A Basic Course in Real Analysis
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Lecture - 25
Some results on limit of functions

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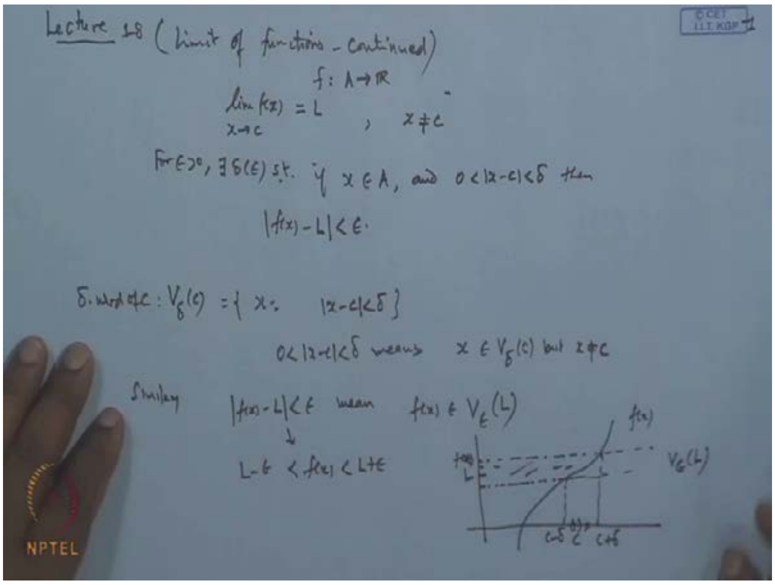

We were discussing the limit of functions and the way we have defined here, we say limit of this function f x when x tends to c is say L, where x is not equal to say c, need not be equal to c. The meaning of this is that for every given epsilon greater than 0; for a given epsilon greater than 0; there exist a delta which will depend on epsilon, such that if x belongs to the set A, A is a any set, f is a mapping from A to R. So, x belongs to A, but x is not equal to c, and x satisfies this condition x minus c is less than delta. So, for a given epsilon greater than 0 there exist a delta depend on epsilons such that for all x which lies between this interval, the value of f x minus L will remain less than epsilon. So, this is the way we have defined this.

Now if we look this definition then it can be easily converted in the form of the neighborhoods, because what is this? The neighborhood of this c or delta neighborhood of c which is nothing but what; the set of those point x such that mod x minus c is less than delta. Now in this if I remove c, then we say the point x is not equal to c. So, 0 is less than mod x minus c less than delta means x belongs to v delta c, but x is not equal to

c. So, it lies in the delta neighborhood of c and the f x minus L; similarly f x minus L is less than epsilon means that f x belongs to the epsilon neighborhood of L. Because the meaning of this is that f x lies between L plus epsilon and L minus epsilon. So, f x lies between this.

It means that if we have this function say, this is our function f x, and here is the point c, the value this is L. So, what he says is if the limit of the function f x when x tends to c be mean for a given epsilon greater than 0. It means if I consider a neighborhood V epsilon L; that epsilon neighborhood of L. Then corresponding to this neighborhood, we can find a delta neighborhood of c; c minus delta c plus delta where c may not be included in it. Then what this limit says is for any arbitrary point x which lies between this interval and different from c, the corresponding image f x will always fall within this line. Then we say limit of the function f x exist.

If it is not, then obviously the limit of the function f x when x tends to f will not exist. It means that, there will exists some epsilon neighborhood of L such that for any x we choose, there will be an x where the limit of the value of the function will lies outside of this range, if it does not exist. So what we conclude is that, we can also write the equivalent definition of the limit in terms of this, we can also express limit concept in terms of the neighborhood.

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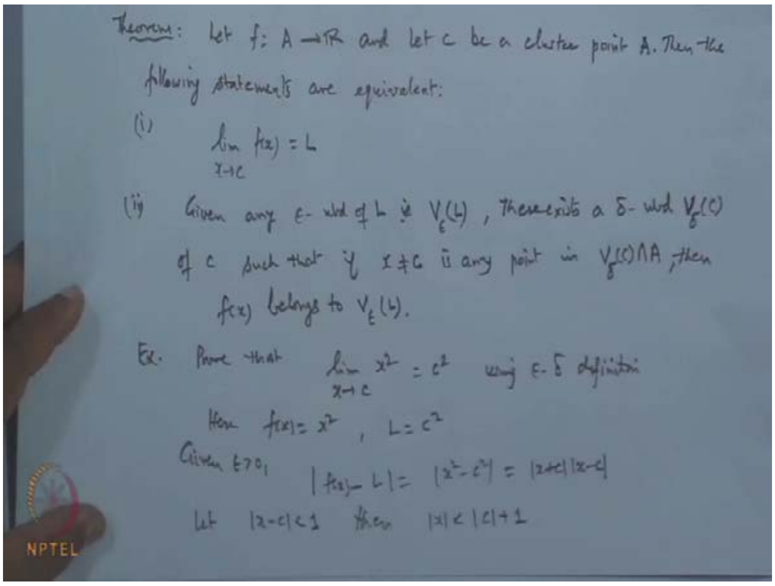

So, this we can write in the form of the theorem. The theorem is let f is a mapping from A to R and let be a cluster point of A, then the following statements are equivalent:

The first statement is limit of the function f x when x tends to c is say L, and second statement says given any epsilon neighborhood of L which we denoted by that is V epsilon L; V L there exist a delta neighborhood denoted by V delta c, delta neighborhood of c such that if x is not equal to c is any point in the common portion of V delta c intersection A in a part of this. Then f x belongs to epsilon neighborhood of L and that what I explained in this figure is that; for any epsilon greater than 0 of or in f c any epsilon neighborhood of L, correspondingly we can find out the delta neighborhood of c such that if x which is different from c lies in this interval then the image will fall here. That is the meaning. So, both are equivalent definition.

Let us take few examples using the epsilon delta definition. Suppose I wanted to show that, prove that the limit of the function x square as x tends to c is c square, we using epsilon delta definition. So, what is our function? So here the function f x is x square, L is c square. So, what we want is for epsilon we will choose first. So, let epsilon be given. So, given any epsilon greater than 0, then identify the delta. We have to identify delta, so that this condition holds. It means we want this difference f x minus L should lie; f x should lie in this neighborhood. So, what is the f x minus L.

So f x minus L, L means c L that is equal to x square minus c square; basically, this is x plus c into x minus c. This will be there. When we take x sufficiently close to c, then the limit of this must be c square. So, how close c is; means any number which is less than 1 we can choose. So, suppose I take; let us take mod x minus c less than 1 first. Why? Because this x; in this case x is lying or close to c which is less. So, if I take this one then the bound for this x plus 1 then we get from here is or mod x can be written as this because mod x minus 1 is less than 1. So, it is greater than equal to mod x minus mod c and then mod x will be less than is mod c plus 1.

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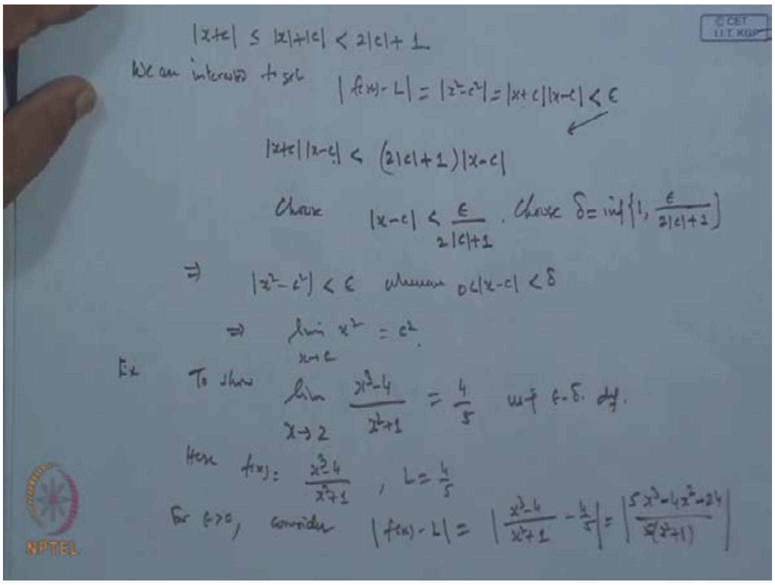

Therefore, mod x plus c; the value of this will be less than equal to mod x plus mod c, but mod x is less than mod c. So, it is mod c plus 1. This we get it from here. Just this by manipulation we are doing, so that we can get the bound for it. Now what we are interested is, we are interested to get this f x minus L which is the same as x square minus c square, which is the same as x plus c x minus c to be less than epsilon. We are interested in this. Now, x plus c bound is already less than 2 c.

So, basically this mod x plus c mod x minus c; when mod x minus c is less than 1, then the bound for mod x is obtained from here. So, we can get this is less than basically two times mod c plus 1; this is less than mod two times mod c plus 1 into mod x minus c. Now this entire thing we wanted to be less than epsilon, this we wanted. So, if I choose mod x minus c to be less than epsilon over 2 times mod c plus 1 and then use this bound here. Then what we get is, this remains less than epsilon. It means this will imply that mod of x square minus c square will remain less than epsilon. Is it right?

So, that is f x minus L is less than epsilon, but what should be the delta. So, it means that if we pick up from here, then choose delta to be the value which is infimum of 1 and epsilon over 2 mod c plus 1. Because you are getting the 2 bound for delta. One is x minus c is less than delta; delta is 1. Here also you are saying x minus c is to this. So, if I choose the delta which is the minimum of these two, then for this particular delta, this result also will hold.

That is for a given epsilon, we can identify this delta that if any x belongs to this mod x minus epsilon less than delta. Then this will continue whenever mod x minus c is less than delta and of course greater than 0 because it is not. So, what they show? This shows the limit of the function x square when x tends to c is c square; that is the answer for it. So, main idea is when it is asked to prove or estimate the limit by using the epsilon-delta definition, what we consider; we start with the given epsilon and then with the help of epsilon, we will identify delta which depends on epsilon.

So, the used idea is that find out f x minus L. Try to get the bound for this in terms of f mod x minus c less than some number. So, this bound can be obtained as this. So, delta we can identify. Once you identify delta; obviously, it depends on epsilon. So, corresponding to this epsilon if I choose this delta, it means if I take this neighborhood x minus c less than delta. So obviously, this delta is also less than 1 as well delta is less than this number. So, we can take the bound. We can definitely find out the bound for this and one can say this limit, this difference is less than epsilon. Hence the proof is here. So this is the way to compute, so the limit in by using the epsilon-delta definition.

Let us take one more example where we are also using some trick to get. Suppose I asked to find to show limit of this say x cube minus 4 divided by x square plus 1 when x tends to 2 is 4 by 5. Suppose we are interested to show this using epsilon-delta definition. Suppose I do not say using delta because this is true. So, you can substitute x equal 2 and one can get the value of this easily; but if it is asked to you like the epsilon-delta definition, then you have to start with the given epsilon greater than 0 and then suitably identify the delta. That is it.

So, what is there? This is our function f x. Here f x is x cube minus 4 over x square plus 1 and L is 4 by 5. So let for the given epsilon greater than 0. Let us consider first, the difference of f x minus L. This f x minus L is x cube minus 4 x square plus 1 minus 4 by 5, and if we find that solve it, then the value will come out to be 5 x cube minus 4 x square minus 24 divide by 5 x square plus 1, this one. Now here, the numerator is one unit higher than the denominator. So, what we do; first, we separate out the factors. Now if you look the denominator numerator x equal to 2 satisfied here; x is 2 means 80 minus 16 minus 24. Sorry. So 40 minus 16 minus 24, it satisfies. It means x equal to 2 will be a factor of this equation equal to 0. So, we can separate out the x minus 2 factors from the numerator.

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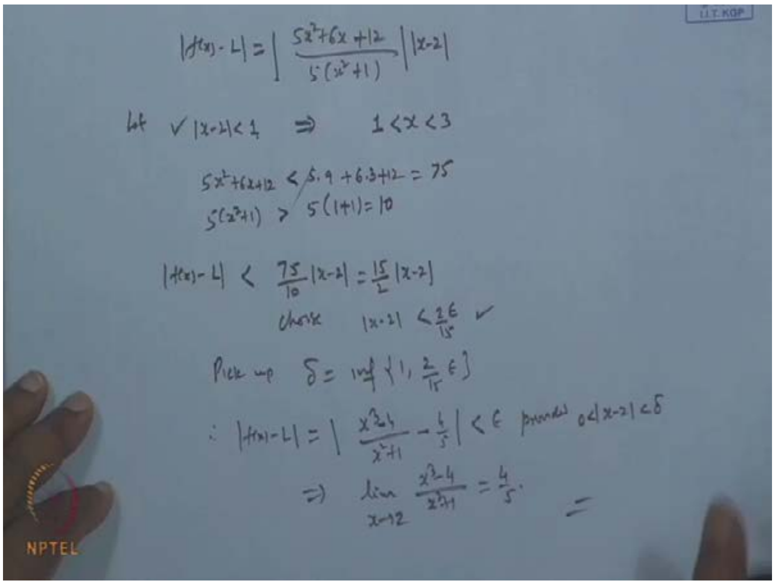

So if I separate out, the things will come out to be mod of f x minus L. This is nothing but mod of 5 x square plus 6 x plus 12 divide by 5 x square plus 1 into mod of x minus 2. Up to here, there is no problem. Now we wanted to estimate the limit of this is. In fact, f x minus L must go to 0 when x is sufficient; 10 into 2, this is our problem. So, it means x minus 2 should be as small we please and x may not be equal to 2 also. So, let us find the bound for this first. Bound for this requires the value of x upper and lower bound for x. So, how to get the upper value?

So let us take x minus 2 is suppose less than 1. Let us take this one restriction. So, if you restrict this one then what will be the x. x will lie between 3 and because x minus 2 is less than 1. So, x is less than 2 or 2 minus x, so again it is. So, it lies between; x lies between 1 and 2 1 and 3. So, this will be the bound for this. Is it not; x minus 2 lies between minus 1 and plus 1. So, minus 1 plus 2 will give this one and we get this.

So, this is the bound for this. Is it clear? We got 2 minus x less than 1, so 2 plus 1. So, that will be greater. So this one; it means x is lying between this. So, what will be the upper bound for this 5 x square plus 6 x plus 12. This will be less than x is 3 bounded. So, it is 5 into 9 plus 6 into 3 plus 12. That is the upper value for this; upper bound for this and that upper bound is we can identify that is equal to what 75 and what is the 5 x square plus 1 because in the denominator; so write down the lower bound of this.

So, it is greater than 5 into 1 plus 1; that is ten. It means this mod of f x minus L can be made less than 75 by 10 mod x minus 2. That is equal to what; is nothing but the 15 by 2 mod x minus 2. So, first bound we have written like this; that is delta 1 you can say. Second bound we will start from here, because this entire thing we wanted to be less than epsilon. So, if I choose mod x minus 2 is less than say epsilon 2 by 15; less than this. So, as soon as you write this thing, then all quantity will remain less than epsilon; so now, picked up the delta as the infimum of 1 and this number 2 by 15, then with this delta.

So, if we take this delta as this, then this condition is satisfied, as well this condition is satisfied; therefore, mod of f x minus L that is equal to what; mod of x cube. The problem was x cube minus 4 divided by x square plus 1 minus 4 by 5. This thing will be less than epsilon, provided the x minus 2 is less than delta and greater than 0 because x may not be true. But here of course is no problem. We can also take 2; then no problem here. So, we can choose this. Therefore, the limit of this x cube minus 4 over x square plus 1 as x tends to 2 is 4 by 5, and that is proved. So, this way we can prove it; that is ours limit problems. So, this is the fourth, third.

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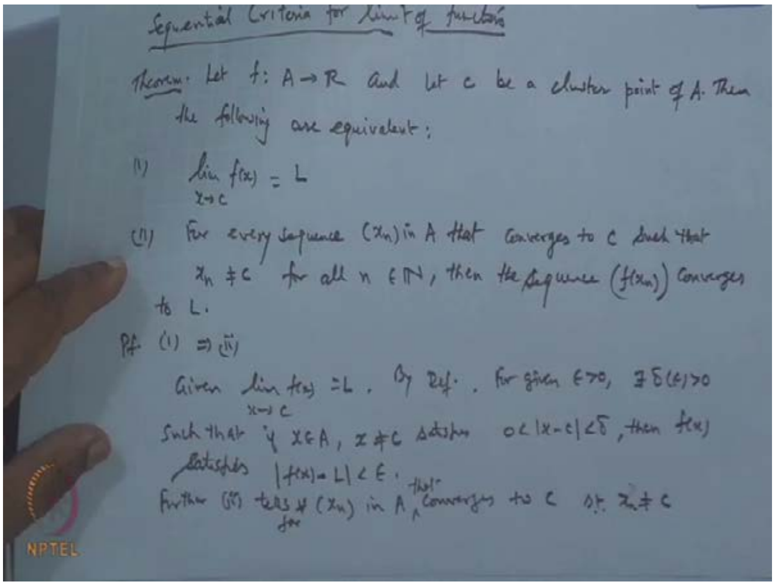

Now there is another way of defining the limit, that is the sequential definition of the limit; sequential criteria for limits of functions. What is this sequence criteria; this is in the form of theorem. The theorem says let f mapping from A to R and let c be a cluster point of A, then the following are equivalent:

The first is limit of f x when x tends to say c is L. Second is for every sequence x n in A that converges to x; converges to c, sorry, such that x n is not equal to c for all n belongs to capital N natural number, then the sequence f of x n; this sequence converges to L. So, both are equivalent in it; means the limit of f x when x tends to c is L means for any given epsilon greater than 0 such that f x minus L is less than epsilon whenever mod x minus c is less and x n is different from c.

This is equivalent to same. There will be a sequence in A which goes to c and then the corresponding image f x n will go to L. That is the proof. So, first we will show; first implies 2. Given limit exist. Limit of f x when x tends to c is L. So, by definition when the limit exist, so by definition; what is the definition is, that for a given epsilon greater than 0, there exist a delta which depends epsilon positive value such that for all x, such that if x belongs to A x is not equal to c and satisfy this condition 0 less than mod x minus c less than delta. Then f x satisfies the condition mod f x minus L is less than epsilon. This is given.

Now further, we wanted to show one implies two, so one is given, that we get this much information. Now let us see the two. From two what is known is, we know the sequence x n that converges to c. So let us identify given further from two, so tells sequence x n in A converges every sequence. For every sequence in A that converges to c; for every sequence x n in A that converges to c such that x n is not equal to c. So when x n converges to c, it means for again for a given say small number say I take the delta. For a given delta we can identify a capital M such that when M is greater than or equal to capital K or capital M, then difference of x n minus c will remain less than epsilon.

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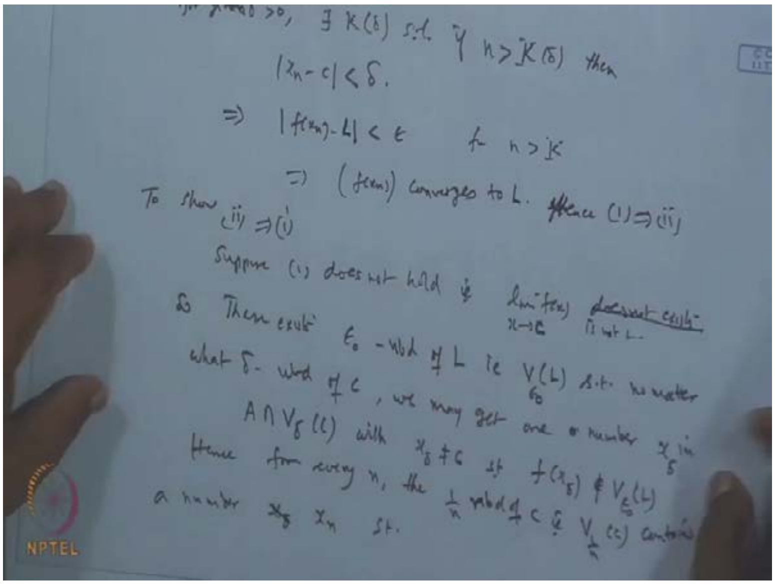

So by definition, so for given delta greater than 0, there exist a K depending on delta such that if n is greater than which depends on delta, then mod of x n minus c will remain less than delta. Is it not; that by definition, when x n converges to c. This is by definition two, but for this particular x n, the result of the previous say, this f x tends to alpha limit says this condition. Then all such x will satisfy this condition the mod of f x minus L less than epsilon. So, this x n will also satisfy this condition.

Therefore, from the previous mod of f x n minus c will also remain less than say f x n minus this is less than epsilon. Is it not; for such x n this will satisfy this condition, and this is true for what; for all n greater than capital K. So it means the limit of f x n, this is L; sorry limit of f x n is nothing but L. So, this shows the sequence f x n converges to L. That is all; second converse part to, so this imply. Hence one implies two. To show two implies one. This we prove by contradiction. Suppose two is true holds, but one is not true. Suppose one does not hold, it means that is the limit of this function f x when x tends to c does not exist or does not go to L, is not L. Suppose this limit is not equal to L. It does not exist or is not L. We can say like this. So, if it is not L or different from L, it means what do you mean by that? It means that if one is not true then there exist.

So, there exist epsilon naught neighborhood of L; that is V L epsilon naught, such that no matter what delta neighborhood of f c. We may we pick up there at least one number, we may get one number say x depends on say delta in the set A intersection V delta c with x delta is not equal to c such that f of x delta does not belongs to the neighborhood; epsilon naught neighborhood of L. That is the meaning of this. When the limit of this does not exist; if I go through again let us say the definition of the limit where we made criteria within this one; first one. When we say the limit of this exists is equivalent to this. It means for a given epsilon that is when we take the epsilon neighborhood of L; correspondingly, we can find a delta neighborhood, such that image of any point inside this and let us think the converse.

Suppose limit is not equal to L does not exist or is not L. It means if we picked up the some epsilon; there will be at least some epsilon naught neighborhood of L will be definitely exist, such that whatever the delta neighborhood we choose, there will definitely one point whose image will not fall inside it. That is the exact meaning of this. So, we get this. This does not fall. It means what; hence for every n the 1 by n neighborhood of c that is V 1 by n c contains a number x delta, this number x delta or x n contains a number x delta that is x n or we can say x n corresponding to this x n.

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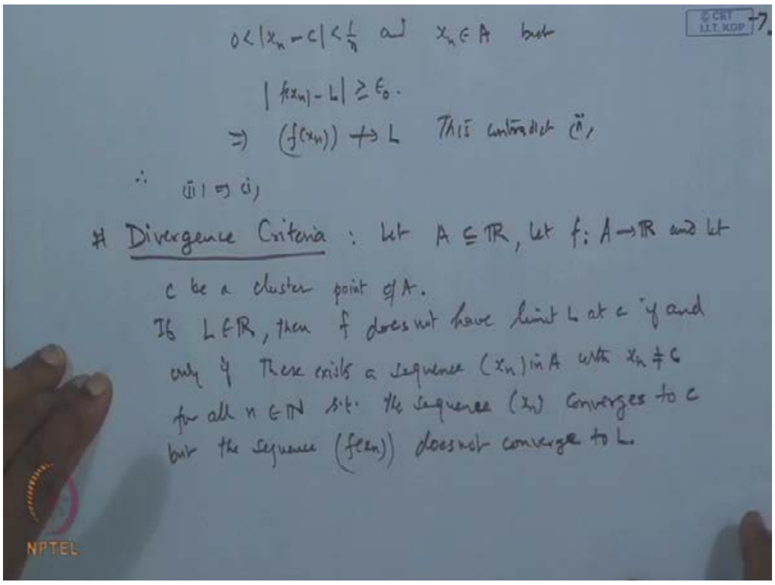

Such that 0 less than mod x n minus c less than 1 by n and x n belongs to A, but mod of f x n minus L is greater than or equal to epsilon naught. That is complete, is it not? So, that does not hold means for any, there exist a epsilon naught neighborhood of L such that whatever the delta neighborhood of c we choose, there will exist one point x n whose image will not fall within the epsilon neighborhood of L. So, that is why. This shows the sequence x n is not, this implies that f x n will not converge, is it not.

So, this implies that the sequence f x n will not converge to any. But that contradiction; that is x n does not converts to, but that is contradiction, because this contradicts the assumption second; because second assumption says that for every sequence x n A that converges to c such that for all N the sequence f x n converges to L. This we are assuming and we wanted to prove one. So, what we have. We have instead of showing that one is two, we are assuming two is two, but one is it does not hold. So, one does not hold that leads the contradiction of the two.

Therefore our assumption is wrong. So if two is two, one will definitely hold. So, definitely two implies one. So, therefore, two implies one. So, this completes the proof. So that is what. Now just like in a sequence we have some criteria for the divergence of the sequence. In a similar way, here also we have criteria for divergence of the limit; that if the limit does not exist, what is the criteria for it?

So, let us see the divergence criteria. Let A be a non-empty subset of R and let f is a mapping from A to R and let c be a cluster point be a cluster point of A. Then the criteria says, if L belongs to R then f does not have limit L at c if and only if there exist a sequence x n in A; there exist a sequence x n in A with x n different from c for all n belongs to the set of natural numbers such that, the sequence x n converges to c, but the corresponding images. But the sequence of f x n means corresponding image is f of x n, the sequence of corresponding images does not converge to L. So, this is the criteria for the divergence. Limit of the function f x when x tends to is not equal to L, it means this will exist.

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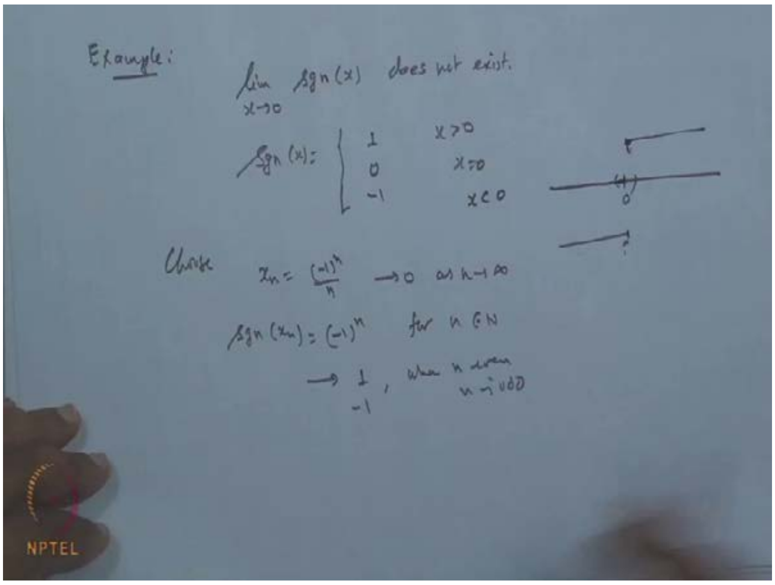

Now let us see few examples where the sequence of the function or the limit of the function does not exist, which are very interesting examples and used very frequently in further study. So, first is the example. The limit of this, obviously the limit x tends to 0 signum of x does not exist. It means what; first the signum. What is the signum of this? Signum function means it is basically sign of the function. So when x is positive this value will be 1; when x is 0 the sign of this we consider just a 0, and when x is negative we put it to a minus 1. So, this is the signum function.

That is the function is obviously, this is our line, so when this 0; when x is 0 the function is 1, x is negative function is minus 1, and at the point 0 we take to be 0. The limit of this signum function when x approaches to 0 does not exist. So, when we approach from the left hand side or we approach from the right hand side, limit will not come out to be the same. In fact before this I should prove that, whenever the limit of the function f x exist the limit will be unique. That I will show just after this. So, here the limit will not be unique. Let us see why? It means if I take a sequence x n which goes to 0, but thus corresponding functions sigma x does not go to that 0.

So, let us choose the sequence x n as minus 1 to the power n; obviously, this sequence will go to 0 as n tends to infinity, but what is the signum of this sequence x n? The signum of this sequence x n when n is obviously even, it is positive. When n is odd, it is negative. So, when the positive n even number, it is a positive value plus 1. When n is negative, it is a negative value with v 1 minus 1 plus 1. So, basically it is the same as this. That is when n is even, you are getting plus 1. When n is odd, you are getting minus 1, and this value limit we get it; so for all n belongs to N.

Now limit of this is does not exist. Because it varies when n tends to infinity, it goes to 1 as well as minus 1, when n sequence approach towards the even when n is odd. So, there are the terms of the sequence which goes to plus 1 even minus 1 odd. So nearby this 0 interval, there is a variation. Variation of two and variation of two it cannot be less than epsilon, because we want this thing to be less than epsilon. I choose epsilon to be half, how the variation can be less than half when the variation is from minus 1 to plus 1; they are exactly two. So this limit does not exist, therefore.

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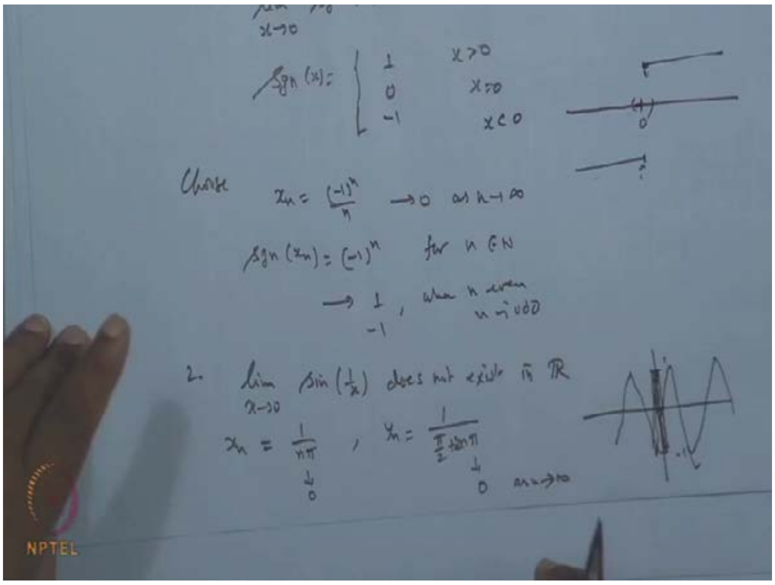

Similarly another example if I take. Suppose I take limit of this x tends to 0 sin of 1 by x does not exist in R. Again sin function; if I look the sin function, the graph of the sin function is something like this. This is 0; it go plus 1 and minus 1 maximum. So, we can get this thing. Sorry, this is like this and something like this. Then coming from here and it is similarly here. But as soon as 0, it fluctuates too much and basically you are getting this thing like this and so on; like this something. This near by 0, it jumps; keep on jumping, going up down, going up and down like this. So, we get the fluctuation and the fluctuation very tricky around the point 0. We are claiming this limit does not exist. It means along the two different paths, if the limit of the sequence has different value and both the paths tending to 0. Then obviously, the limit will not exist.

So, let us take the limit x n. Choose x n to be 1 by n pi, and suppose another sequence y n if i take 1 by pi by 2 plus 2 n pi. Now both the things are tending to 0. This is also tending to 0; this is also tending to 0 as n tends to infinity. So, these sequences are converging towards 0. That is mod of x n minus 0 is less than delta. Now for this sequence what is the f of x n.

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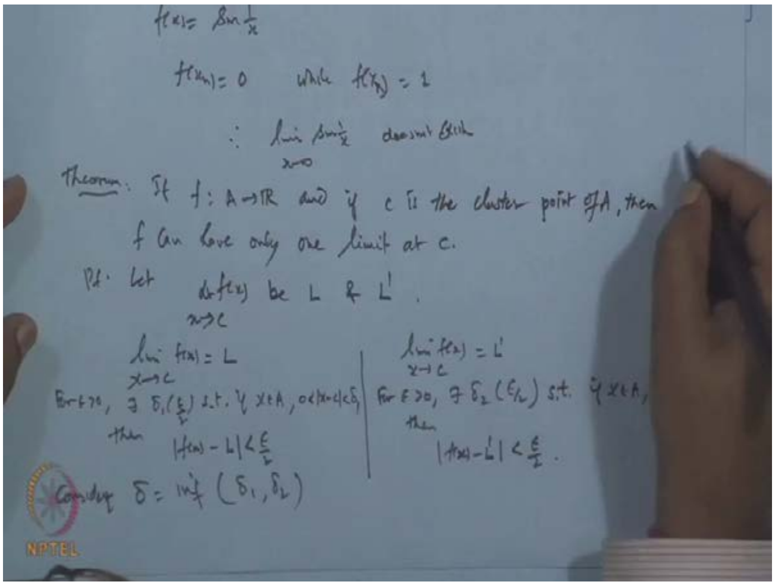

If I take f of x n, f x is sin 1 by x. So, what will be the f of x n; f of x n is 0 because when x n is 1 upon n pi, the corresponding sin of n pi and sin integral multiple of pi is 0. While f of y n; this is sign of 2 n pi plus pi by 2. That is always sin pi by 2. So, this will be 1. So, it means the f along one point it is 0, sequence along another point the limit of this is 1. So, it does not exist.

Therefore, limit of this does not exist. Now here as I told that we are assuming the limit is unique, is it not? Limit cannot be discarded. If the limit of the function exists, it will be unique. So, let us prove that result is and all these exercise which we have depends on this only. So this we should prove it earlier, but however. If f is a mapping from A to R and if c is the crystal point of A, then f can have only one limit at c. We cannot have the two limits.

So proof is, again we will prove by contradiction. Suppose there are two limits. Then finally, we will see that these two limits are not different; they are identical. So, let the f x of this when x tends to c; if suppose let this limit be L and L dash. So, limit of this f x is L when x tends to c; limit of f x when x tends to c is also L dash. So, apply the definition now. So for a given epsilon greater than 0, there exist a delta 1 depending on epsilons such that if x belongs to A, and 0 is less than mod x minus c less than delta 1. Then mod of f x minus L, let it be epsilon by 2 here. Delta 1 it depends on epsilon by 2 suppose; f x minus L is less than epsilon by 2 for all x this.

Similarly here for the same epsilon greater than 0, there exist a delta 2 which depends on epsilon by 2 such that if x belongs to A and x lying between x minus c less than delta 2. Then mod of f x minus L dash is less than epsilon by 2 for all such. Now consider mod L minus L. Consider delta to be the infimum of delta 1 and delta 2. So if I replace delta 1 by delta, then this result will also hold. If I choose delta 2 by delta, again this result will hold.

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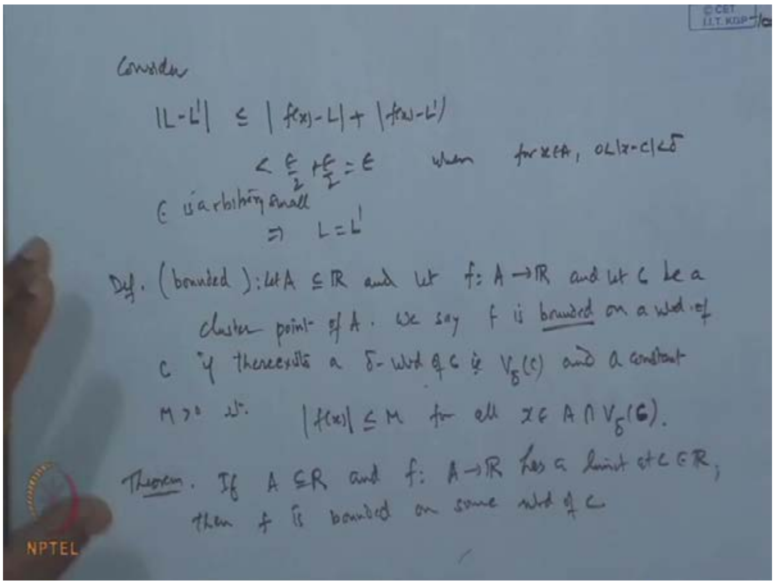

So now, consider mod of L minus L dash. Now this is less than equal to; add and subtract this. So, we can write f x minus L plus f x minus L dash. But f x minus L is less than epsilon. So, for all x belongs to A and 0 less than x minus c less than delta; this is also less than epsilon by 2, this is also less than, that is this also less; when this lessens.

So what it shows; that this implies L is equal to L dash, because epsilon is arbitrary small. So, we can get L equal to L dash; that is the limit will be unique.

So if as functional value f x, when x tends to c has a two different limit along with two different paths which are approaching to c, then the function will not have a limit at that point. So that is about, and then such a case we say it is a diverging function; diverges at x c. Because divergent does not mean, that function limit of this must go to infinity only. It is also one of the criteria when the limit of the function f x when x tends to c, say infinity. It means the function itself is not defined at infinity at the point and the limit is coming to be infinity. So, it is not finite.

But even if it is finite and having different along the different path, still we say, then we say the function f x does not have a limit at x equal to c. So, that is the criteria for this. Now this we have discussed. Now there are few more results over the limits and that we have seen in parallel to; this will be parallel to our theorems as we have discussed in case of sequences. So, before that let us see the definition of the boundedness. Let A is a non-empty subset of R, and let f is a mapping from A to R, and let c be a cluster point of A.

We say f is bounded on a neighborhood of c if there exist and if bounded on a neighborhood; if there exist a delta neighborhood of c that is V delta c, delta neighborhood of c and a constant capital M greater than 0 such that mod of f x is less than or equal to M for all x belonging to A intersection V delta c. If this is true, then we say the function f x is a bounded function; is bounded in the neighborhood of c. If the function has a limit, then it will always be bounded. So, the result is if A which is non-empty subset of R and f is a mapping from A to R and let A has a limit at c belongs to R. Then the function f is bounded on some neighborhood of c.

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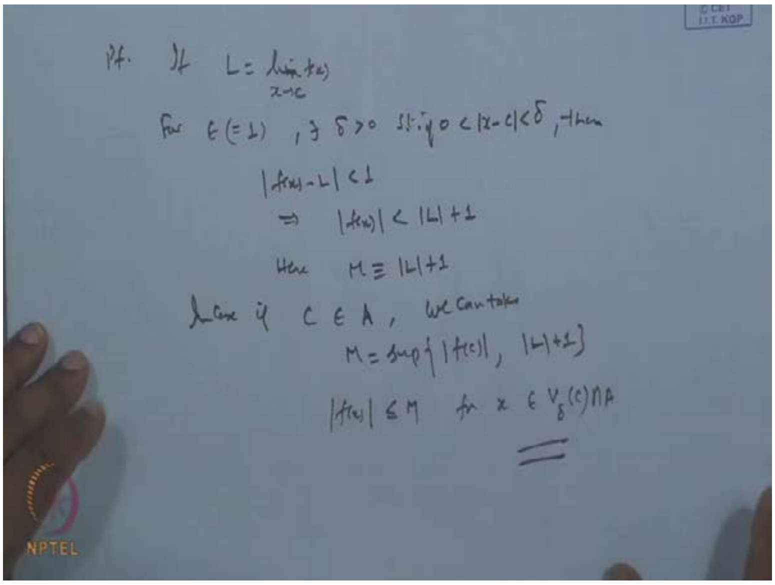

Let us see the proof of this result. So if the limits of the function f exist, then the function f must be bounded in some neighborhood of c. So, suppose f is the limit. Let if L is the limit of f x when x tends to c, this is given. So let us by definition, for a given epsilon I choose epsilon to be 1. For a given epsilon greater than 0 because it is already get 0, there exists a delta which will depend on epsilon positive such that 0 if 0 is less than mod x minus c is less than delta.

Then by definition mod of f x minus L is less than 1. Now from here, can you not say the mod of f x is less than mod L plus 1. So, I choose M to be. So, here M can be chosen as mod L plus 1. Then for all x in the neighborhood of this, it is there. In case if c also belongs to A; in case if c belongs to A, then what we do is we choose, we can take M to be the supremum value of mod f c and this bound. So if I choose this, then all this function f x will satisfy this condition for all x in some neighborhood of c, that is intersection A. So, this was proven.

Thank you very much. This is, we will continue again for few more results on this.