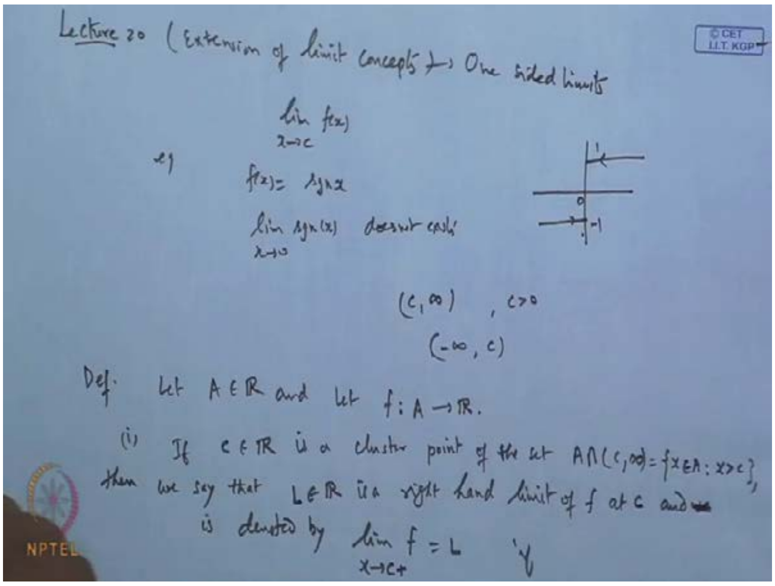
**A Basic Course in Real Analysis**
**Prof. P. D. Srivastava**
**Department of Mathematics**
**Indian Institute of Technology, Kharagpur**

**Lecture - 27**
**Extension of Limit Concept (One-sided Limits)**

In the last lecture, we have discussed the limit of the function f x when x tends to c.

(Refer Slide Time: 00:27)



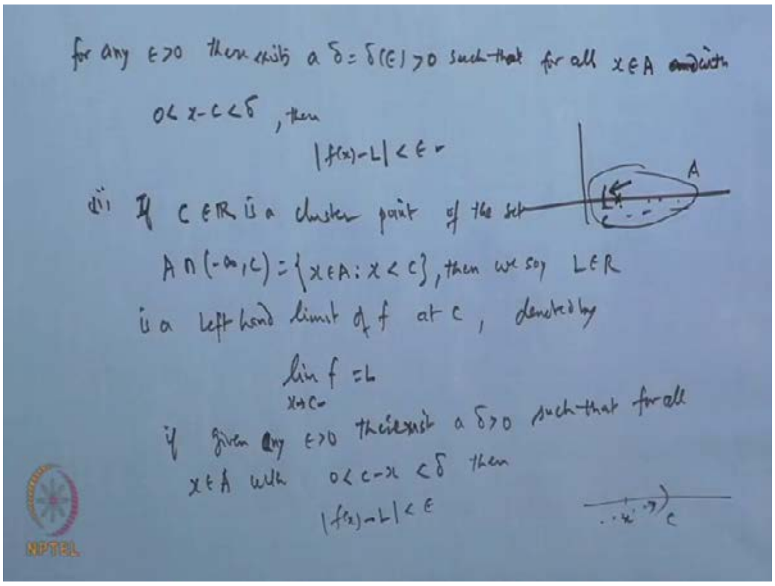

And, in that case, we have seen many examples, where the limit of the function f x does not exist. And the reason for this, either the function is not defined at the point c – means it goes to infinity or minus infinity or may be it has a different limit along a different path; means when we take the right-hand side, the limit comes out to be different than the left-hand side limit. Then we say the function does not evaluate. For example, if we take the function f x at the signum of x, we have seen the limit of this function f x signum of x when x tends to 0 does not exist, because the reason is very simple. When you take the left-hand side, the limit approaches to minus 1; right-hand side the approach to plus 1; and at the point 0, it is 0.

However, if we consider only the set A over which the limit is taken, which is either in the right-hand side of this 0 or may be the left-hand side of the 0; say if I take the c greater than 0 – means c infinity interval if I take, and then find the limit of the function f x over the interval c infinity, where the c is positive, then the limit of this function right and left exist, because there is no scope for going below of this. Similarly, when you take minus infinity say c to minus infinity; then also, the left-hand limit exist. So, in such a case, though the limit does not exist, but we can say, partially, the limit of the function exist when you approach either from the left-hand side or from the right-hand side, individually, limit exist and maybe different; that is different matter.

Now, in such a case when we define the limit of the function, we call it that thing as the left-hand limit. When you approach the point from the right-hand side, then it is called the right-hand limit. When it goes from the left-hand side, this one goes to here – left-hand and right-hand limit. And then we see, if both the limits coincide, then only we can say, the function has a limit A in general; otherwise, we can also introduce the concept of... We will introduce the concept of left-hand and right-hand limit, which we can say is an extension of the limit concepts. So, that is also called one-sided limits. So, let us see first.

Let A belongs to R and let f is a mapping from A to R. Then one – if f is in c; if c is in R, is a cluster point of the set A intersection c infinity; that is, the set of those point x, where belongs to A such that x is strictly greater than c. Then we say that, L – a real number belongs to R; L belongs to R is a right-hand limit of the function f at the point c. And we write, is denoted by limit f x tends to c plus is L.

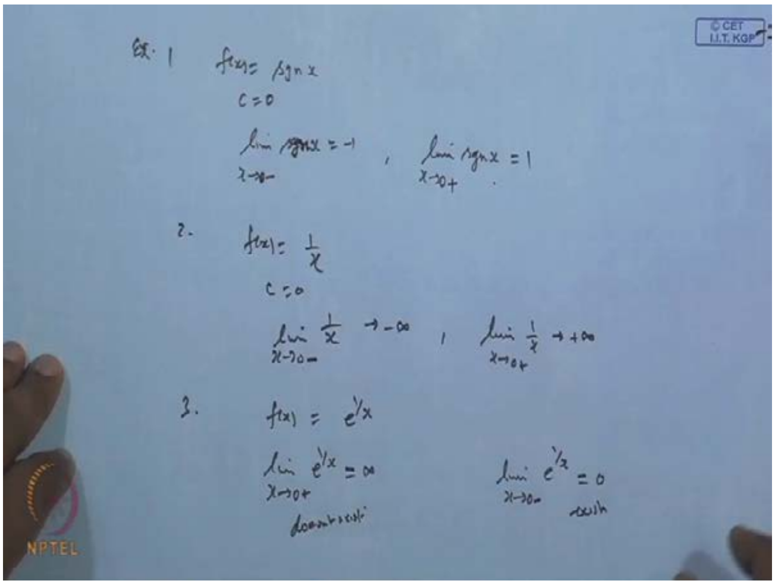
(Refer Slide Time: 05:28)



For any epsilon greater than 0, there exist a delta, which depends on epsilon greater than 0 such that for all x belongs to A and 0 is less than x minus c less than delta; x belongs to A and as well as this condition. Then the mod of f x minus L is less than epsilon. So, for all x... We you can write – with, 0 is less than x; then this one. So, this is called the left-hand limit. So, here this function is there. This is the interval c, infinity; c is this point and then infinity here. So, we can choose here. And then A intersection... A is some... If we take the A intersection c; say A is suppose somewhere here; this is A; then A intersection c will be this set. So, if we picked up the point x here, which are greater than c; and, if their functional value f x minus L lies between this interval; then we say that, the limit of the function f x from right-hand side exist and equal to L; that is, we are approaching to c from the right-hand side – to c.

Then, second, we define – if c is a point in R is a cluster point of the set A intersection minus infinity to c; it means this is the set of those point x belongs to A; where, x is strictly less than c. Then we say L belongs to R is a left-hand limit of f at c denoted by limit f when x tends to c minus is L. If given any epsilon greater than 0, there exist a delta greater than 0 such that for all x belongs to A with condition is 0 less than c minus x less than delta; then we have mod of f x minus L is less than epsilon. So, this is the left-hand limit of this. So, we are approaching the c from the left-hand side; that is, this is the interval c. And the approach is from this side. So, when you take the x point here, which approach to c from the left-hand side and if this condition holds, then we say the function has a left-hand limit at the point c. So, the left-hand limit and right-hand limit this way we define.

(Refer Slide Time: 09:51)



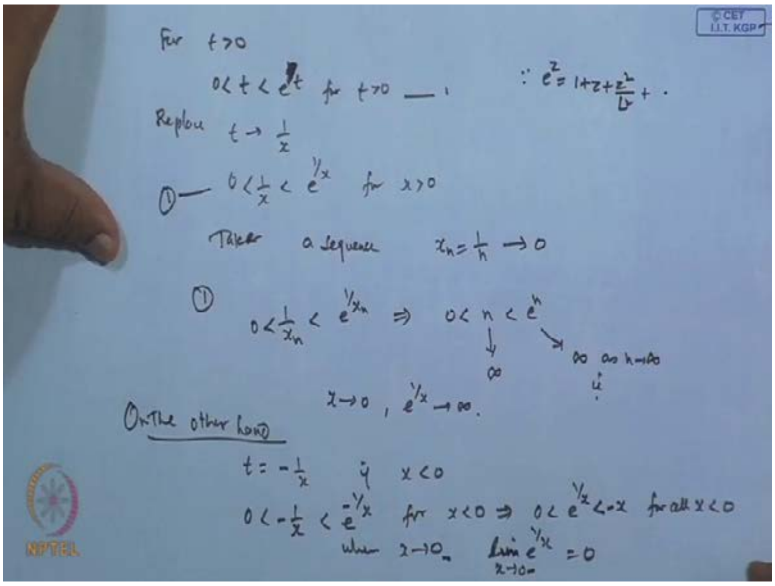

And, we have seen the example that, if we look the function sin x, the limit of this left-hand limit comes out to be... Example is, if we take the function f x equal to signum of x and the c is a point 0, then we say, the limit of the function f x, that is, signum of x when x tends to 0 minus is minus 1 from the left-hand side. While the limit of this signum of x signum – sgn of x as x tends to 0 plus; this is plus 1. So, this is the right-hand limit; this is the left-hand limit for this one.

Now, when we introduce the concept of the left-hand or the right-hand limit, we look the set A and intersection with the c infinity if the right-hand limit; or, intersection with minus infinity to c if the left-hand limit. And then if the limit comes out to satisfy this condition, then we say limit. However, it is not necessary that always we will have either left-hand limit or right-hand limit or maybe both; it may so happen that, we may not get neither left-hand limit nor the right-hand limit. Hence, limit will not exist; or maybe sometimes we get the left-hand limit, but not the right-hand limit and vice versa; or, otherwise or sometimes, we can get both the limits, but the values are different. So, this is the case when both the limit exists, but they are not equal.

The second example if I look, the function f x, which is 1 by x; then what we say is when c is a point 0, then we say the limit of this function 1 by x when x tends to 0 minus; from the 0, it goes to minus infinity. And limit of this 1 by x when x tends to 0 plus, it goes to plus infinity. So, limit does not exist here of course, and they are different. They diverge.

Third case is if we look the function f x equal to say e to the power 1 by x; suppose if I take this function e to the power 1 by x. And then when we take the limit of this function e to the power 1 by x when x tends to 0 plus or limit of the e to the power 1 by x when x tends to 0 minus, what we will see here that, when x tends to 0 plus, the limit will be infinity; while in this case, the limit comes out to be 0. So, left-hand limit exist, but the right-hand limit does not exist, comes out to be finite, is infinity. The reason of this is simple. Why it is so?

(Refer Slide Time: 12:58)

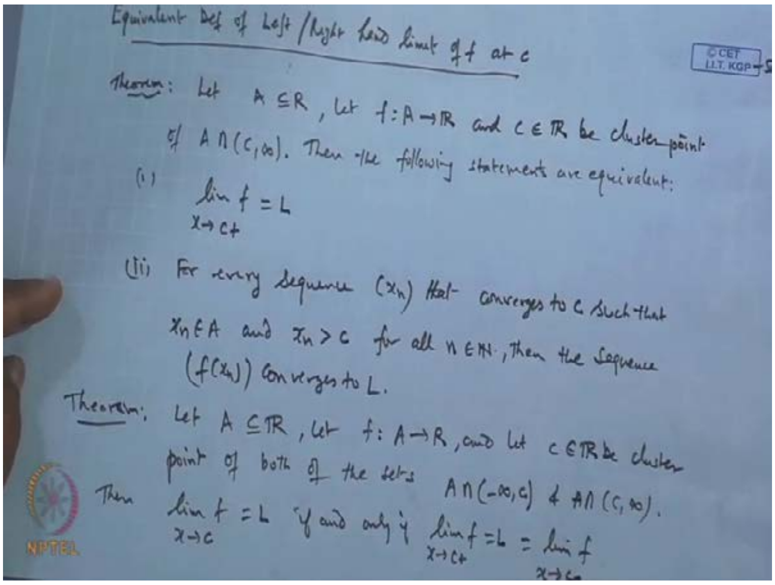



Because if we look that any t; for any t greater than 0, we have 0 less than t less than e to the power 1 by t, because the expansion of e – because e to the power z or x is 1 plus z plus z square by factorial 2 and so on. So, obviously, when you write... If 0 is less than equal to t less than t... And this is true for all t greater than 0. So, when you replace t by 1 by x, what we get is 0 less than 1 by x less than e to the power 1 by x for x greater than 0. Now, if we take a sequence x n say 1 by n, which goes to 0; this x n goes to 0. So, what about this 1? From equation 1, what we get? The corresponding sequence 1 by x n, which is less than e to the power 1 by x n greater than 0 will imply that 0 less than n less than e to the power n. And since this tends to infinity, therefore, this limit will go to infinity as n tends to infinity; that is, when x tends to 0, e to the power 1 by x will go to infinity.

Now, on the other hand, in this expression, if we replace t by say minus 1 by x if x is negative; then minus 1 by x is of course positive over here. Then we get from here is 0 less than minus 1 by x less than e to the power minus 1 by x valid for x to be negative. But this implies just a manipulation. And this will give you the result 0 less than e to the power 1 by x, which is less than minus x for all x negative. So, when x tends to 0, then what happens to this? From the left-hand side, this will be 0. So, this limit comes out to be 0, because... So, this implies limit of this e to the power 1 by x when x tends to 0 from negative side is 0.

Therefore, this limit is... So, we have seen the three examples: one is when the left-hand limit and right-hand limit – both exist, but they are different; second case – when none of the limit neither left-hand limit nor right-hand limit exists, it comes out to be infinity or minus infinity; while in the third case, we have only the left-hand limit exist, right hand limit does not exist. So, the concept of the left and right-hand limit basically the existence of the left or right-hand limit basically depends on the function. And when both these limits coincide, then we say the function has a limit at the point x in 0. So, that is what.

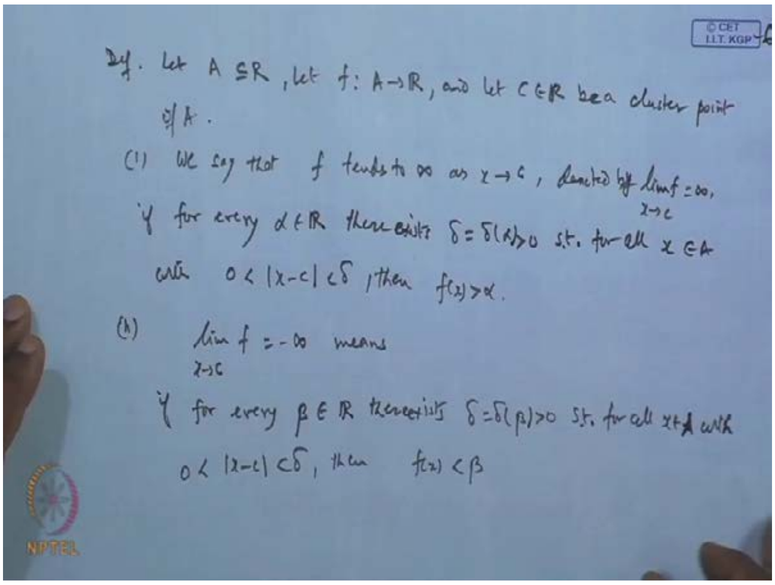
(Refer Slide Time: 16:44)




Now, the equivalent definition in terms of the sequences; this is the equivalent definition of left and right limit – left oblique right-hand limit of the function f x at c. This is in the form of theorem. Let A is be a subset of R and let f is a mapping from A to R; and, let c is an element of R be a cluster point of A intersection c, infinity. Then the following statements are equivalent. The first statement says limit of the function f when x tends to c plus; that is, the right-hand limit of f is suppose L; then it is equivalent to the second statement that, for every sequence x n, that converges to c such that x n is in A and always x n is greater than c means always towards the right-hand side of the c for all n belongs to capital N. Then the sequence of its functional value, that is, f of x n; this sequence converges to L. If this is obtained, then we say, the right-hand limit of the function f is L. So, this is the equivalent definition in terms of the sequences. Similarly, we can write it further left-hand limit; if the limit of the function f from left-hand side exist means, there exist x n converges such that x n is strictly less than c for all n... And then f of x n converges to L. In a similar way, we can write it.

Next, very interesting result, which I told earlier also. The relation between these extension of the limits and the limits. What theorem says is, let A is a subset of R – non empty subset of R; and, let f is a mapping from A to R; and, let c be a cluster point both of the sets; that is, A intersection minus infinity c as well as A intersection c infinity – these sets; then limit of the function f as x tends to c exist and equal to L if and only if limit of f when x tends to c – means right-hand limit of f is L; which is the same as the left-hand limit of the function f at c. If both the limits coincide and equal to L, then we say, the limit of the function exists and equal to L. So, this will be the definition for this. Then we come to... Now, here.... So far, we have taken only the concept of the limit when c is finite; limiting value is finite; L is also finite. So, now, we will take the case when limiting value is finite, but limit comes out to be infinity. So, what will be the form of the definition when the limit L comes out to be infinity or minus infinity?

(Refer Slide Time: 21:39)

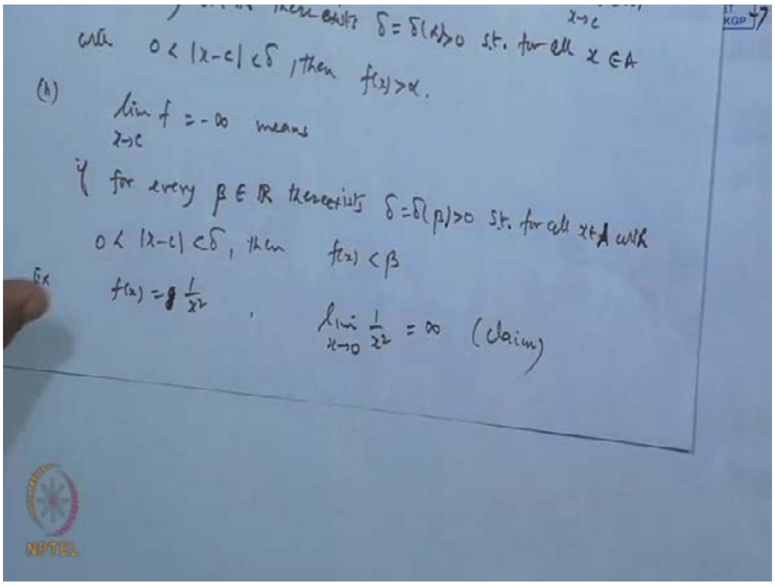



Now, this is definition. Let A, which is a subset of R; and, let f mapping from A to R; and, let C belongs to R be a cluster point of A. Then we say that, f tends to infinity as x tends to c denoted by limit f when x tends to c is infinity if for every alpha belongs to R, there exists a delta, which depends on alpha positive such that for all x belongs to A with the condition... because it is lying between from both sides with x minus c; 0 is less than mod x minus c less than delta; in this neighbourhood, either from left-hand side of c or right-hand side from c; and, within the neighbourhood of delta neighbourhood of c. Then the functional value of this f x is greater than alpha.

And, since alpha is arbitrary, it means the value of the function f x cannot be bounded; it is unbounded. When we say, the limit of the function f when x tends to c is infinity means when x approaches to c, the function f is not bounded. This is equivalent to say that, if I choose a delta neighbourhood of c, then the point in this delta neighbourhood of c will exceed to any given number alpha. And this once you decide any alpha, you can identify a delta such that when you choose the x in this delta neighbourhood of c, then the corresponding value of the function at this point will be greater than alpha. So, that way, we say the limit of the function f x when x tends to c is infinity.

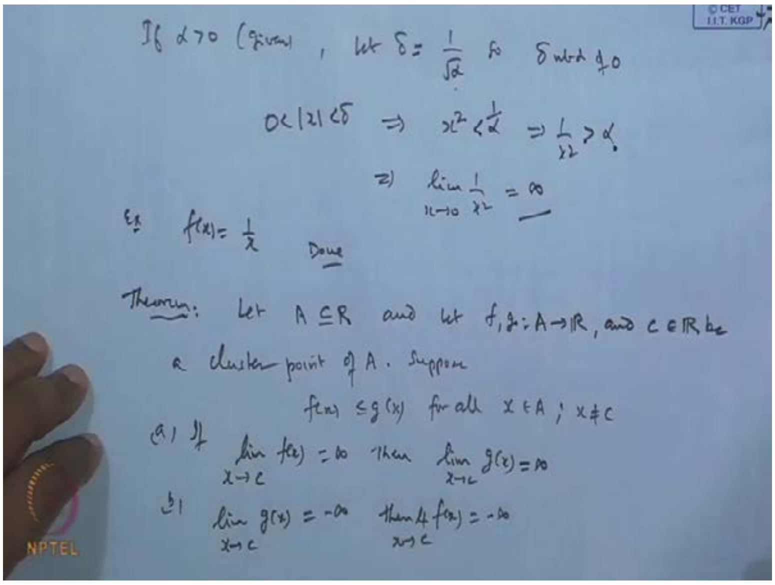
Similarly, we define the limit to be minus infinity as follows. As we say f is the limit of the function f when x tends to c is minus infinity means if for every beta belongs to R, there exist a delta, which depends on beta positive such that for all x belongs to A with 0 is less than mod x minus c less than delta, we have then f of x is less than beta. So, when we say the limit is minus infinity, it means then x is approaching to c either from the left-hand side or from the right-hand side. The functional f x is approaching to minus infinity; that is, it is unbounded towards the negative side. Then that is equivalent to say is that, whatever the number you pick up, there will be a delta neighbourhood of c such that the value of the function will be still lower than that number beta. And this shows the function f x will go to minus infinity when f is sufficiently close to c.

(Refer Slide Time: 26:13)




Now, let us see an example following. For example, the function f x say 1 by x square. Now, if we look the limit – limit of this function 1 by x square when x tends to 0. Now, what happen? When x tends to 0, either from the right-hand side or from the left-hand side, both will go to 0. What if x tends to 0? So, it will go to infinity. So, we claim this limit is infinity. This is our claim.

(Refer Slide Time: 26:50)

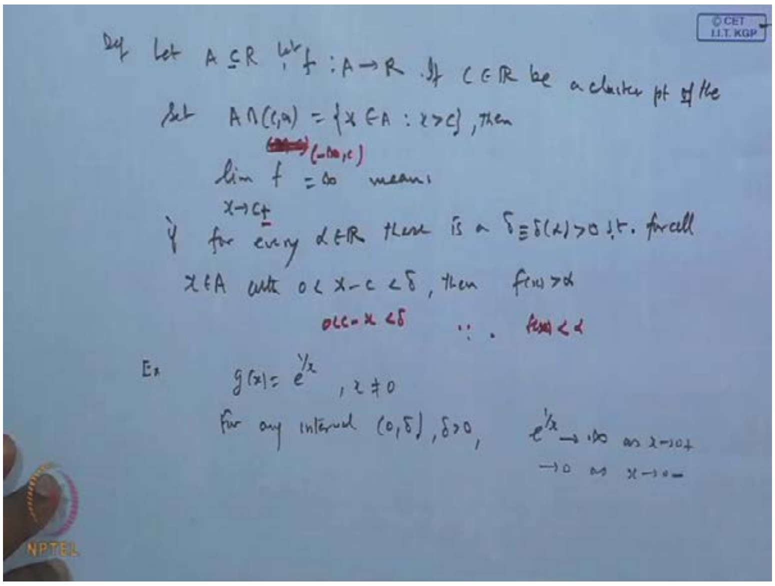



To justify it, we write like this. If alpha is greater than 0 is suppose given; and, let us find delta as 1 by root alpha. So, delta depends on alpha. Now, once we take, it follows that, delta neighbourhood of this. So, delta neighbourhood of 0; delta neighbourhood of 0 means mod of x is less than delta greater than 0. Now, then this implies that x square must be less than... This delta is 1 by root alpha. So, x square must be less than alpha. And hence, 1 by x square will be greater than alpha; no, this is 1 by alpha. So, it is greater than alpha. It means when x is sufficiently close to 0, the value of this can exceed to any number alpha for a given alpha.

So, whatever the alpha is arbitrary. So, when you choose any alpha, we can find a neighbourhood of 0 such that the value of this exceed by that number. It shows the limit of this 1 by x square when x tends to 0 is infinity; that is what (( )) Another example if we take; suppose I take the... 1 by x also we have seen; that is OK. So, take the... 1 by t... f x equal to 1 by x. This also we have seen. The limit of this when x tends to 0 from right-hand side is infinity, left-hand side plus infinity. So, this is already done. So, need not...

Now, just like we have the Skew's theorem, in case of the limit, similar type of result we can also have in case of the right-hand or left-hand limit. So, that results in the form – let A, which is subset of R; let f and g, which are the mapping from A to R; and, let C belongs to R be a cluster point of A. Suppose that f x is less than equal to g x for all x belongs to A, that is, for x belongs to A; but x is not equal to c. Then limit of this function f x – if limit of f x when x tends to c is infinity, then limit of g x when x tends to c will also be infinity. And b part is, if the limit of g x when x tends to c is minus infinity, then limit of this x tends to c will be minus infinity. Of course, this is obviously one can say… So, we are just skipping the proof for it and (( ))

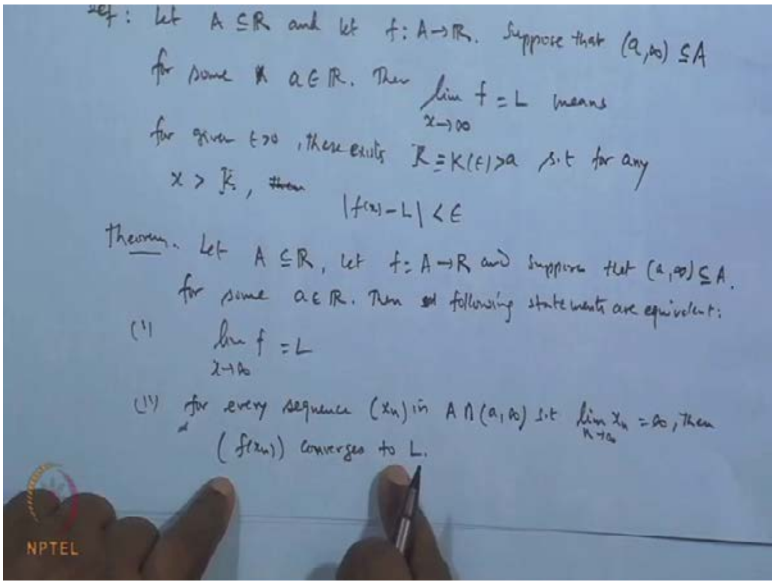
(Refer Slide Time: 30:38)




Now, so far we have taken when L is infinity and then we have chosen this part; now, one-sided limit of this definition. Let A, which is subset of R; and, let f is a mapping from A to R; and, if c belongs to R be a cluster point of A of the set A intersection c infinity; that is, the left-hand limit and right-hand limit – we are giving the concept here. x belongs to A, where the x is greater than c; then limit of this f when x tends to c plus is infinity – means if for every alpha belongs to R, there is a delta depending on alpha positive such that for all x belongs to A with 0 less than x minus c less than delta, then the f of x is greater than alpha for this one. So, this is the left-hand concept of the limit. Limit of the f... Right-hand side is infinity means that if I picked up a set A intersection c infinity and c is the cluster point of this set, then we say, the limit of the function when c approaches towards the right-hand side or right-hand limit of A is infinity means that we can identify a delta corresponding to every alpha such that the f of x will exceed by alpha for all x satisfying this condition.

Similarly, when it is the negative side, then we say, simply if it is negative say minus, then what changes here is the corresponding c – it will be infinity c, that is, minus infinity c. And then from here minus infinity c; and from here, it will be just c minus x 0 less than x less than delta. Then it will come to be f x less than alpha; like that. So, similarly, the changes will be like this accordingly. We are not giving that. Then for example, is this say 1 by delta; g x equal to e to the power 1 by x when x is not equal to 0. Now, this we have already seen that, if we take for any delta for any interval 0 delta, where delta is positive, the right-hand side limit of this function tends to this right hand when x tends to delta, right-hand side tends to infinity as x tends to 0 plus. And this limit goes to 0 as x tends to 0 minus. This we have already discussed in limit.

(Refer Slide Time: 35:01)

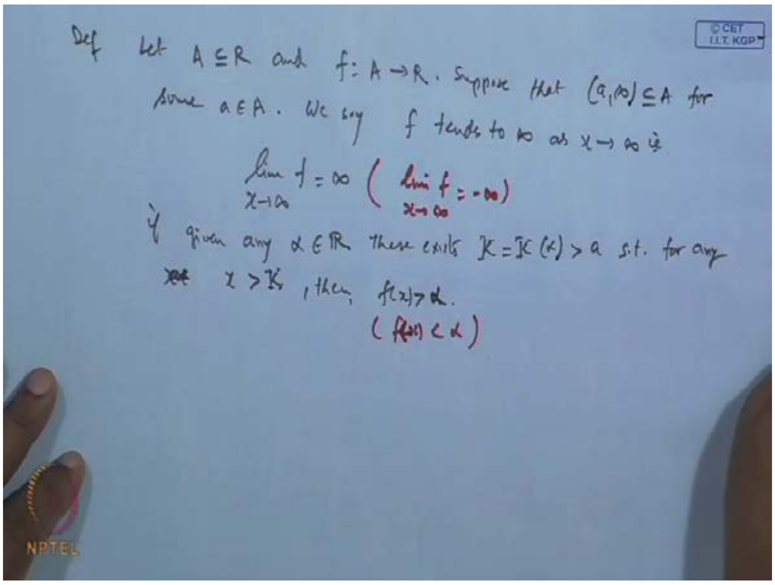



Next concept when c... One of the limiting point c is not finite. If it is infinity, then also we can define the limit as follows. So, limit at infinity. We can say, the limit at infinity concept. Limits at infinity – when we have the limiting point c is infinity, we define as follows. Let A be a non empty subset of R and let f is a mapping from A to R. Suppose that A infinity is contained in A for some A belongs to R. Suppose this interval. Then we say, then the limit of this f when x tends to infinity L means that, for a given epsilon greater than 0, there exist a k, which depends on epsilon and greater than A number – this k greater than A such that for any x greater than k, f of x minus L is less than epsilon. Then we say, the limit of this function f when x tends to infinity is L. When x is sufficiently large – means at the infinity, we are finding the behaviour of the function f x.

What is the behaviour of the function at the point at infinity? This shows the limit. Then plus infinity, minus infinity also in a similar way we can write.

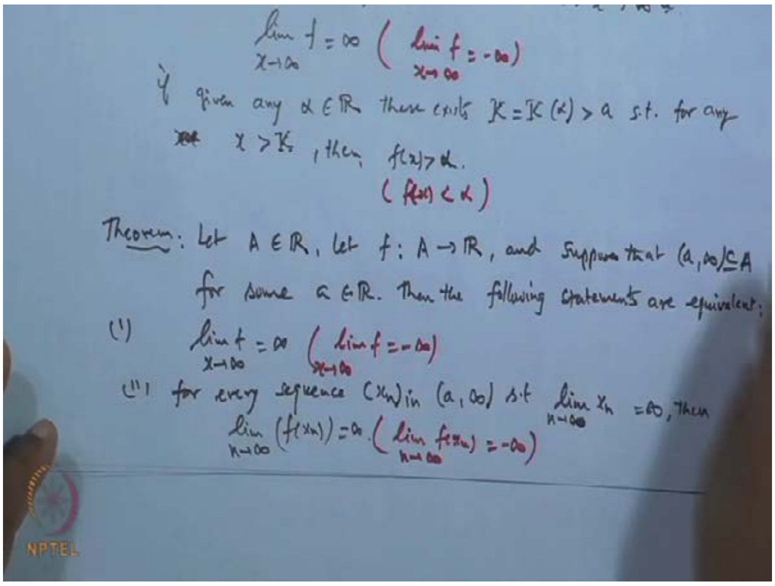
Another result.Let A which is subset of R and let f is a mapping from A to R. And suppose that a, infinity is contained in A for some a belongs to R. Then the following statements are equivalent. The first statement says, the limit of this function f when x tends to infinity is L; and, second statement says that, for every sequence, x n in A intersection a, infinity such that limit of x n as n tends to infinity is infinity. Then the sequence f of x n converges to L. So, this is the equivalent definition in terms of the sequence. When we say, limit f as x tends to infinity is L, then in terms of the epsilon delta definition, we have taken that, for a given epsilon, we can identify a k such that when all x greater than k, the difference f x minus L will be small. In terms of the sequence, we can say, there exist a sequence x n, which are tending to infinity and then the corresponding functional values will converge close to L. So, that is an equivalent definition for this.

(Refer Slide Time: 39:56)



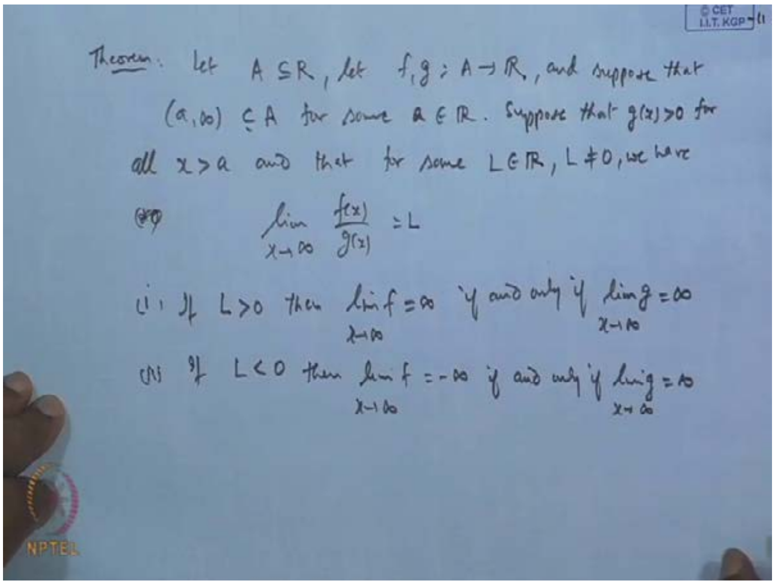

Now, when both the limits plus c is also infinity, a is also infinity; then we define the concept as follows. Let A, which is a subset of R; and, let f from A to R. Suppose that a, infinity; this is contained in A for some a belongs to A. Then we say f tends to infinity as x tends to infinity; that is, we write like this – limit of f when x tends to infinity is infinity; if given any alpha belongs to R, there exists a k – depends on alpha greater than a such that for any x greater than k, then f of x will be greater than alpha. So, this is the concept when both the... means c is also infinity and L is also infinity. Similarly, if we take limit of f as x tends to say infinity is minus infinity; x tends to infinity means for any alpha, there exist k depending on alpha such that k alpha is greater than 1; then for any x (( )) this, it will show f of x is strictly less than alpha. So, that will be the criteria – when x tends to infinity, f is minus infinity; these two.

(Refer Slide Time: 42:39)




Again, in the sequential form we can say like this. So, equivalent concepts in the sequential criteria; that is, in the form of theorem. Let A belongs to R and let f is a mapping from A to R; and, suppose that a, infinity is contained in A for some a belongs to R. Then the following statements are equivalent. The first statement is, limit of the f as x tends to infinity is infinity. And second statement says for every sequence x n in a, infinity such that limit of that sequence x n when n is sufficiently large, is infinity; then the limit of f of x n as n tends to infinity will be infinity; limit of this. So, this is the equivalent concept. If suppose we want that one – limit of this is minus infinity; if the limit of f when x tends to infinity is minus infinity, equivalently, we can say here is for every sequence x n in a, infinity such that limit of the x n is infinity; then limit of these f of x n will be minus... Then here limit of f of x n as n tends to infinity will be minus infinity. This is the equivalent way; equivalently respective way or you can say this. So, concept is this one.

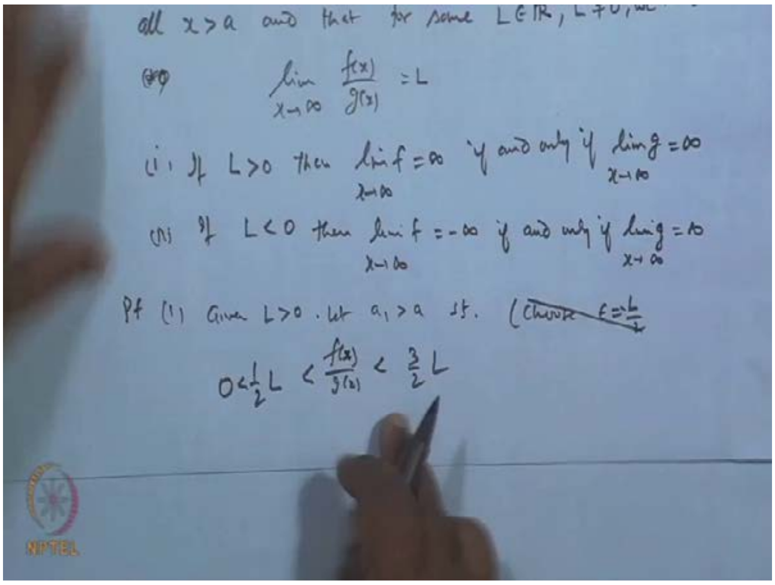
(Refer Slide Time: 45:07)




Now, one more result, which will help in getting sometimes limits. Let A, which is subset of R; and, let f g are the mappings from A to R. And suppose that a infinity of an interval... a, infinity is contained in A for some a belongs to R; and, suppose that g x is always positive for all x greater than a. So, these are the restrictions we put in. And that for some L, which is in R, L is not equal to 0. We have first, is the limit of this f x by g x when x tends to infinity is say L. Then depending on L, we can identify the limit for f and g. So, first condition says, if L is positive, then limit of f as x tends to infinity is infinity if and only if limit of g as x tends to infinity is infinity. So, if L comes out to be greater than 0, then both these functions f and g when x is sufficiently large, will be infinity. The value will come out to be infinity.

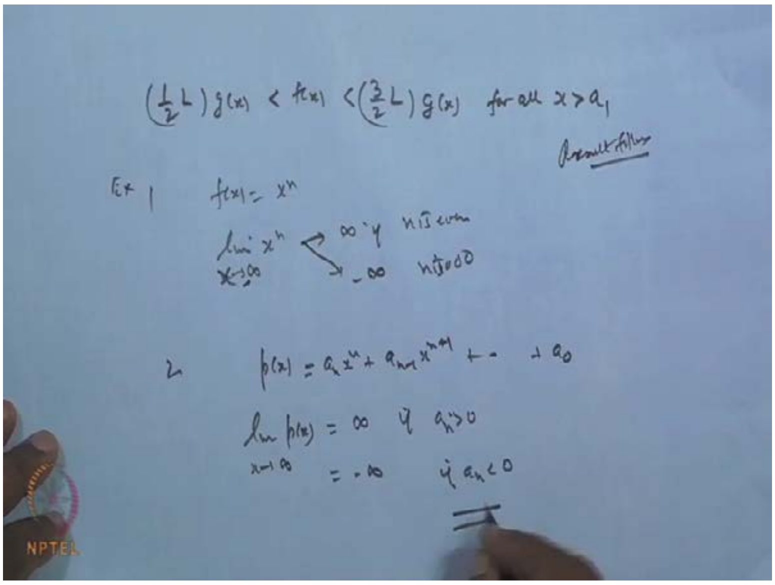
And, second is, if L is negative, then limit of f as x tends to infinity – if this is minus infinity if and only if limit of g is plus infinity; that is, if L is negative and limit of g is positive, then limit of f will be minus infinity. If limit of f is minus infinity, then limit of g will be infinity if L is negative. But if L is positive, then if f limit is infinity, g will also have limit infinity. And similarly, vice versa; if g has limit infinity, L will also have limit infinity.

(Refer Slide Time: 48:15)



See the proof of this; it is not difficult. Say first, given that L is positive. So, once L is positive, then... And a belongs to this. So, let us take a 1, which is greater than a – belonging to the interval a to infinity; this interval is a... such that f x by g x, because this limit is L. So, if I choose the epsilon as L by 2, then we can write this thing as f x by L is lying between 3 by 2 L greater than... because f and g both are given to be a positive; g is also positive. So, if this we can write it f x and g x as greater than half L, then x is sufficiently large, which is positive. Since L is (( )) we can... Suppose there exist a 1 such that this condition holds, this is our (( )). Therefore, what will... So, multiply by g x, because g is given to be positive.

(Refer Slide Time: 49:43)



We say here is multiply by g x; we get half L into g x will be less than f x will be less than 3 by 2 L into g x. And this is true for all x greater than a 1. So, obviously, this will be true. Now, take the limit. What is now changed is if at all f is infinity, then g is infinity. So, here if f is infinity, g will be infinity; if g is infinity, f will be infinity; if f is infinity, from here g will be infinity. So, result follows. So, this will be the...

Now, let us see some examples based... We have x to the power n and other form. We have already discussed that, if function f x is x to the power n, then limit of this function f x, that is, x to the power n when n tends to infinity – when x tends to plus infinity or x tend to minus infinity; that way. So, if I take x tends to minus infinity; then when n is even number, this will go to plus infinity. This will go infinity if n is even, because x is tending to minus infinity, but n is even. And when n is odd, then this will go to minus infinity. So, we can get. And this can be (( )). Similarly, if the polynomial p x is given say a n x to the power n plus a n minus 1 x n minus 1 up to say a naught; then limit of this p x when x tends to plus infinity, this will be equal to infinity if a n’s are positive; and, p n will be negative minus infinity, if a n’s to be negative. So, this can be (( )). So, in case of fraction also, we can do it this way.

Thank you.