a plus delta for all n greater than equal to m. And for such point, which lies in this 1, satisfy this condition. So, using 1, what we get is, mod of f x n minus f x naught continuous at x naught. So, here x naught, actually I have taken limit is x naught. Let us take x naught is here. So, here is x naught. This is also x naught because in place of the, so f x naught, this will remain less than epsilon for all n greater than equal to m. Therefore, limit of f x n as n tends to infinity is f x naught. Hence, Heine’s definition follows. Hence, by Heine’s definition f is continuous at x naught.

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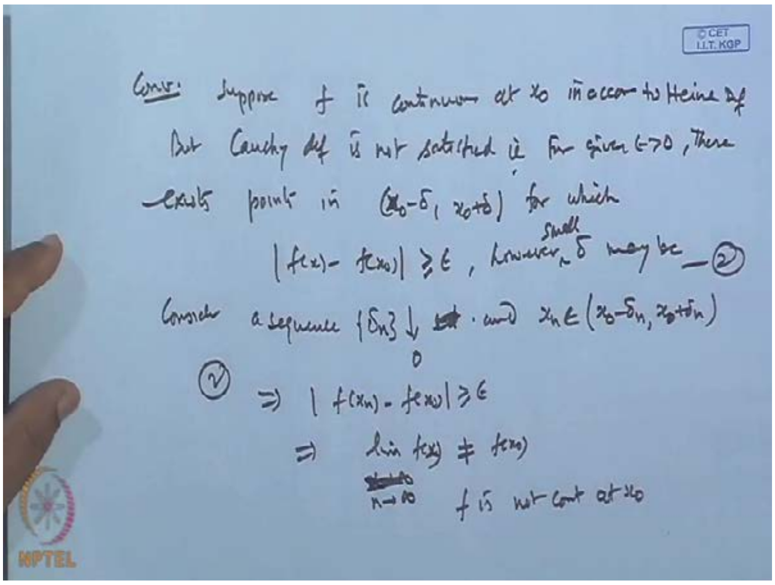

Conversely, suppose Heine definition – suppose f is continuous at x naught in accordance to Heine's definition; suppose this. Suppose we say, it is continuous in accordance. But, Cauchy's definition is not satisfied; that is, for a given epsilon greater than 0, there exists points in the interval say x naught minus delta, x naught plus delta for which the fluctuation – mod of f x minus f x naught exceed by any given number epsilon, howsoever is small delta may be. So, what do you mean by this? This means if you take a sequence which is in this, then all these sequences.

Let us take a… Consider a sequence delta n or decreasing nature, which converges to 0 and x n be a point belongs to this x naught minus delta n, x naught plus delta n; if we take this point in this sequence of the point in this interval, then according to the second, implies that mod of f x n minus f x naught will be greater than equal to epsilon. This shows, the limit of the f x n as n tends to infinity is not equal to f x naught. So, function f is not continuous at x naught or it contradicts the definition. Therefore, this assumption that, Cauchy definition is not very satisfied is wrong here for this.

Thank you very much. So, this was the equivalence of this.