A Basic Course in Real Analysis
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Lecture - 24
Raabe's Test, limit of functions, Cluster point

So in the last lecture, we have discussed several test for judging the absolute convergence of this series of real numbers. And those sets mainly we have discussed the nth root test, comparison test, ratio test, integral tests. Now, even we see that these test are good enough to judge the series to be convergent, absolutely convergent or not, but still there is a case of failure.

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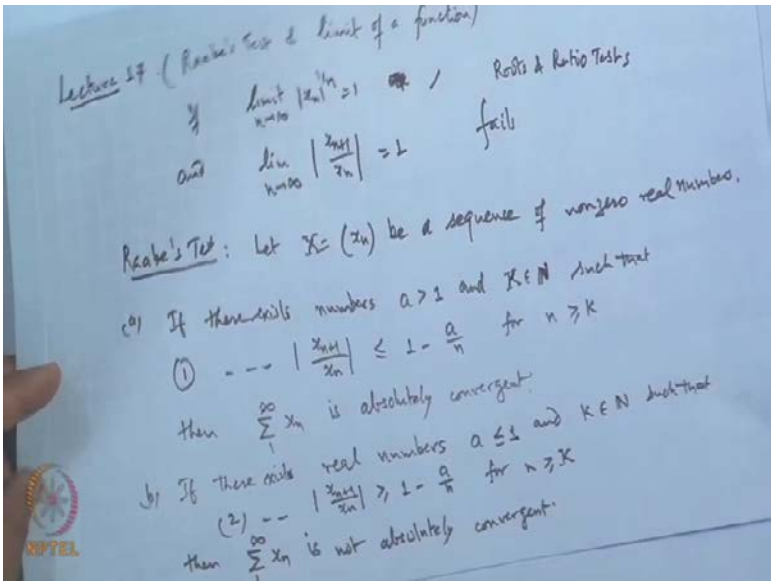

When the limit of x n to the power 1 by n limit of x mod x n to the power 1 by n as n tends to infinity if this limit comes out to be 1 or limit n. Similarly and limit of this mod x n plus 1 over x n ratio test as n tends to 1 comes out to be 1. Then both nth root test and ratio test fails, root test and ratio test fails. We are unable to decide whether the given series is absolutely convergent or divergent or it may be convergent, may be divergent depending on this.

So, the further test which will help in this direction is given by Raabe's, which is known as the Raabe's test and that can be applied only when we see that this test fails. We cannot also, but initially when we start with that, we always consider step by step. Go it step by step and then if required then we come for the higher order, higher level tests. So Raabe's test says, that is let X which is x n be a sequence of non-zero real numbers. Now first one: If there exist numbers a which is greater than 1 and K which belongs to N the set of natural number, K is a positive integer such that mod of x n plus 1 divided by x n under mod is less than equal to 1 minus a by n for all n greater than equal to k. Then the Raabe's test says, the series sigma x n n is 1 to infinity convergence is absolutely convergent.

Now, on the other hand if there exist real number there exist real numbers a which are less than or equal to 1, and K is a positive integer such that mod of x n plus 1 divide by x n under mod if it is greater than equal to 1 minus a by n for all n greater than equal to k. Then the series sigma x n 1 to infinity is not absolutely convergent. It means it will be divergent sequence, divergence series absolutely convergent. The proof of this follows as it is. Suppose let us give this number. Suppose this is equal to 1; this is 2.

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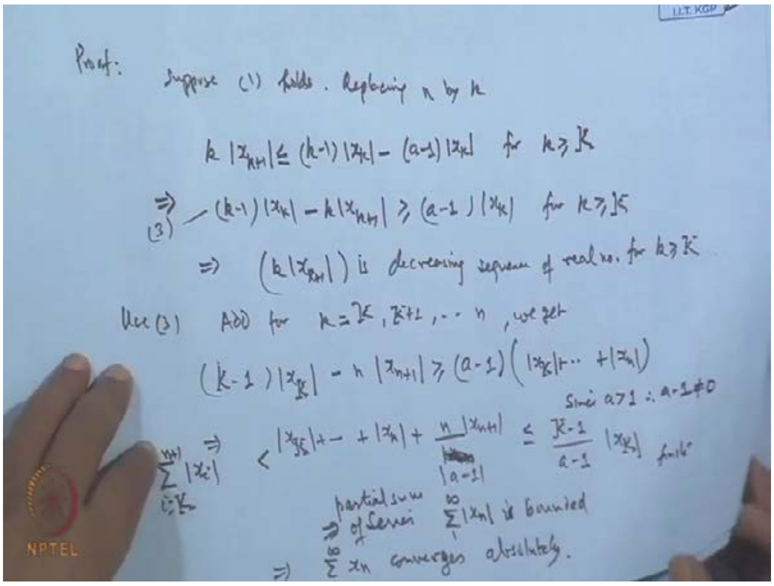

Suppose 1 holds, then we have to show the series is absolutely convergent. So 1 holds means mod of x n plus 1 by x n is less than equal to 1 minus a by n. There exist a number a greater than one such that this is true for all n greater than equal to k. So, let us replace n by k is small case suppose, then what we get is here; that k into replace n by small k is nothing no problem and then we multiply this. So, k into mod x k plus 1 less than equal to k minus 1 into mod x k minus a minus 1 mod x k. This is true for all k greater than equal to capital K.

What I did is I just used this in place of n let us say small k. So, when we multiply by this k you are getting a k times mod x k plus 1 is less than equal to 1 minus a into mod x n. This can be manipulated like this. It can be written as k minus 1 mod x k minus n because mod x k get cancelled from here, and basically you are getting this k minus a into mod a mod x k. So this way we now again reorganize it. So, this implies that k minus 1 into mod x k minus k times of mod x k plus 1. This term is greater than equal to a minus 1 into mod x k, and this is true for all k greater than equal to capital K.

Now a is strictly greater than 1 in the case 1, first case; so a is great. So, this is a positive quantity. This is also a positive quantity because we are choosing x n as a sequence of the non-zero real number. So, mod of x k will be positive quantity. So, what it shows. This shows that k times of mod x k plus 1, basically is what is a decreasing sequence. So, what we get is.

So, this implies that the sequence k into mod x k plus 1. This sequence is a decreasing sequence of real numbers for k greater than equal to capital K. Now if I add this quantity, let it be this quantity k minus 1. Let it be 3, this is third. Now if we add this from; use 3 and add for small k as capital K capital K plus 1 up to say n. So, when you add this will automatically telescopically this full term will get canceled; and finally we get capital K minus 1 into mod of x capital K minus n mod of x n plus 1 is greater than equal to a minus 1 into mod of x capital K plus mod of x n. Now, since a is greater than 1. So, a minus 1 is non-zero. So, a minus 1 is different from 0. We can divide by this.

So, this implies that a mod of x capital K plus up to mod x n then plus n by mod x n plus 1; sorry n times mod x n plus 1 divided by this a minus 1 this a minus 1. This term you are taking here and divide by a minus 1 is less than equal to capital K minus 1 divided by a minus 1 mod of x K. Now a is any number, is it not. So, a minus is a fixed number. K is also fixed. So, this is basically a finite quantity. This is finite bounded. Now here if I look this one, n is greater than mod a minus 1 because once a is fixed then n can be taken so large so that n over a minus 1 is greater than 1. So, basically this entire thing is greater than the series sigma mod x i i is capital K to n plus 1 is greater than this. So, what they show; that this series when n is sufficiently large this is bounded series.

So, this implies that the series sigma mod x n. this series is a bounded partial sum of the series this is bounded. The partial sum of this series is bounded 1 to infinity and the series and the term of this series k into x k plus 1 is a decreasing sequence of real numbers. Is it not? So, this is a bounded sequence and monotonically decreasing sequence of real number, because once it is decreasing mod x k minus 1 will also be decreasing k times. So, this is a bounded sequence. Hence every bounded sequence is a convergent one monotonic. So, this shows that the sequence, this implies the series x n 1 to infinity converges absolutely. So, this is the proof of this part one. This is part a.

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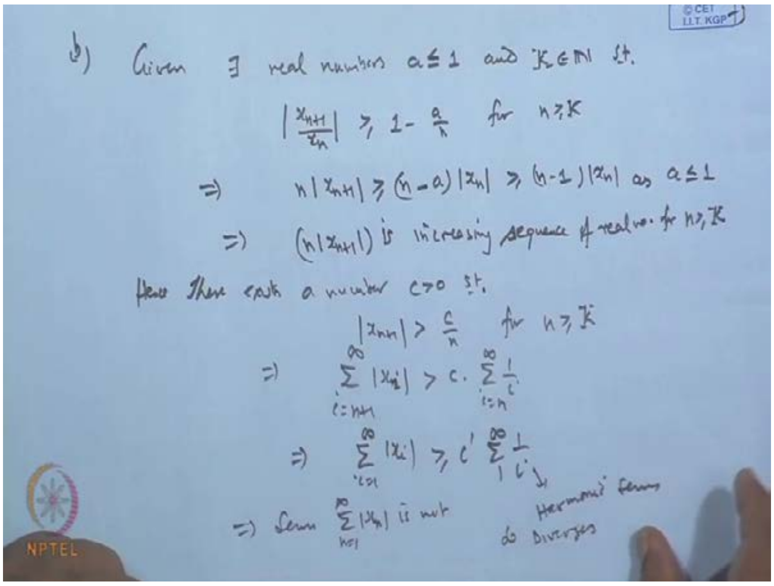

Proof for part b: Now in this case, we given that this relation. Relation given that there exist real numbers a less than or equal to 1, and k a positive integer such that mod of x n plus 1 by x n is greater than equal to 1 minus a by n for all n greater than equal to k. So, what this shows? This shows that if we multiply this n into mod x n plus 1, this is greater than equal to n minus a into mod x, but what is, but a is less than equal to 1. So, minus a is greater than equal to minus 1. So, it is greater than equal to n minus 1 mod of x n as a is less than equal to 1. So, n of mod x n is greater. Then this implies the sequence n into mod x n plus 1. This sequence is an increasing sequence of real numbers for all n greater than equal to capital K.

So, once it is increasing. So, if we take any c, then n can be obtained. So, that is greater than equal to c. So, and there were just a number c. So, there exist a number c greater than 0 such that this mod x n plus 1 is greater than c by n, and this is true for all n greater than equal to capital K. Therefore, the sigma of this x i i is n to say infinity. Now this is a harmonic series. So, what we do is that if I take i is equal to 1 to infinity; that is sigma i is equal to 1 to infinity mod of x i is greater than equal to; so same terms we are adding here. So, some c replace by c dash is greater than equal to sigma 1 to infinity 1 by i. Is it not? Some few terms we are adding here.

So, that can be written as some c dash sign or may be plus something, but this is a harmonic series. So once it is harmonic, it means it will be diverging one. So, diverges. So, once it is divergent, this series will be diverging one. So, this implies that series sigma mod x n n is 1 to infinity is not absolutely convergent .Therefore, original series will not convergent absolutely. So, that is the reason for this. Now the equivalent has the corollary of this because it is a difficult to get this ratio.

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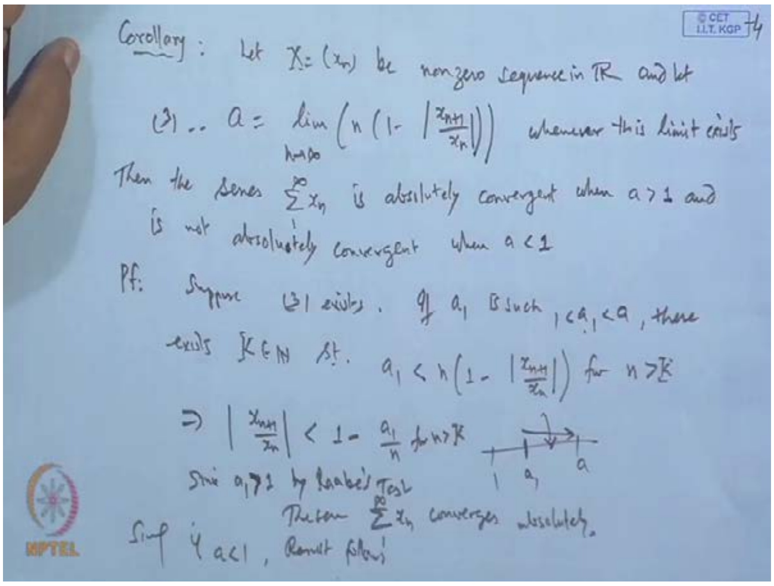

So in terms of the limit, we can easily get this test which is applicable and so in this corollary we can say, let X which is x n be a non-zero sequence in R set of real number, and let a is the limit of this; limit of n 1 minus mod x n plus 1 by x n. Mod of this as n tends to infinity, suppose a. Whenever this limit exist. Then this series sigma x n 1 to infinity, this is absolutely convergent when a is greater than 1 and is not absolutely convergent, when a is strictly less than 1. For a equal to 1 again test fails, which we are unable to get. So we have to apply some other tricks for that, for a is less than 1.

The proof runs as usual. Suppose the limit in this, let it be this 3. Suppose third exist, limit exist in 3. So, it means the limit of this is a where a is strictly greater than 1. So, we can identify. So, there exist. So, if a 1 is a number, a 1 is such that which lies between a 1 is less than a. For this a 1 there exist some k, a positive integer such that a 1 is less than n times of 1 minus mod x n plus 1 by x n, for all n greater than capital K. This is because this limit is a, and a is strictly greater than 1.

So, if I choose a number in between 1 and a, it means there will be a sequence of the term of this series which will exceed by a 1. Then only the limit will go to a. There are the sequence because limiting is going to a. This is 1, here is a. The limit is going tending to this. So, what I am doing is I am choosing one a 1 here. So an n can be obtained, a k can be obtained such that when you take all n greater than k, all the terms of the sequence will fall here. That is the n greater than a 1.So, if it is so then what happens to that.

Therefore, the series when this converges; so we get from here that therefore, this implies what, x n plus 1 over x n. This will be when you transfer it what you are getting is less than 1 minus a 1 by n, but a 1 is what and this is true for all n greater than k, but a 1 is strictly greater than 1. So by Raabe’s test since a 1 is strictly greater than 1. So by Raabe’s test, the series sigma x n 1 to infinity converges absolutely; similarly for the other. Similarly if a is less than 1, the result follows. In a similar way you can prove that this result is true. Now we have seen the corollary and the result. Now this is corollary is weaker basically. The result is very strong; that is this ratio. This type of thing is stronger than this one, which can be seen easily by the following example.

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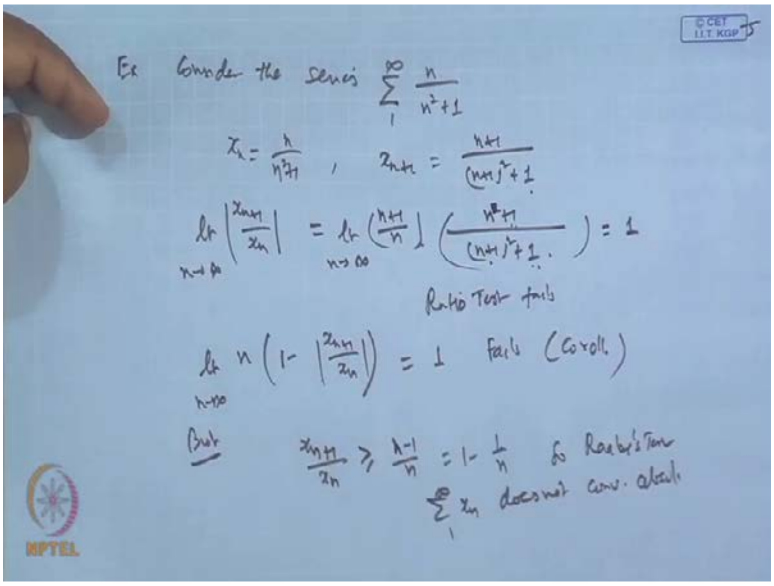

Suppose I take this problem. Let us consider the series sigma n over n square plus 1. Test for the convergence, absolute convergence. All the terms are positive, test for the convergence of this series. Now if we apply the results here. Suppose I take here the x n is n over n square plus 1. So, x n plus 1 becomes n plus 1 n plus 1 whole square plus 1. So, when you take the x n plus 1 by x n mod of this, then what happens to the x n plus 1. Then this comes out to be what; n plus 1 by n. This term and then n square plus 1 n plus 1 whole square plus 1 and then limit of this as n tends to infinity, limit of this as n tends to infinity. Now limit of this is 1 and again when you divide by n square, the limit will come out to be what; this is 1, and then this limit x n plus 1 by x n n over n plus 1; t is 1 or not. Let us see n square plus 1 so 1 then 1 plus 1 is 2. So, why it is n plus 1?

Calculation limit of this is x n it is n over n plus 1. So, if you take this n plus 1 n plus 1. So, divide by n plus 1 whole square. Divide by n plus 1. So, we get from here is n plus 1 square. So, what we are getting is 1. Divide by this n plus 1 square. This is sorry, that is why it’s a mistake. This is n plus 1 not n. So, divide by n plus 1 square here. So, this limit is 1; this limit is one. Product will be 1. So, the ratio test fails. Now if I take the Raabe’s test n into coronaries of the line minus x n plus 1 over x n this thing and then take the limit of this as n tends to infinity; then again you will see this limit comes out to be 1. We can just solve it and we get the limit to be one. So, again fails. We are unable to get, but this is by corollary.

But if you look the x n plus 1 by x n what you are getting; x n plus 1 by x n. This is nothing but what; is greater than or equal to n minus 1 by n. If you just see the ratio of this and find out the term you will get this is greater than equal to; that is 1 minus 1 by n and this is 1. So, this diverges. So, by Raabe's test the series sigma x n does not converge absolutely. So, this shows that though the Raabe's test difficult to find out this type of ratios and get this derivation, but it is much stronger than the coronaries of this Raabe's test. So, but practically that corollary is much more easy to apply rather than to this. So, that is what; so this all about this convergence or absolute convergence of the series.

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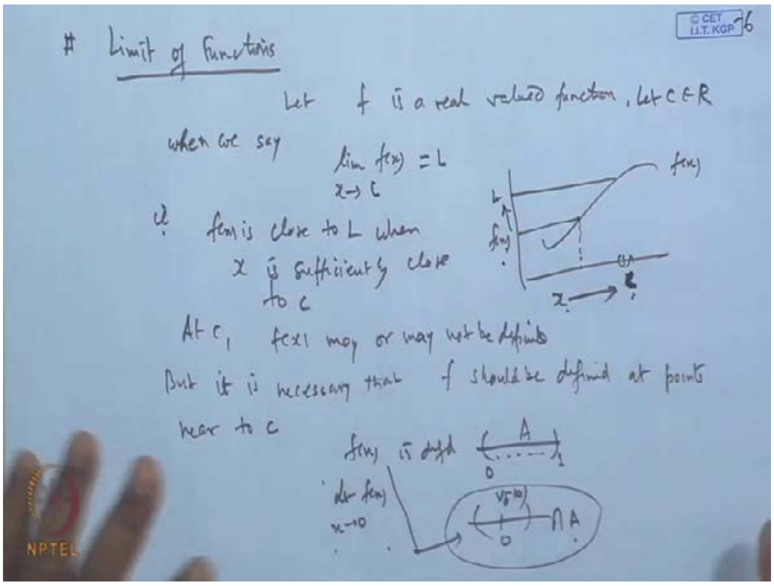

Now will switch over to limit of functions; so next topic you say limit of functions. First what is the idea of this? Suppose f is a real valued function. Let f is a real valued function; that is f x when x belongs to R and f x is also real. When we say the limit of this f x when x tends to some number c, where c is any real number say r. When we say the limit of this is L. The meaning of this is that if this is function f x, I am assuming to be continuous function of course and this is our c, sorry c and x is a point is converges to c. Then correspondingly the functional values f of x is tending to L. When x approaches to c, the limiting value of this f x is L. At the point c, the function may or may not be defined the, but still the limiting value of this f x we say it is L.

It means the difference between L and f x is very very small when x is sufficiently close to c. It means that is the f x is close to L when x is sufficiently close to c. That is the meaning of the limiting behavior of the function f x at the point c. At the point c, the function f x may or may not be defined, is not necessarily defined. But what is used is important part. But it is necessary that f should be defined; f should be defined at points near to c, then only limit of the f x has some meaning. If the function is not defined at the point closer to c, then we cannot approach the value of f x when x is approaching to c, because they are on the so many points where function is not defined. We cannot find out the f x.

So, the important part or the necessary part is that, in order to get the limit of the function at a point c then necessary condition is or necessary part is, the function must be defined at all points which is very very close to c. That is the important point. Second point which we get, that at the point c, the function c may or may not be available in the set A; may or may not be available in the set, still we can find the limiting value of f x when x approaches to c. Suppose I would say the function f x, say f x is defined over the interval say 0 to 1 and we find the limit of the f x when x tends to 0.

So, though the function is not defined; the 0 point is not available in the set A. But still this has a meaning, because the function is defined at every point in the interval. So what it says is the point c may or may not be point of A, but even if A the limiting of the f x is possible. We can ignore that point, that whether it is in c or not. What we are interested is that we will find out the neighborhood of the point A, neighborhood of the 0; this is the 0. We will find the neighborhood of the 0 with the suitable radius delta and then find the intersection with A. In this, all the point in this set; consider the functional value at these point and take the limit when x approaches to 0. If this limit exist and equal to L, then we say the limit of the function exist. So, that is what.

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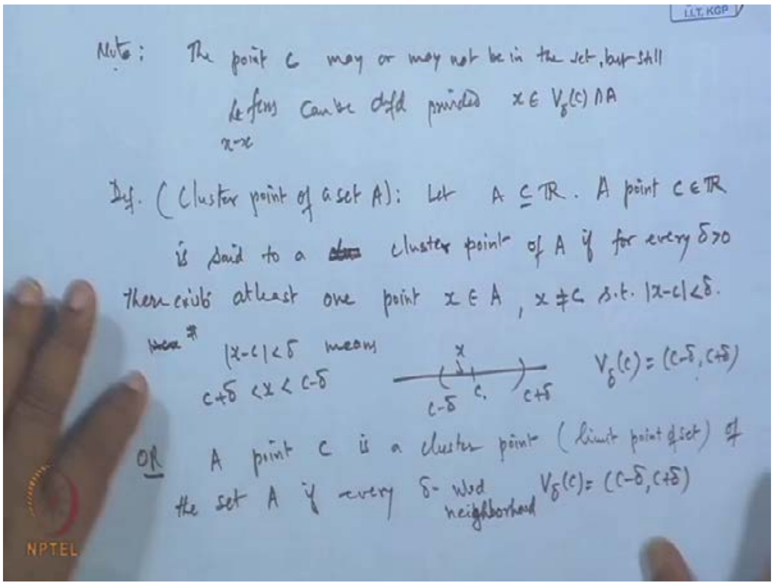

So note we can write it, note like this. The point c may or may not be in the set be in the set, but still the limit of f x when x tends to c can be defined provided x tends is provided. x belongs to the neighborhood of c intersection. Function is defined for this. So, that is the point. So, before going for this limit let us see the concept of the cluster point, because always this limiting value that we will consider c as a cluster point for this set. So, let us see the definition cluster point. Cluster point of a set A. Let A be a non-empty subset of R. A point c, a real number in R is said to be a cluster point of A if for every delta greater than 0, there exist at least one point one point x belongs to A, which is different from c such that mod of x minus c is less than delta.

So, what is the meaning is that a point c is called the cluster point of a set A, if for every delta greater than 0 there exist at least one point x such that x minus c is less than delta. So, this meaning is x minus c mod of this less than delta; obviously, means that x lies between c minus delta and c plus delta. It means there exist a neighborhood around the point c is a neighborhood. This we denoted by v delta c, a neighborhood which is c minus delta c plus delta. So, a neighborhood can be obtained in which at least there exist one point x different from c, and this is true always for h delta. Then we say c is the limit. So, c is the cluster point of this set A. If when we picked up the neighborhood of delta, then at least one point x will belongs to this different from c.

In other words we can say a point or we can also define in terms of the neighborhood. A point c is a cluster point. Cluster point is also called the limit point; cluster point or may be limit point of a set. Cluster point of the set A if every delta neighborhood; nbd neighborhood every delta neighborhood, and we dually use that word nbd; neighborhood v delta c; that is c minus delta c plus delta.

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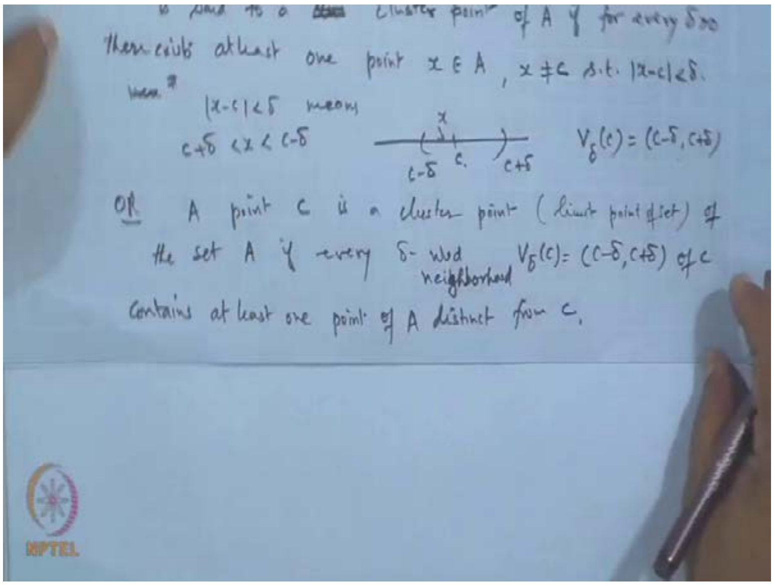

Every neighborhood of this of c contains at least one point at least one point of A distinct from c. Then we say c is a cluster point of this.

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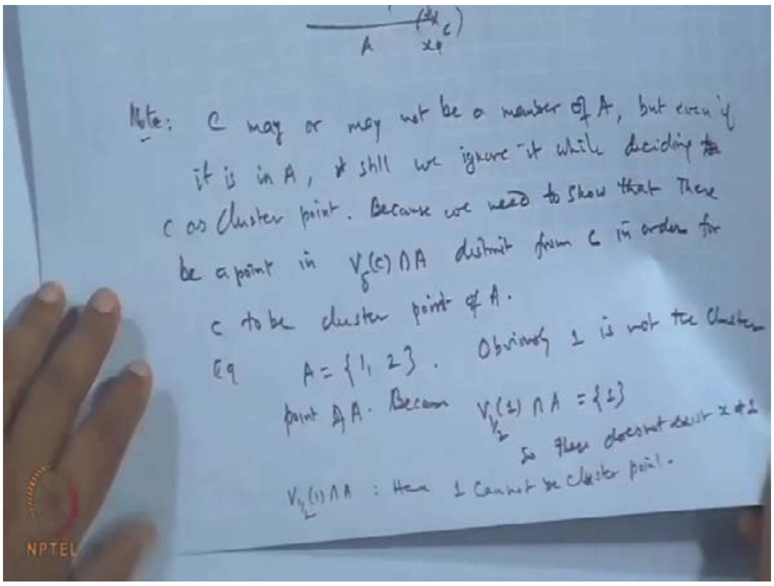

So basically, this is suppose set A. c will be somewhere here. Now this will be the cluster point. If we picked up a neighborhood around the c, then there must be some point x of A different from c available. Whatever however small radius you choose, then we say c is the cluster point. That point c may or may not belong to A. So that is the important part is c may or may not be a member of A. If it is in the set A, then also we can ignore it. But even if it is member of A, it is in A. But still we can ignore it while deciding the cluster point, while deciding c as the cluster point. What is more important is, that we simply picked up whether it is a cluster point or not. What we could see here, we require; that is because we need to show that there is or there be a point in the delta neighborhood of c intersection a.

That is important part, distinct from c in order for c to be a cluster point. Cluster point of A that is important. For example, suppose I say A is a set having two elements 1, 2; then 1 is not a cluster point. Obviously 1 is not the cluster point, why; cluster point of A. Because what we see if I take the neighborhood of 1 with center set half with radius half and find out the intersection with A, then this set contains only the single point 1. So, there is no other point. So there does not exist x, which is different from 1 in this set in this collection; V 1 by 2 one intersection A. Hence 1 cannot be cluster point. Similarly two cannot be cluster point on this. So, this way we can identify for them.

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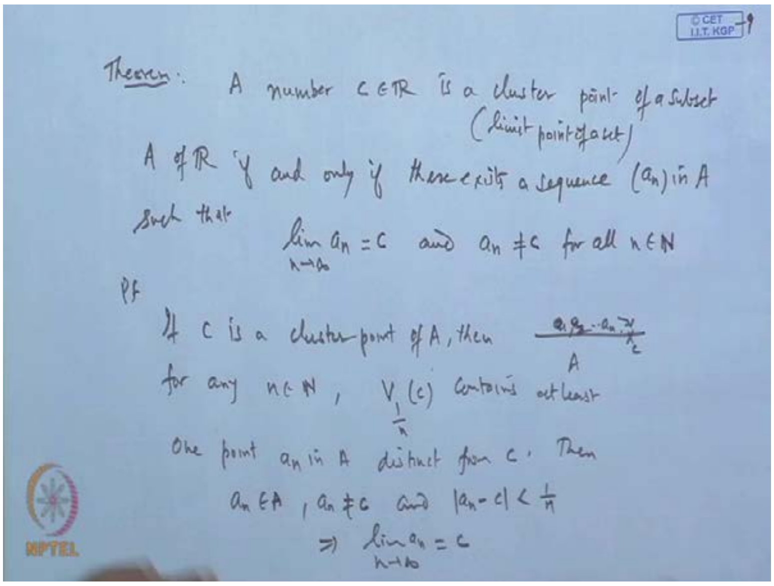

Now there is a relation and we can also define the cluster point in terms of the sequences. So, this result is which connects the concept of this convergence of the sequence and this cluster point. A number c belongs to R is a cluster point of a subset, cluster point or limit point same; limit point of a set. Cluster point of a subset A of R, if and only if there exist a sequence a n in capital A such that limit of a n as n tends to infinity is c and a n s are different from c for all n belongs to capital N set of natural number. So, that is the important part is.

Proof of this: In place if c is a cluster point of a subset A of R then there will be a sequence available in A, if this is our c and here is A. So, there will be a sequence a 1 a 2 a n and so on, which will converge to c, which will go to c, but a n s will be different from c. That is if I draw a neighborhood around the point c, then we will get the a n s different from c. For any neighborhood, we get at least one a n s available in this. So, let us see the proof. If c is a cluster point of A, then by definition if it is then by any delta neighborhood of c must includes the some point which is different from c.

So, delta I am choosing to be n, then for 1 by n. Then for any n belongs to N, the 1 by n delta neighborhood of c; delta becomes 1 by n contains at least one point a n which is different distinct in a n and in A distinct from c. That is it by definition of the cluster point. So, what we get; then we get a n s. These are in A; a n s are different from c and the distance of a n minus c is less than 1 by n and this is true for each n. There will be some a n d. So, this implies the limit of a n as n tends to infinity is c.

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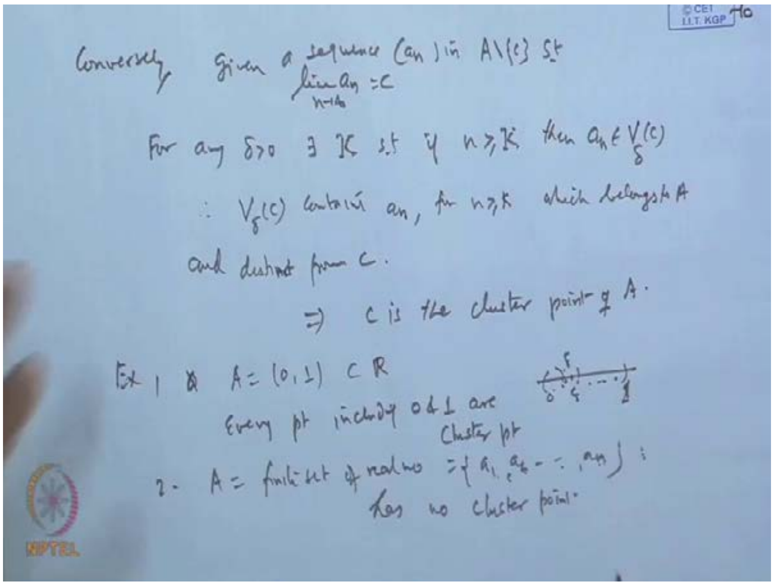

Conversely, if suppose the limit of a n is given to be c. Given their limit of a n s as n tends to infinity c. There exist sequence a n such that limit is n. So, conversely given a sequence a n in the set A minus this c, such that limit of this a n is c. This is given here. So by definition of the limit, then for any delta greater than 0 there exist a k; a positive integer k such that if n is greater than equal to K, then a n s will belongs to the neighborhood of c, with delta neighborhood of c. That is by definition of this. For any delta, we can find out this. After certain stage, all the terms of the sequence will belongs to this delta neighborhood c. Therefore, delta neighborhood that V delta c, delta neighborhood of c contains the point a n s for all n greater than k and different from which belongs to A and distinct from c.

So, this implies that c is the cluster point of A. Is it not? So, c is the cluster point of A. It means we can in terms of the sequence also we can define the cluster point, in neighborhood or cluster. Now let us see few examples. Suppose x I take open interval or A be the open interval 0, 1 which is subset of R. Now this is a complete interval 0 to 1. Every point will be the cluster point. Why? Because when you take any point here c; any point of this say, I take c here. Then one can identify any neighborhood around the point delta neighborhood if I choose; then one can identify the points of this set which is distinct from c. So, c becomes the cluster point.

So, and this is true for any c is arbitrary in between 0, 1. So, every point in this is a cluster point. As well as, so every point including 0 and 1 are cluster point. 0, 1 is also cluster point for this; but 0 and 1 both are not available in A. So, similarly closed interval 0, 1, we can choose like this say finite set of real numbers. Then finite set of real numbers means suppose a 1 a 2 a n.

These are the set element of the set, but it does not fix any limit point; has no cluster point. Why? Because if suppose some point is a cluster point, say any point whether in between or outside. If suppose say a 2 is the cluster point. Then there is a gap between a 2 and a 1, a 2 and a 3. So if you draw a neighborhood around the point a 2, you would not get a point other than a 2 inside this. Basically no point will be available. So, it cannot be a cluster point. If a point is outside of this set, again there is a gap between a n and this and we cannot get a point in the neighborhood delta; this is distinct from this. So, point of the set. So, finite set does not have a limit point; no cluster point.

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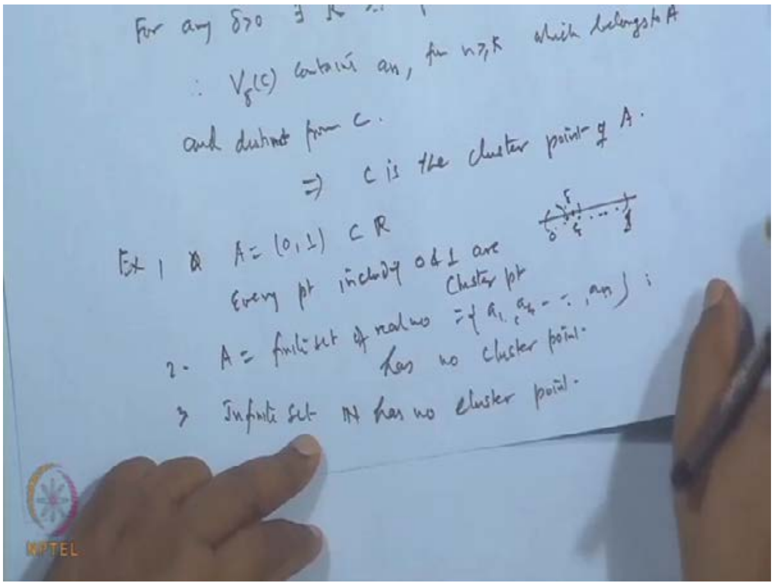

Similarly infinite set capital N of natural number has no cluster point. The same reason is the same.

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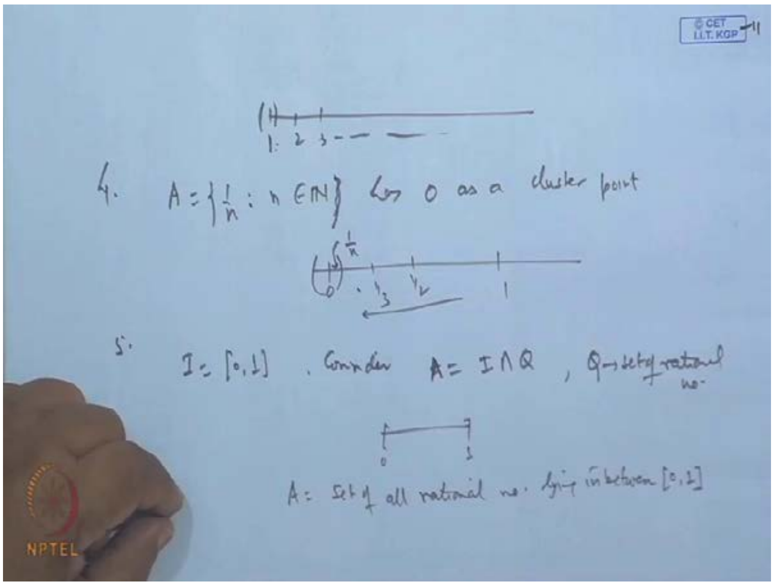

Because this is our natural number 1 2 3 and so on; now we cannot take or we cannot get any neighborhood around the point which contains the point of the set other than that point. Suppose I take 1 and I say 1 is the cluster point. Then if I draw a neighborhood like this; now this neighborhood does not contain a point of the set other than 1, because basically no point is available. So, 1 cannot be a cluster point; similarly 2, 3 and so on. So, this infinite set has no cluster point.

Then fourth example: The set A and if I take the set 1 by n where n is a natural number. Now this has 0 as a cluster point. Why? Because this is our 0, here is 1, then half, one-third and like this. So, the sequence is decreasing and it crusted around the point 0, means if I draw any neighborhood around the point 0. There are so many points will be available here after a certain stage, which are close to 0 and different from 0. So, 0 becomes the cluster point for it. Then if we take I and close interval 0, 1 and consider a set A which is I cross intersection Q. Q is the set of rational numbers. It means the interval 0, 1 is chosen and then from there, your finding the inter section with Q.

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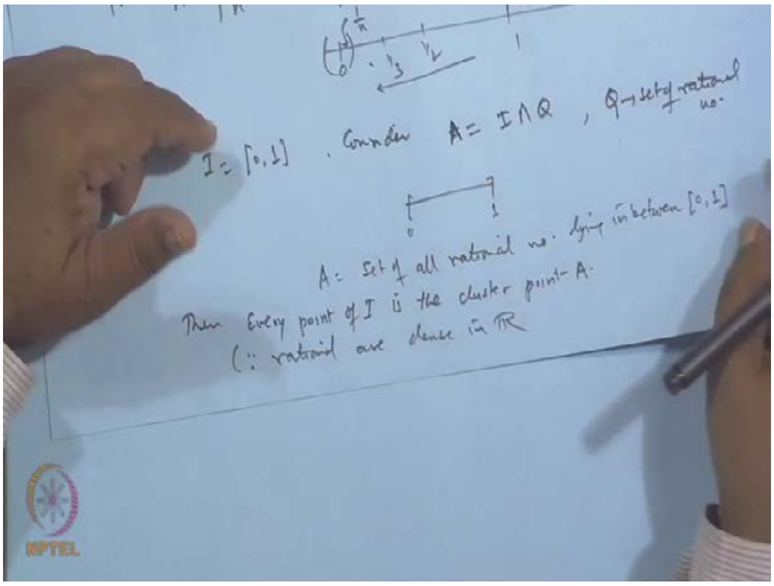

So, A is the set of all rational numbers lying in between 0, 1; this close set 0, 1. Then every point in I is a cluster point of A. Then every point of I is the cluster point of this set A. Every point in I or I is a crystal point of this. Why? This is because of the result which we have seen that between any two rational numbers, there always a rational number infinite. There is a rational number and can be find out between the two rational number; so if I picked up any point here say 0, then draw the neighborhood.

Now since between 0 and 1 rational numbers are dense; this is the reason is because rationales are dense in R; close all of rational number is the entire real line. So once it is rational dense, if you picked up any number, you will always get in vicinity another rational number and which is distinct from that number. So, every point of 0, 1 becomes the close cluster point for this set A. So that is one.

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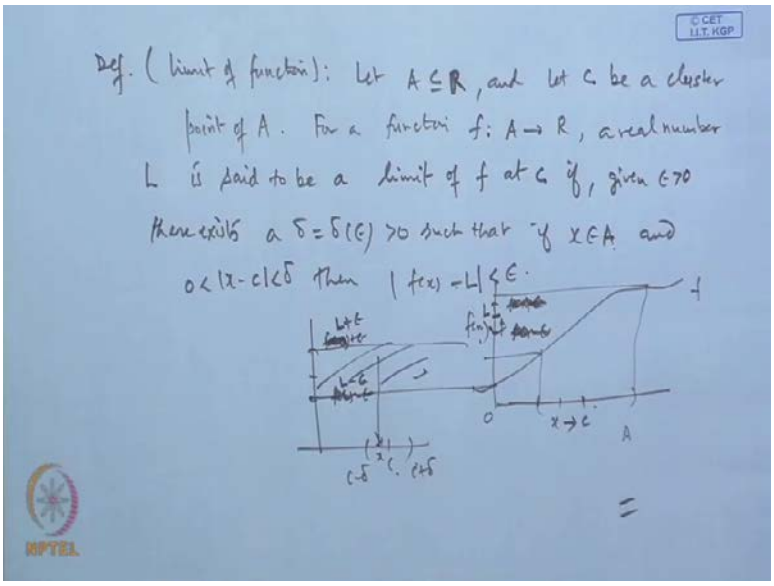

Now we come to that re-concept of the functions. Now this is important part, definition of the limit. Let a or limit of the function, that is important. Let A be a subset of non-empty subset of R and let C be a cluster point of A. For a function f which is a function from A to R, a real number capital L is said to be a limit of the function, limit of f said to be limit of f at c, if given any epsilon greater than 0 there exist a delta which depends on epsilon of course. Delta depends and positive. If there exist a delta greater than 0 such that if x belongs to A and 0 less than mod x less c less than delta, then mod of f x minus L is less than epsilon. That is what is shown. So, what is the meaning of this is, this is our C is the cluster point.

C is the cluster point of the set. Suppose this is the set A and c is a cluster point for this set. A function f is defined from A to R. So, this is our function f which is defined carrying A to R. This is R suppose. Here everywhere the function is defined, so we get this. This is f x, range of this. So, what he says is, that if we say this L is the limit of the function f x at the point C; that is if I take x and x tends to c the corresponding function f x and this goes to L. Limit of f x is L means that if we take any epsilon greater than 0, then there exist a delta such that this for all x belongs to this and f x minus L is less than epsilon.

So, if I take L epsilon neighborhood around the point L that is here f x minus epsilon f x plus epsilon. If you choose this, that is this our f x minus epsilon f x plus epsilon. Here is c, c minus delta c plus delta. So, for the given epsilon greater than 0, there exist a delta such that whenever we take any point x different from c in this neighborhood, the image of this will fall within this range; within the L minus epsilon to L plus, sorry this is L minus epsilon L plus epsilon and L minus epsilon this will be. So, with for within this range, then we say that this limit of the f x is L.