A Basic Course in Real Analysis
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Lecture - 26
Limit Theorems for functions

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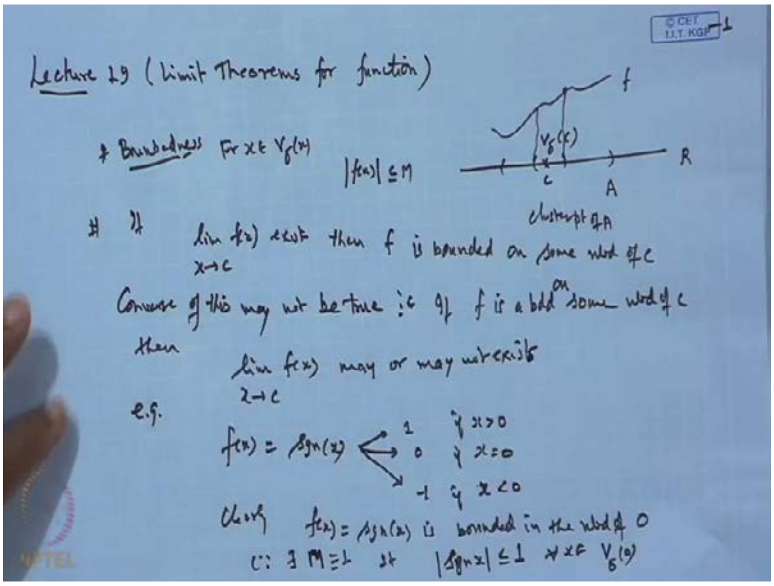

So in the last lecture, we were discussing about the boundedness of the function in the neighborhood of a point, and we have seen that if R is a line, and A is suppose a set A which is sub set of R and c is point in R which is say a cluster point of A, and the function f is defined on this interval A as on this set A which is subset of R. Then we say the function f is bounded in the neighborhood of a point c means; if there exist in neighborhood, delta neighborhood of V delta c and a constant M such that the values of the function f x for x belonging to this V delta x is less than equal to M.

Then we say the function f is bounded over a neighborhood around the point c; and one more result which we have seen, that if the limit of the function exist, if limit of the function f x when x tends to c exist an element then f is bounded on some neighborhood of c. This also we have seen the result; so these two things, boundedness of the function and this. Now this result we have proved only one side. If the limit exists at the point c; that a limit of the f x when x tends to c exists, then only we can say function is a bounded function. What about the converse part? The converse means if f is a bounded function on some neighborhood of say point c. Can you say the function has a limit x c. The answer is no.

So, the converse of this may not be true always; that is if f is a bounded function on some neighborhood of c, then limit of this function f x when x tends to c may or may not exist. For example, if we take the function f x signum of, say, signum of x which we have already discussed signum function, and we have seen that signum x which has a value x value 1 when x is greater than 0; equal to 0 when x is 0, and has a value minus 1 if x is negative. Clear. And this we have seen that this function, the range of this function is bounded. So in the neighborhood of the 0, it is basically a bounded set because we can find a constant M. M is say 1, where all the values of the function clearly f x which is signum of x is a bounded function in the neighborhood of 0.

Because we can identify a M because there exist an M say equal to 1 such that the value of this signum x, it will always remain less than equal to 1 for all x belongs to the delta neighborhood of 0. But as we have seen, but limit of this functions signum of x when x tends to 0 does not exist. Because when x tends to 0 from the positive side, the value will come out to be 1. From the negative side, the value will come out to be minus 1. So, limit is not unique or we cannot identify a delta, such that the difference between f x minus the number say 0 cannot be less than a smaller value epsilon. So, it does not exist. So, what we say is that the converse of this is not; however, if we say the function, if it is an unbounded function in some neighborhood of the point c, then obviously function will not have a limit at that point.

Because when the limit does not exist, there are two possibilities. Either the function is not defined at the point where the limit is required; it means the function goes to infinity or minus infinity when x is approaching to C, or at the point C the function is not at all a finite value. A second case may be the function has a finite value, but when limiting value of this has two values or more than two values along a different path. So, in that case the limit will not be unique. So, we say the limit does not exist. So when you say f is unbounded, it means at the point c the function is not defined at that point. So, it is just like 1 or 1 minus x. At x equal to 1 the function it tends to infinity, in fact it is not defined; it goes to infinity unbounded function.

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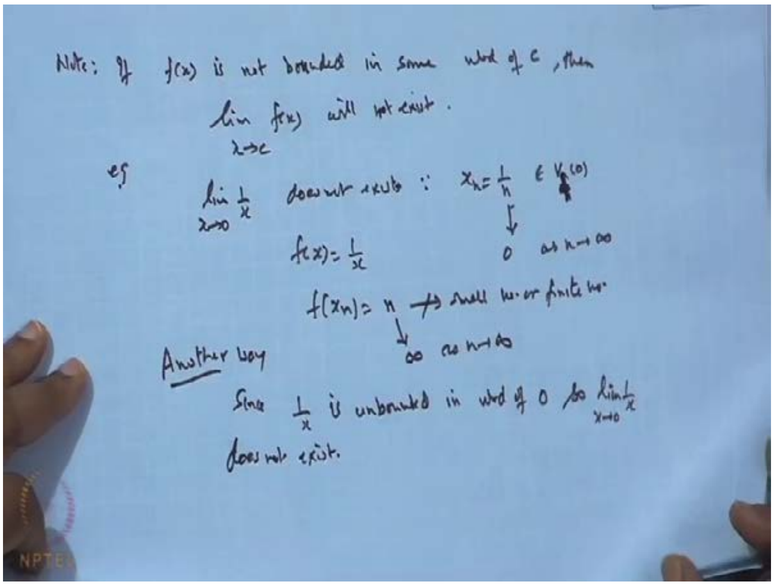

So, we can say as a remark or note: If the function f x is not bounded in some neighborhood of say C, then limit of this function f x when x tends to c will not exist. So, that is clear. So, that will be one of this and the reason is for example, if we take the limit of 1 by x when x tends to 0. Suppose I wanted to know whether the limit exist or not; so this limit does not exist. Why? Because we have already seen; because if we take a sequence x n say 1 by n which is belongs to the some 1 by n neighborhood or delta neighborhood of 0 say delta is such. So, that is the limit; all the terms of the sequence x n belongs to it after certain a stag, say delta neighborhood of 0.

Then this sequence tends to 0 as n tends to infinity. So this is sequence, but what about the function? Function f x is giving to be 1 by x. So, what is the functional value? The functional value is comes out to be n, which does not go to the smaller quantity, a smaller number or finite number. In fact it will go to infinity. It goes to infinity as n tends to infinity. So, limit does not exist. But this can also be justified from here. Another way of justifying: Let since 1 by x, this function is unbounded in the neighborhood of 0. So, limit of this function 1 by x x tends to 0 does not exist. So, this is one way. Now we have few results, theorems which are parallel to the theorem sequences.

Just like in case of sequence we have established some results. The sum of the limit of the sum of the sequence if the sum of the limits difference product and like this; similarly, the similar results holds good in case of the function. In fact we have seen one result. That if the limit of the function f x when x tends to c exists say L, then it can be equivalently written in terms of the sequential form. That is we can get the sequence x n which goes to c and x n is different from c such that limit of this f of x n will go to L. So, all the results which are valid; the proof of all the results are exactly parallel as we have done in case of the sequences; however, we can also establish that prove with the help of epsilon-delta definition. So I am just stating the results with a proof, because it goes runs as earlier as follows on the same lines as we have used in case of sequences.

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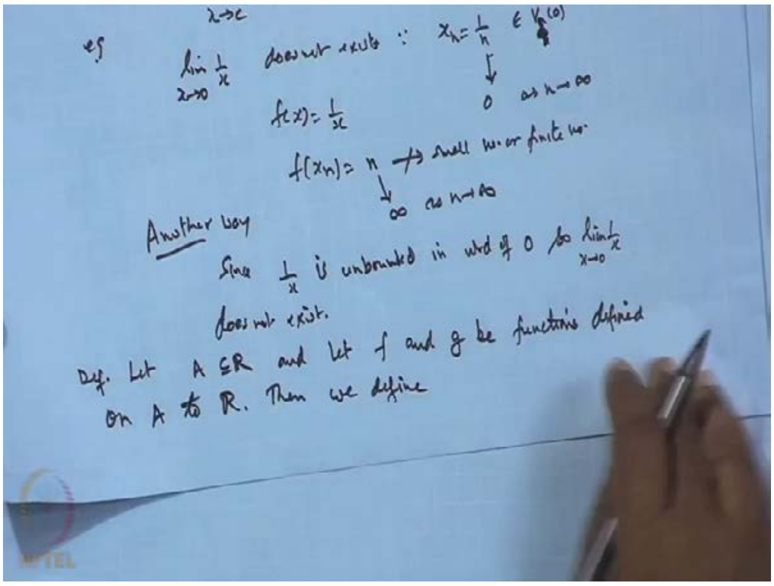

So results are, before going for the results, we define of course it is a very obvious thing, but I will complete it. Let A is non-empty subset of R, and let f and g be functions defined on A to R. Then the addition, subtraction, etcetera, we define as follows.

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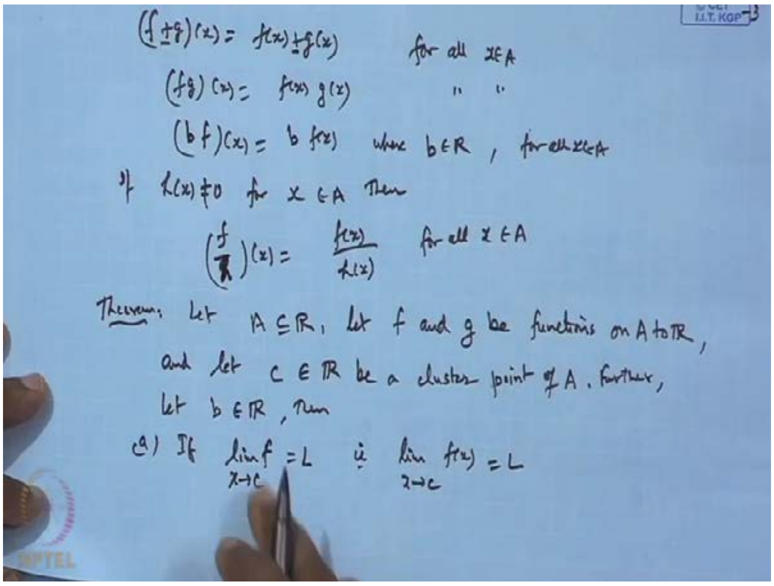

Then we define the, sum of the two functions f plus g x as f x plus g x. The difference of this is defined like this, and the product of this is f x into g x. If b is a constant, then b of f x is defined as b of f x where b is some real number R for some constant R real number; and if h x is not equal to 0 for x belonging to A, then we can define the sequence quotient f by h x as f x over h x for all x belonging to A, and this is also true for all x belonging to A.

So, this way we are defining. So, similar results, theorems also limiting theorems can also be written like this. Let A which is subset of R, and let f and g be functions on A to R, and let c which is point in R be a cluster point of A. Further, let b is a real number; any real number belonging to R, then the following result we can say; if the limit of f when x tends to C is L. Meaning is that is limit of f x when x tends to c is L. So, I am dropping the x away. So, that limit is this and limit of g when x tends to c is M say.

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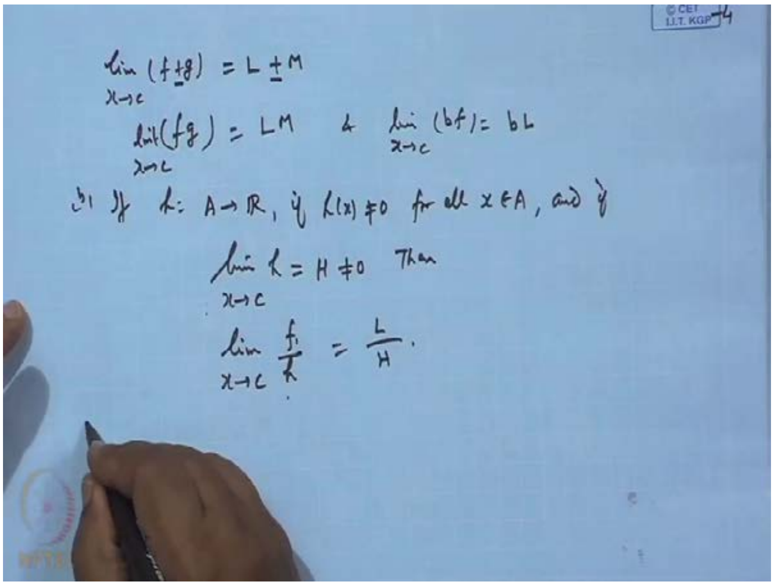

Then the limit of this sum f plus g x tends to c will be L plus M, limit of the difference will be L minus M, limit of the product f g as x tends to c is L M, and limit of this b f x tends to c is b times L; and further if h is a mapping from A to R a function and if h is not equal to 0, h x is not equal to 0 for all x belonging to A, and if the limit of h exist; limit of h when x tends to c is h which is different from 0. Then the limit of this ratio f over h when x tends to c is nothing but L by H.

So, proof is ok. Now here we will make one remark. The remark is in the part b, we have put a restriction that limit of h x when x tends to c should be different from 0. Because if it is 0, then this limit is not defined; however, even that suppose f is also 0, h is also 0, then we cannot define the limit of f over h. Because that comes out to be a indeterminate case. The limit may or may not exist. If it is suppose limit L is also 0 and h is also 0; in that case, when you substitute the value or take the limit which is come out to 0 by 0.

You cannot say the limit is 1 or limit is 0. It may be anything. So, all may not exist also. So what we say is then whenever this limiting behavior of the function L by H is different from 0; and particular in the case of ratio when they are different from 0 h is different from 0, then only we can say the limit exist. Because if h is different from 0, even L is 0, then limit will come out to 0. But if L is also 0 and h is also 0, then the problem occurs. So, we put this restriction on it. So, that is what we may not.

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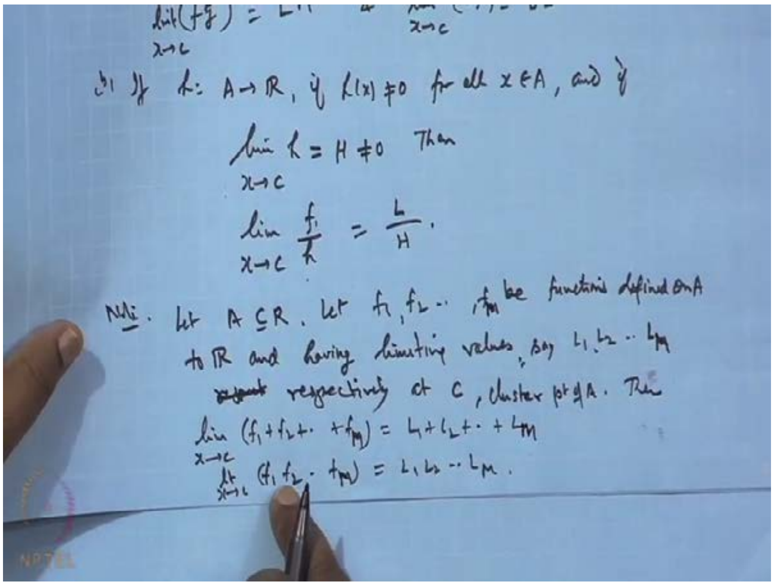

Another note; So, we can say note: Suppose they are all the functions which are defined on A, and having let f 1 f 2 say f n be a functions defined on A to R, and having and see with the cluster point of this, and having the limiting values say L1 L 2 L n respectively. When at the point limiting value at the point c which is a cluster point of A, then these limits of this f 1 plus f 2 plus f n when x tends to c will be the sum of these limits; and similarly say f n, so here n and when you find the product of this, again when you take, and here it is I would say it as the, so here it should not be; suppose it is M.

So, here is also M, this is M. Otherwise it will confuse because limiting value is taken once again. So, when x tends is then M and this is M and this comes out to be L 1 limit of this x tends to c comes out to be L 1 L 2 L M. And as a result when all L 1 L 2 L n are same, then it comes out to be x takes to the power n when these are all identical as a particular case.

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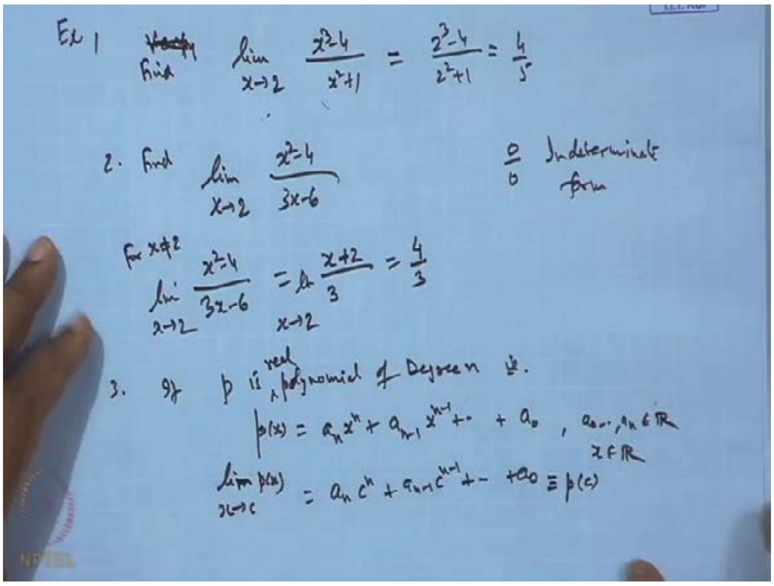

Now let us see this few example based on these previous results. Suppose we wanted to show say limit of this; verify the limit of this x tends to say 2 x cube minus 4 over x square plus 1 or find the limit or find the limit of this. Now it is not asked to use the epsilon-delta definition. So, what we do is, because it is of the form f x by h x h x is x square plus 1 which is different from 0 in the neighborhood of 2; so obviously, we can use the result, we can find out the limit of this function, and then value. So, when getting the limit of this functions substitute x tends to 2 means it will go to 2 cube minus 4 over 2 square plus 1 and that comes out to be 4 by 5.

So just substitute it, provided everything goes well. If I take the limit of this say x square minus 4 over 3x minus 6 when x tends to 2, find the limit of this? Now here when you substitute x is 2, the denominator is vanishing. So, it was not defined for that. Similarly, when you take x equal 2, the numerator is also obtained. So basically, it is coming to with the 0 over 0 form; when it should be placed x equal to 2, which is known as the indeterminate case, indeterminate form.

That we will discuss later when we go for the L' Hospital's rule you will know that, how to compute this value of such a ratio which comes in the form of indeterminate. But here this is a function, very simple function. We can also use some trick to find out the limit of this. Obviously, when we say the x tends to say 2; the numerator is 0, denominator is

0. It means there must be one factor involving x minus 2 in the numerator, as well as x minus 2 in the denominator. Then only numerator and denominator both are tending to 0.

So, remove that x minus 2 factor; so if I write the function x square minus 4 3x minus 6 for x different from 2, then what happen? We can write this thing as x plus 2 over 3, just because x minus 2 get cancelled, and then when you take the limit of this now as x tends to 2 it is the same as limit x tends to 2 which comes out to be 4 by 3 and that are solved. So, that way we can easily find. Now, third is if suppose p is a polynomial of degree n, that is p x is of the form real polynomial of degree n. So, it is of the form say a n x to the power n a n minus 1 x n minus 1 plus a naught, where a naught a 1 a 2 a n; these are all real numbers x belongs to R.

So, p x is a polynomial of degree n. If we are interested in computing the limit of this p x when x tends to c, then obviously it is the sum of these functions, because individual each one is a function. So, when you take the limit of this as x tends to c then x to the power n will go to c to the power n. So, it will come out with the a n c to the power n plus a n minus 1 n minus 1 plus a naught, which is equivalent to the polynomial p at the point c, a constant p c. So, limit of the polynomial p x when x tends to c is here.

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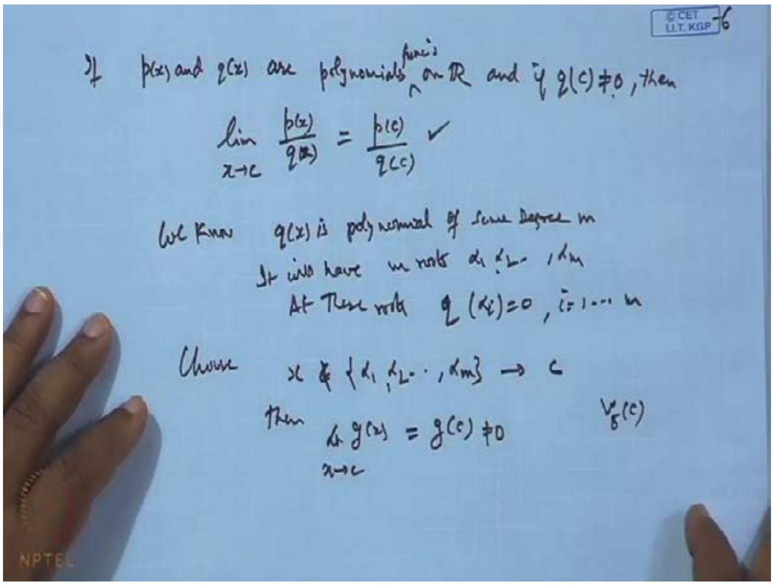

Then if suppose we have a two polynomial p x and q x. If p x and q x are the polynomial then polynomials on a function R and if the polynomial at the point c; q polynomial at the point c is different from 0 as a value different from 0, then the limit of this ratio p x over q x when x tends to c is equal to p c over q c. So, what we do here is we sorted out for the point. p x is a polynomial of any degree say n and q x is a polynomial of degree say m and it is given that q at the point c is different from 0. But that does not solve our problem. Why? Because we know q x is a polynomial of some degree, say degree say, suppose I take say m; it means it will have at most m roots. So, it will have m roots. Some may be repetitive, some real, some compared like. It may have m roots.

It means at these roots alpha 1 alpha 2 alpha m; at these point at these roots, the value of q must be 0. I am assuming all the roots be distinct and real, say distinct and real. So, q must be 0. So it means when you approach to c, then path should not contain these point; because if it contains this point, then you cannot find the limit of q x when x tends to c. So, what we do is we separate out. We choose x which is not in this set alpha 1 alpha 2 alpha m which is different from this, and then this x is tending to c; so obviously, if I choose all the point in neighborhood of the delta neighborhood of c, which does not involve these points.

Then the limiting value of q x when x tends to c must be greater. Than the limit of g x when x tends to c in that case it is equal to g x and g x already given to be non-zero; so this one. Once it is non-zero, then you can apply this result and get this. So, when you are very careful when you find the limit of this p x over q x. The point should not be available where the q has this one roots. Those point in that neighborhood or the roots of the q x must be out. Then only you can get the limit of this. So, that is the one thing which we have.

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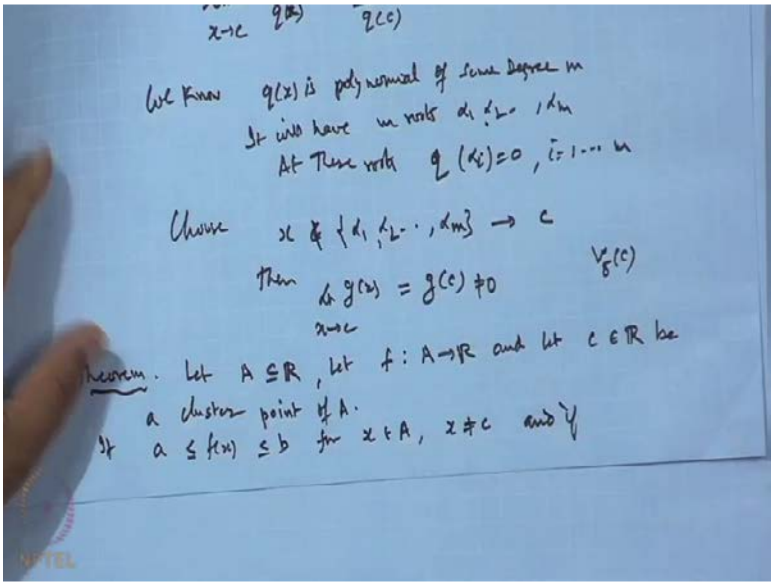

Now this result is also interesting. The result says let A which is subset of R, and let f which is a mapping from A to R, and let c belongs to R be a cluster point of A; and suppose if A is less than equal to f x less than equal to b, means the all the images of f in the neighborhood of c lies between these f x is for all x belongs to A and x is not equal to c. All the point f has a image over the set A excepted c which lies between a and b.

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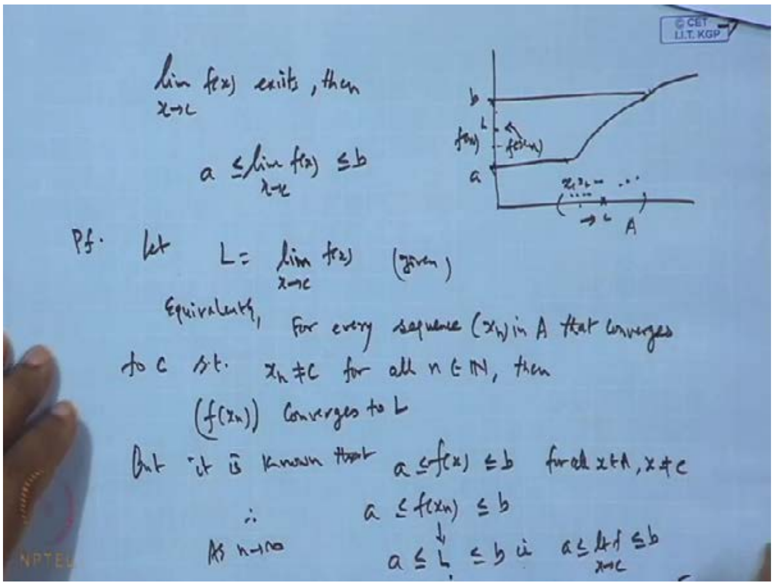

Then and if limit exists and if limit of this f x as x tends to c exist, then the limit of this f x when x tends to c will lie between a, and b; means the lower bound cannot be f x limit of f x should be lie between the two bound a, and b. So what it says is, if suppose we have this set A. Here is some c and this is a function say f. There it has a lower bound a and this is a upper bound b for this function may be something. So what it says is, that if all the images f x lies between a and b and function is also have a limit at the point c, then the limiting value of this say L will lie between a and b. That is what.

So, proof is simple. Assume that given f is the limit. Let L is the limit of the function f x when x tends to c. This is given. Because limit exist means this will be given. Now as we have seen that limiting epsilon-delta definition is sequential way, conventional way is also the same. These means we can also define the limit of the function by means of sequence, and the sequence limit says that this result if I go through, what this result says. This is equivalently we can say for every sequence x n in A that converges to c such that x n is not equal to c for all n belongs to N rational number, then the sequence of the functional value f of x n; this sequence converges to L.

So, when we say the limit of the function f x when x tends to c is L. The equivalently we can also say, that there will be a sequence x n in A; means x 1 x 2 x n these are available in this or from here also, which are different from c. Then the corresponding images f x 1 f x 2 f x n, this sequence will converges to L. This is the way, equivalent definition of this. So using this definition, we can now prove like this. So, let us suppose a sequence x n in A exist that converges to c, but all the point x n is different from c then the corresponding f x n will converge. Now what is given is, that f x will always lies between. This is given; f x always lie between a, and b but given. It is known that the f of x will always lie between these two bound for all x belongs to A and x is of course not different to c.

Now here x n is in A, but they are different from c. So for such x n, this value f of x n will also lie between a, and b, is it not. So, when this two bound is there. So, when you take the limit of n as n tends to infinity. This sequence f x n will go to L and this will now lie between a, and b. So, limit of f x n when n tends to infinity; that is a is less than equal to limit of f when x tends to c is less than equal to b and that is proved the result, is it not. That proves the result.

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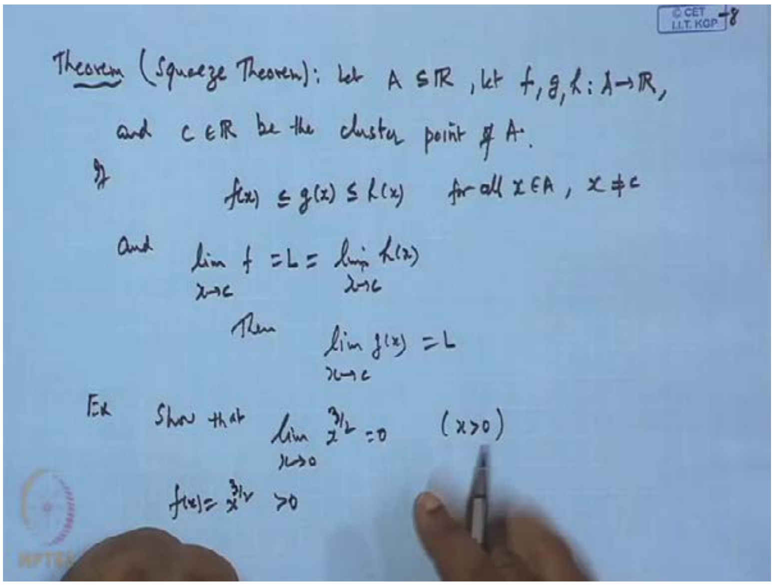

We have another result which is the Squeeze theorem. Just like as in case of a sequence we have shown, then similarly for functions also we have the theorem which is known as the Squeeze theorem. What this theorem says? Let A is a subset of R non-empty subset of R. Let f, g, h is a mapping from A to R and let c be the cluster point c belongs to R be the cluster point of A. Now if this inequality holds f x is less than equal to g x is less than equal to h x holds for all x belonging to A except say c x is not equal to c and if the limit of f as x tends to c is L which is limit of h x when x tends to c. Both the limits exist and have the same value say L. Then this Squeeze theorem says the limit of g x when x tends to c will also be L. So, this was proved earlier also and even it follows from the previous results also, because the limit of this is L, limit of this is b.

So a, and b are there and this g x always lie between a, and b for all a. Is it not? So therefore, by Squeeze theorem, by previous result also, limit will exist and since both are equal left and a, and b are; so this limit will also be L or otherwise also g x minus f x you can write it and h x minus we can get the results for it like this. So this is nothing to much prove. Let us see the use of application of the Squeeze theorem. Here we can apply the Squeeze theorem to get the limit of a function easily, quickly. Suppose I would say, show that limit of x to the power half when x tends to 0 is 0. We have already seen when x to the power n when n is a positive integer, then it can be written x into x into x up to n and limit x tends to 0 it will be 0, but when this power is not integer then x raise to it as a fraction. Then you cannot write in the form of x into x or something like that. We do not.

So, we have to show that. Of course this will be discussed when we did for the Cantor's theory another; that x to the power alpha, alpha is rational, alpha n is a sequence converges then alpha will go. That is already discussed, but still we are using this Squeeze theorem we can prove this; so how. So, suppose f x is basically x to the power half for all x, this is we want this as 3 by 2 let it be. For this x is greater than 0, so that this is greater than 0. Now f x is 3 by 2 when x is positive it is positive throughout.

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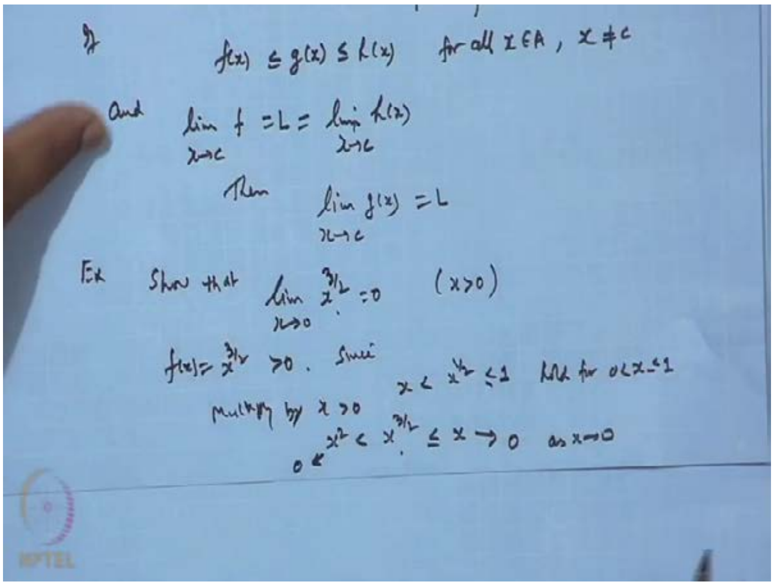

Now we have this as since x is less than x raise to the power half which is less than 1 or may be at the most equal to 1 holds for 0 less than x less than equal to 1. This result is true. So, if I further multiply by x because x is positive. So, it will not reverse the inequality. So, what we get is x square is less than x raise to 3 by 2 is less than equal to x Now, apply the Squeeze theorem. Here the f x is this which tends to 0. This g x h s is also which tends to 0 as x tends to 0. Therefore, limit of this has to go to 0. So, this proved.

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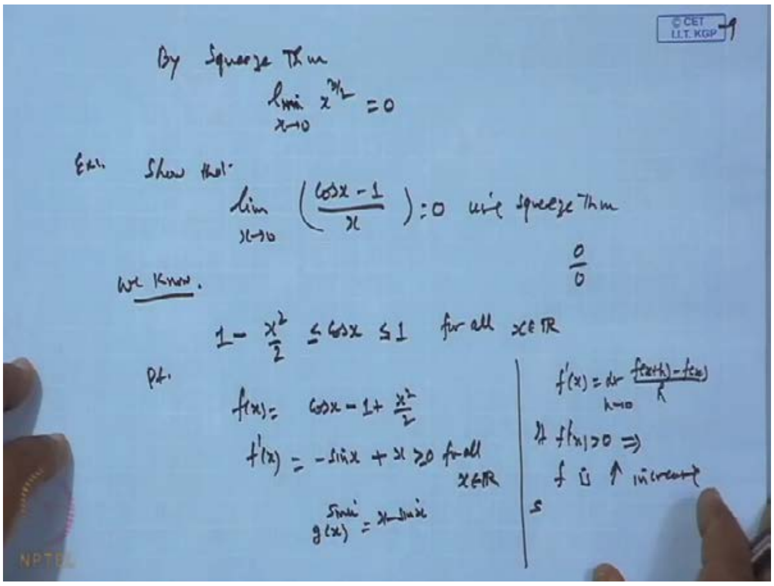

So, this shows that by Squeeze theorem limit of this x raise to the power 3 by 2 as x tends to 0 is 0. Similarly second example: Suppose I say prove that limit of this or show that limit x tends to 0, cosine x minus 1 by x; this limit is 0 using Squeeze theorem. Now here, we cannot use that p x by q x form. Why? Because q x is 0, p x is also 0. So basically when we substitute x is 0, then it comes out to be the 0 over 0 form. So we cannot apply the earlier previous result, that limit of f x is L, limit of g x is L, then limit of f by g is L by M. This we cannot use it. So, let us use the Squeeze theorem. What the Squeeze theorem says, that we have to identify this bound for cosec and sin.

Now if so it we know or this we can prove it later on what this result is to; minus x square by 2; 1 minus this is less than equal to cosine x is less than equal to 1 for all x belongs to R. In fact this result is valid. The reason is like this. Of course which we have not discussed it which will come when we discuss the differentiability and the application of derivatives. But still let me just finish. When we say the derivative f prime say x, this is the meaning of this is f x plus h minus f x by h when x tends to 0. This is if this limit exist we say derivative function is differentiable and has a derivative f prime x. Now if this function, if this is positive greater than 0; then obviously, this has to be greater than 0, this has to be greater than 0.

The reason is because if it is not, so then the negative sequence of the positive quantity; when you take the limit greater than 0 it may be at the most 0. But it never happened that a sequence has a negative term and limiting value is always greater than 0 is not possible. So from here if f dash is greater than 0, then I will show it when we go for the differentiability. This will implies that function f is an increasing function. So, that is the one which I will prove it when we go for the derivative, application part of the derivative. But here I am using this one. So, use this thing. If I construct this function f x as cosine x minus 1 plus x square by 2; now if I take the derivative of this, derivative of this becomes minus sin x plus x. Now x minus sin x, this will always be positive for all x greater belongs to R or may be equal to 0. Why?

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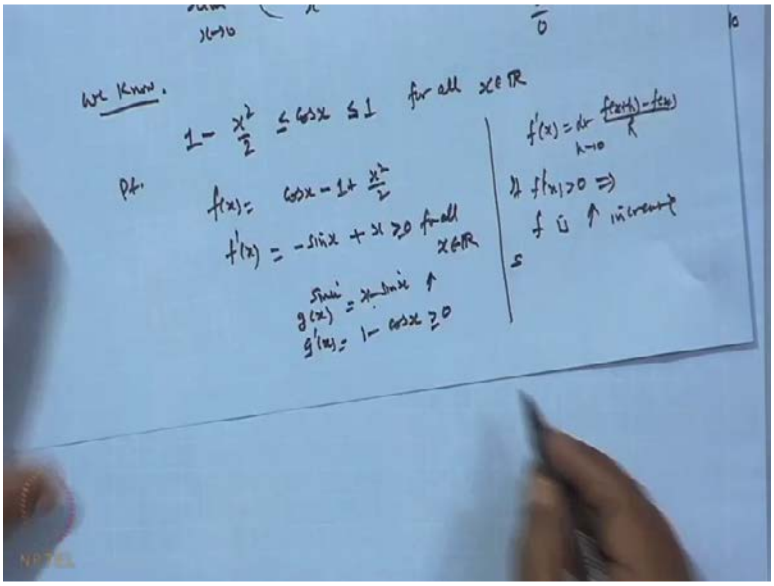

The reason is, that since x minus sin x if I take this function as g x and then differentiate it g prime x we get one minus cosine x. The cosine x value will always be bounded by 1. So, it will always be greater than or at the most equal to 0. So, g is an increasing function.

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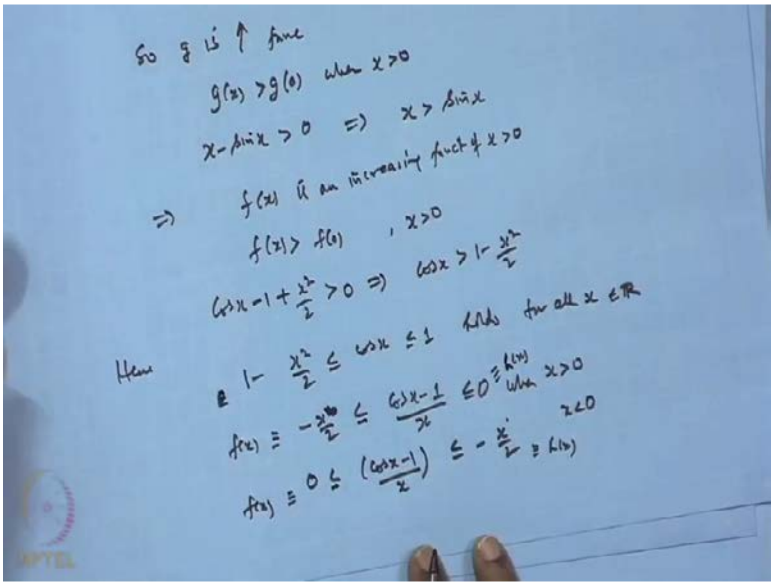

So, once it is increasing function, the value of the function g; so g is increasing function. So once it is increasing, then the value of g will greater than 0 g when x is greater than 0. So when x is greater than 0, we get x minus sin x is greater than 0 because the functional value g is 0. So, this implies that x is greater than sin x. It means sin x will always be less than x. So, this is greater than 0. So, this implies the function f is an increasing function of x when x is positive. So when it is increasing function of x, then the value of the f x is greater than f of 0 when x is 0. So when x is 0, f x is what? Cosine x minus 1 plus x square by 2 is greater than f of 0. When you take f of 0 then x is 0 means 1 minus 1 is 0; so this is 0.

This implies that cosine x is greater than 1 minus x square by 2. So, this side is proved. Cosine x is always be less than equal to 1 because it is bounded function, bounded by 1. So, this result is true. Now for x to be negative, in a similar way we can show it the inequality holds. Is it not? So even if x is negative, what happens to this? We can write this plus sin x and this. So, we can further show that this holds; means it is increasing function we get. So if it is true, then from here. Hence 1 minus x square by 2 is less than equal to cosine x is less than equal to 1 holds for all x belonging to R. So, if it is there then we can write it for.

So let x is positive, then we can write this for as minus x square by 2 is less than equal to cosine x minus 1 by x which is less than equal to 0 and when x is negative, then we can write as 0 is less than equal to cosine x minus 1 by x less than equal to minus x square by minus x by 2, x we are dividing. Now when x is positive, then there are nothing change; you can just transfer it 1 here. So, cosine x minus 1 is greater than this and divide by x we get this one, 0. When x is negative the inequality reverse. So, inequality reverse means 0 will come over here and this will come this side and we get this result. Now take the limit of this. So, what happen by this Squeeze theorem? Here this is our f x when x is 0; sorry this is our f x this one is Squeeze theorem says what? This is f x g x and h x. So, this will be f x, this is our h x, and in this case this is our f x and this is our h x. Now if I apply the Squeeze theorem, the limit of f x when x tends to 0 is 0. Here it is already 0. Limit of h x when x tends to 0 it is 0. So, here it is 0.

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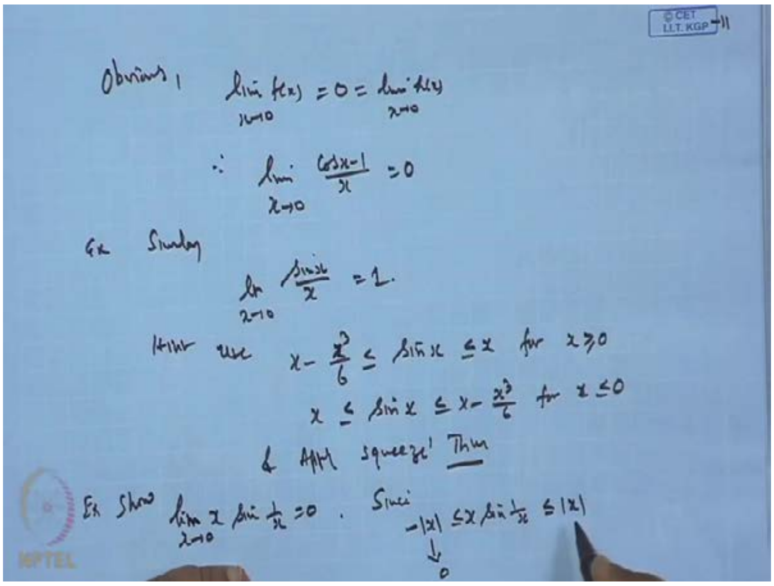

So obviously, limit of f x when x tends to 0 is the same as the limit of this h x when x tends to 0. Both come out to be 0. Therefore, limit of this cosine x minus 1 by x when x tends to 0 will come out to be 0. So, that is the one and similarly, the similar result says we can show for our sin x. Similarly we can prove that limit of sin x by x as x tends to 0 is 1. This I am just dropping, but hint is use this inequality x minus x cube by 6 is less than equal to sin x, which is less than equal to x for x greater than equal to 0 and x minus x is less than equal to sin x, which is less than equal to x minus x cube by 6 for x less than equal to 0. So use this and apply Squeeze theorem, you will get this result quickly. So, that is it.

Now let us see one more last example which we wanted to show the limit of x sin 1 by x. So, that limit of this is 0. Now again we will apply this Squeeze theorem, sin of z will always be bounded by minus 1. So, clearly we have this result. Since x sin 1 by x, this will always be dominated by this two bound. We can say this thing is true because sin 1 by x is lying between minus 1 and plus 1. So, when you take x positive x negative will lie between minus mod x to plus mod x and then this part is tending to 0, this part is tending to 0. So, by Squeeze theorem limit of this will go to it. So, by Squeeze theorem this limit will go to 0 and that is we wanted to prove it. So, and this result are used very frequently. These limits while constructing a function; for a continuous function, differentiability function, etc., there may be used as counter examples also to show the function is continuous, but not differentiable and like this way.

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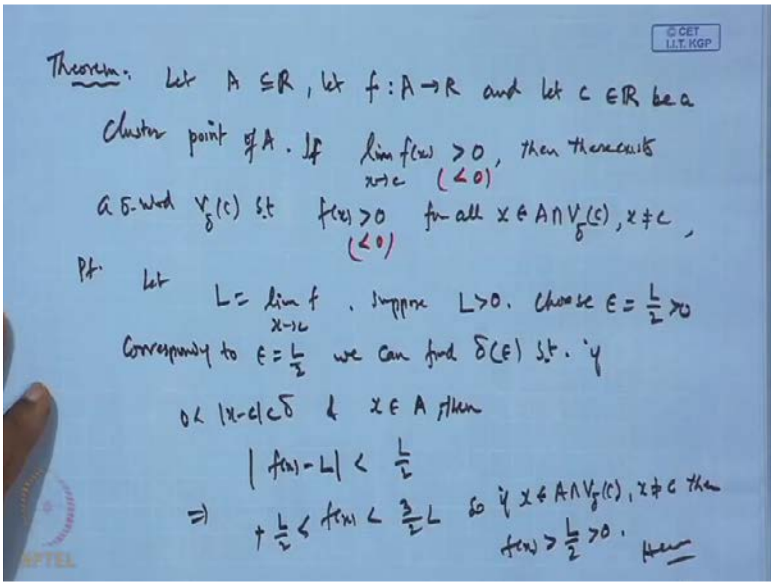

Now we have the one more results, and that result in the form of theorem. The theorem says, let A which is sub set of R, non empty sub set of R. Let f is a mapping from A to R and let us c is in R be a cluster point of the set A and suppose f is if limit of this f x when x tends to c is greater than 0, then there exist a neighborhood V delta c, delta neighborhood of c such that f will be greater than 0 for all x belongs to A intersection delta neighborhood of c and x is not equal to c.

So what it says is if the function has a positive limit in some neighborhood of the c then f x as a positive limit when x approaches to c then there would be neighborhood when the function is also positive. Similarly, if the limit of the function is negative, then the correspondingly there exist a neighborhood such that this will remain less than 0 for that. The proof is this. Let L is the limit of this function f when x tends to c. Suppose I would say this and suppose that L is greater than 0. Then choose epsilon to be L by 2 because we can take any epsilon greater than 0. So, I am choosing epsilon.

Now use the epsilon-delta definition. So, corresponding to this epsilon which is L by 2 we can find a delta which depend on epsilon of course, delta such that if 0 is less than mod x minus c less than delta and x belongs to A, then the f x minus L remains less than epsilon. Epsilon is what L by 2. Therefore, this implies what; f x lies between 3 by 2 L and greater than what; minus L by 2, f x is lying between f x minus L. So L plus L by 2 and then it will lie between minus L by 2 plus L by 2 that is L by 2. So, it will lie between this one. So what we say is, that f x will always be greater than this. So, there exist it follows. So if x belongs to A intersection V delta c x is not equal to c, then f x will be greater than L by 2 which is positive and holds; hence the result, hence the proof.