A Basic Course in Real Analysis
Prof. P. D. Srivastava
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 23
Tests for absolutely convergent series

In the last lecture, we have discussed the concept of the Absolute convergent series, conditionally convergent series, and also we have discussed the grouping of the series and re-arrangement of the series.

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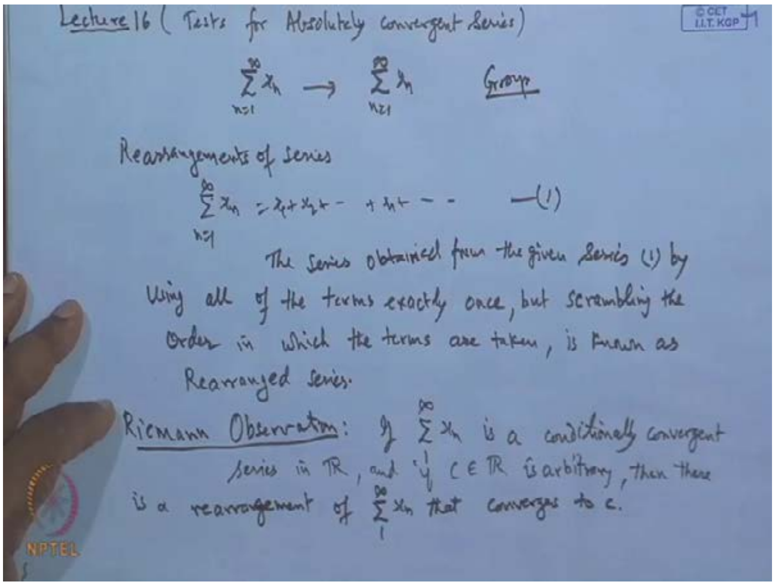

So, grouping of the series, we mean, if a series is given sigma x n of real numbers, n is say 1 to infinity a series. Now, if with the help of this, if we construct another series sigma y n, n is 1 to infinity, where the terms of the series, order of the terms of the series are is kept fixed, we are not disturbing the order, the first element remain at the first place, second element remain in the second place, but what we are doing, we are grouping the finite number of terms.

And then the new series, so obtained will be known as the series grouping or grouped series of the previous one. And in that case, we have also seen a result that if a series sigma x n is convergent, and has the sum s, then the corresponding grouped series will also be a convergent series, and will have the same sum at the sum; it means by grouping the terms of the series the new series, so obtained will not change is corrector, if the original series is convergent. The newly constructed series by grouping the elements will also be convergent and the sum will remain the same.

So, this is the case of the grouping, but in case of the rearrangement of the series; rearrangement, we have defined like this rearrangement of series, suppose a series is given sigma x n, n is 1 to infinity, where x 1 plus x 2 plus x n and. So, on and if we construct a series another series y n from the given series x 1, such that we are using all the terms of the series only once by using all the terms exactly once, but scrambling the series obtained from the given series say one, from the given series by obtained from the given series by using all of the terms, exactly once, but scrambling the order in which the terms are taken are taken. So, the series obtained from the given series one by using all of the terms exactly once, but scrambling the order in which terms are taken is known as rearranged series.

So, in this case, we are free to interchange the position of the terms, and then taking the new series, and then considering the new series. So, obtained now, as in the case of the earlier when grouping of the series does not change the nature of the series, but in case of the rearrangement of the series the nature is may change, even the some changes, if the series is convergent having a sum s, then if after rearranging the terms the series will remain convergent, but the sum will differ.

So, this was observed by Riemann and the Riemann basically; this is the Riemann observation; what Riemann has observed; what he says the observation, if the series sigma x n n is 1 to infinity is a conditionally convergent series is conditionally convergent in R set of real numbers, of course series of real numbers, and if c is any point real number belongs to R say is arbitrary, then there is a rearrangement of the series sigma x n of the series, that converge this to c. So, this was the observation made by Riemann, it means, if a series is not absolutely convergent series, but it is a say conditionally convergent series having infinitely many positive and infinitely many negative terms, then in that case, if I rearrange the terms of the series and get say another series, then such a series can be will have a sum different from the previous one.

In fact, if I want a sum to be c, then an arrangement can be possible. So, that the rearranged series will converge to the value c, and this observation can be justified by

Riemann is justify like this, that is first the condition is the series should be conditionally convergent; second condition which he has impose is the there should be infinite number of positive terms, infinite number of negative terms, and then what he did is, he first consider the first positive terms, whose sum positive terms and the sum of that series of the positive terms does not converging to does not exceed why greater than c, sum of the positive term greater than c, then later 1 he consider the some of the negative terms, which greater than c, and like this, he has he is able to show that for any given c one can make a rearrangement.

So, that series will converge to the same point c. So, this was the observation. Now, if such a series is given, which is conditionally convergent, but not absolutely convergent then; obviously, the series will give problem, when we interchange, or when we shifting the position of the points that is the terms are shifted or interchanged then you won't get the unique sum; however, this case is not there in absolutely convergent series. So, this next result shows, if a series is an absolutely convergent series, then what ever the rearrangement, we make the series will remain convergent, and will have the same sum as the original one.

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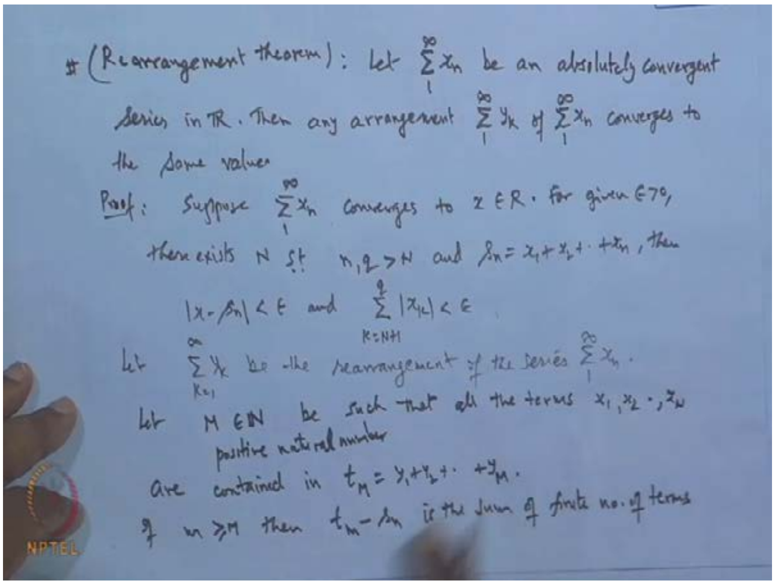

So, this result is given in the form of theorem, which is known as the rearrangement theorem, the theorem says like this let sigma x n 1 to infinity be an absolutely convergent series in R, then any arrangement sigma y k 1 to infinity of sigma x n of sigma x n converges to the same value, see the proof of this.

So, this is very interesting result, that if you are dealing with the absolutely convergent series, then we need not to bother in that the rearrangement of the series will give the different sum, it will not give the same value as the previous one earlier. So, proof, suppose the series sigma x n 1 to infinity is convergent series and converges to the value say x belongs to R. So, by definition, if the series is convergent then sequence of this partial sum will go to x, when n is sufficiently large.

So, for a given epsilon greater than 0, there exists a positive integer capital N such that, when n q, n and q both are greater than n and s n be the sequence partial sum say x 1 plus x 2 plus x n sum of the first n terms of the series, then the x minus s n is less than epsilon, and sigma of this mod of x k, when k varies from n plus 1 to q remains less than epsilon, that is if the series converges, then by definition sequence of this partial sum will go to 0, it means the reminder terms of series will remain less than epsilon.

So, for any q which is greater than N, this is the basically the first remainder terms, where finite sum, finite terms on the remainder remaining series it will be remain less than epsilon. So, this is true, now let us take the rearrangement of the series, let sigma y k, k is 1 to infinity be the rearrangement of the series sigma x n with the rearrangement of the series sigma x n now, if I choose let m belongs to the positive natural number; this is the a positive natural number capital N, and here is the capital N is just a sum positive integer n.

So, set of this natural number. So, capital M is a positive integer capital M be such that all the terms, say x 1 x 2 x n are contained in say are contained in sat t n sum y 1 plus plus y 2 plus y n, this is a rearranged series and what I am doing is I am taking the sum of the first m terms, since it is a rearrangement it means, we are just changing the order x 1 x 2 x n.

And the new series is shown, but the first m terms which we are choosing involves this x 1 x 2 x n plus few more times, of course is there in that, so obviously it follows so; obviously, when m is greater than n. So, if m is greater than or equal to m, then in that case the t n; this sum minus s n, because this sum will definitely involve x 1 x 2 x n. So, when you take the minus.

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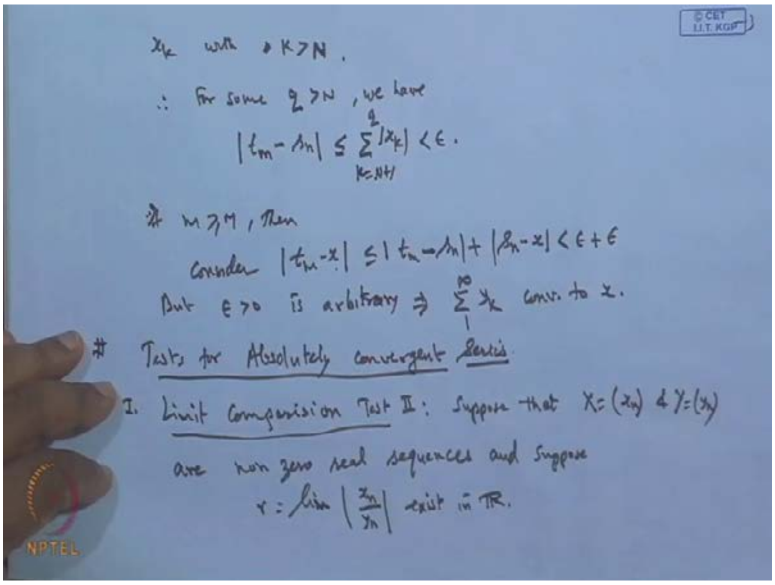

So, x 1 x 2 x n will out, and this is will be is a sum of finite number of terms x k is the sum of finite number of terms x k with k x k with k with k greater than capital N, because those x 1 x 2 x n will get cancel, and since m is greater than n, and this sum contains all x 1 x 2 x n. So, those term will vanish will cancel and the remaining term will definitely is start from n onward. So, this is x k, when k is greater than n like this. So, this will be there, hence for some q therefore, hence for some q which is greater than n, we have t m minus s n mod of this now, this will remain less than equal to sigma k equal to n plus 1 to q mod of x k, because t m minus s n will involve those term x k, when k is greater than n.

So, we can find q, such that this sum of these terms will remain less than or equal to sigma of the, but because the series sigma x n is convergent series. So, this condition holds. So, this will remain less than epsilon. So, this will be less than now, we wanted that series the series sigma y n is convergent, we want this series to be convergent, converges to the sum x. So, let us find the t m minus x x is there. So, consider. So, if m is greater than equal to m, then consider mod of t m minus x now, this will be remain less than equal to mod of t m minus s n plus mod of s n minus x now, t m minus s n is absolutely less than epsilon.

So, this is less than epsilon and x is the sum of the series. So, s n will go to x. So, this will remain epsilon, for all m greater than equal to M. So, this is (( )), but epsilon is arbitrary small. So, when (( )) this shows, when epsilon tends to 0, this t m will go to x therefore, this implies that the series converge sigma y k, k is 1 to infinity converges to x. So, that is proves the result, that in case of the absolutely convergent series the rearranged series.

So, obtained will remain the convergent and will have the same sum as the original one. So, we are mostly interested in those series, which are absolutely convergent because the nature of the series to, if it is convergent then we need not to bother for the rearranged series, because it whatever the way you sum up, the sum will remain the same. So, let us go for the some few tests, for the absolutely convergent series. So, test for absolutely convergent series, we have already seen. So, many test for a convergence of series of real numbers, and one condition for this is comparison test, we have seen and another root test, we have seen and then Cauchy convergence criteria is also there for the convergence of the series and like.

So, here we will simply state the few results without proof, because the proof follows runs on the same lines as we have done earlier for a general case. So, let us see the first result says which is the limit comparison test; first one we have seen this is the second test, we have seen, what this says suppose that a series a sequence X equal to x n and Y equal to y n are non zero real sequences and suppose limit of this exists, limit of mod x n over y n say is equal to R exist in R.

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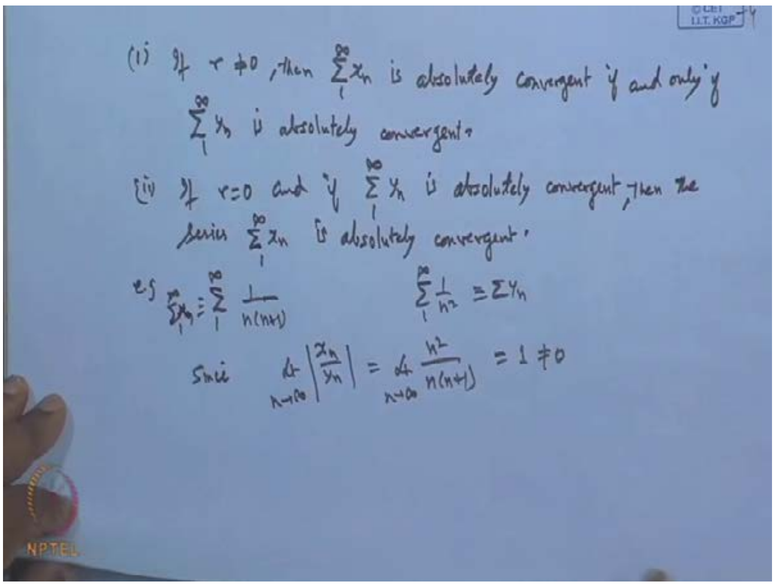

Then, what is result says if r is different from 0, if r then the series sigma x n 1 to infinity is absolutely convergent if and only if the series sigma y n is absolutely convergent, and second result says if r is 0, and if the series sigma y n is absolutely convergent, then the series sigma x n is absolutely convergent. So, this what is suppose the two series are given, one is the sigma x n, other one is the sigma y n, the result says if x n y n both are sequence of real numbers, and if this limit x n over y n as n tends to infinity, limit of this exists, and suppose it is r, then the r, if it is different from 0, then nature of this series sigma x n, and nature of this series will be the same.

So, for the absolute convergence is there, that is if a series sigma x n is absolutely convergent, then sigma y n is absolutely convergent and vice versa now, if r is 0 then in that case sigma x y n is absolutely convergent will imply the sigma x n is absolutely convergent, but not the other way around. So, just given series, if we are able to construct the series y n in such away, so that the ratio limit of the ratio exists, then one can identify whether the series is absolutely convergent or not this is the one.

Then another test is, I think examples we have already seen suppose I take a series sigma 1 by n square, say suppose then 1 by n square, and if you take say n n plus 1, say for example, if I take the sigma 1 by n n plus 1, say 1 to infinity, we want this series to be testing this series all the terms are positive of course, then what we do construct the sigma 1 by n square, this is equivalent to sigma y n; this is equivalent to sigma x n 1 to infinity now, this series nature of this series we know it is a convergent series because a sigma 1 by n to the power p now, what happen, if we take the x n the same x n over y n mod of this, limit of this as n tends to infinity, what is this limit is nothing but what n square over n n plus 1 limit as n tends to infinity.

So, if I take n outside, then we get basically limit is 1 different form 0. So, here r is different from 0 therefore, both the series will have the same nature. So, this series is a absolutely convergent; therefore, this is also convergent. So, that way we can find similarly, for the r is 0 we can get it, the second test which is root n ratio test, second test which is the root test.

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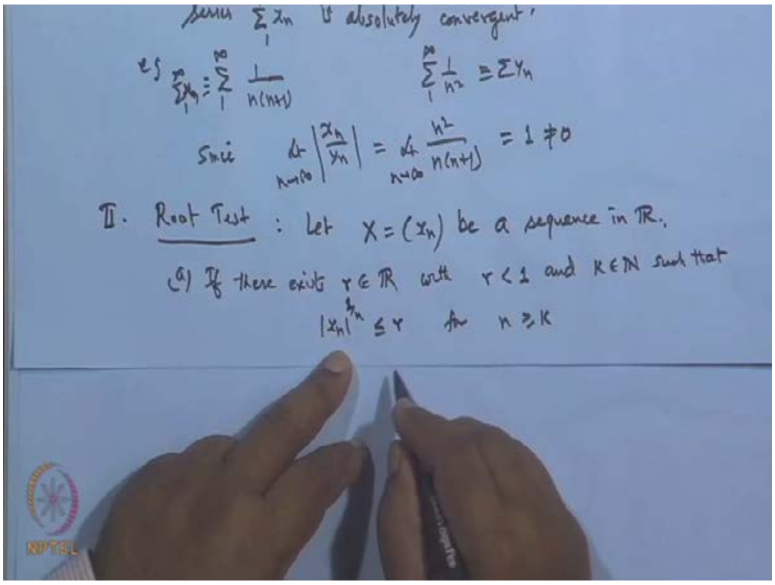

Let x, which is say x n be a sequence in R real number, then first, if there exists R if there exists R in the set of real number means R is a real number with r less than 1 and a positive integer k belongs to set of natural number n, such that mod of x n to the power 1 by n, this mod is less than equal to r for n greater than equal to k, may be the few term this condition may not be satisfied, but after a certain stage the mod x n to to power 1 by n remains less than that number r which is less than 1.

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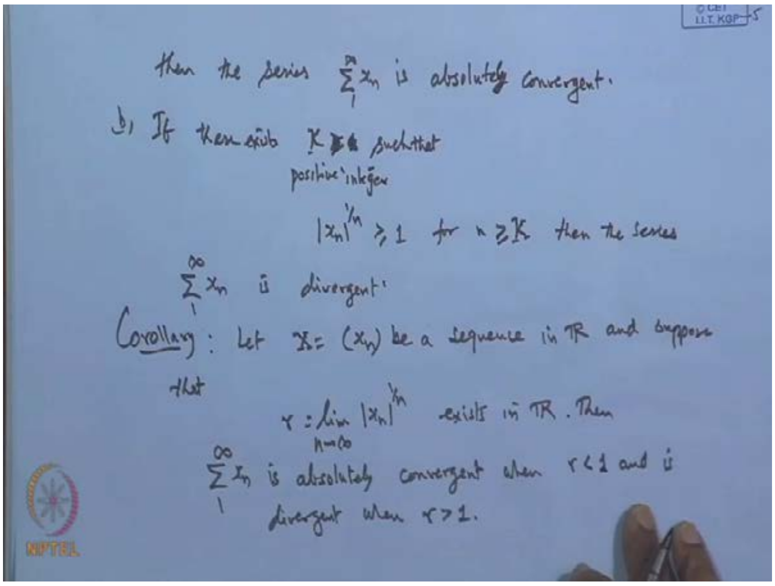

Then the series sigma x n 1 to infinity is absolutely convergent, this is what and second part, if suppose, if there exists k, if there exists a positive integer k greater than 1 belongs to n of course, k belongs to n positive integer I will say positive integer k positive integer k belongs to n. Such that may be equal also there is no problem belongs to n such that or you can remove this, there exists a positive integer k such that mod of x n power 1 by n is greater than equal to 1, after this n greater than or equal to k, then the series sigma x n is divergent.

So, again this is the n eth root test is the parallel to our root test for the general sigma x n, when the mod of x n to the power 1 by n are if lying between 0 and 1, then this is convergent greater than 1, then diverges is it not. So, mod x n is greater than or is strictly less than 1. So, again this proof will be the same, we are just dropping now, since as a corollary of this earlier case also we have seen the limiting instead of choosing, because this inequality to identify such an r is a difficult one.

So, what we do, we wanted to avoid this part. So, instead of this, we can take the limiting value and as a corollary, we can say of this result is let x, which is x n be a sequence in R, and suppose that the limit of this x n mod x n to the power 1 by n as n tends to infinity exists, and equal to R exist, then the series sigma x n is absolutely convergent, when R is strictly less than 1 and is divergent, when r is greater than 1. So, for r is equal to 1, we can say anything about it, because if we take the sigma 1 by n, then r is 1 series diverges, if I take sigma 1 by n square then also r is 1, because a series converges. So, for r equal to 1 conclusion cannot be drawn is it. So, this was now this is next.

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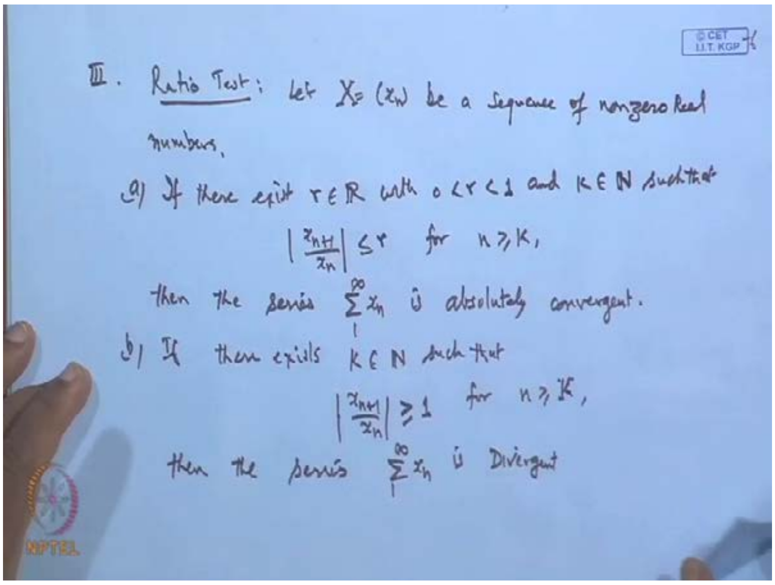

So, this is the root test, ratio test; third test is ratio test let X which is x n be a sequence of non zero real number, then the first says, if there exists an r, if there exists r belongs to the set of real number capital R with 0 less than r less than 1 and k belongs to n set of natural number n, such that mod of x n plus 1 by x n; this is less than equal to r for n greater than equal to k, then the series sigma x n 1 to infinity is absolutely convergent, and part b says, if there exists k belongs to the set of natural number n such that mod of x n plus 1 by x n is greater than or equal to 1 for n greater than equal to k onward, then series sigma x n 1 to infinity is divergent, again as a corollary of this result is the limiting value.

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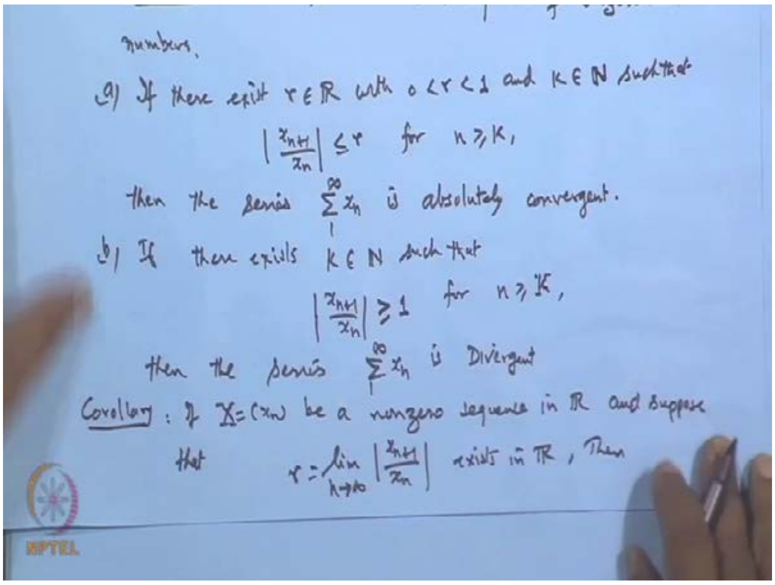

So, again we say the corollary to this is, let x n be a sequence of non zero terms, if x n be a non zero sequence of real numbers, and suppose that the limit exists, limit of mod x n plus 1 over x n as n tends to infinity exists and say R.

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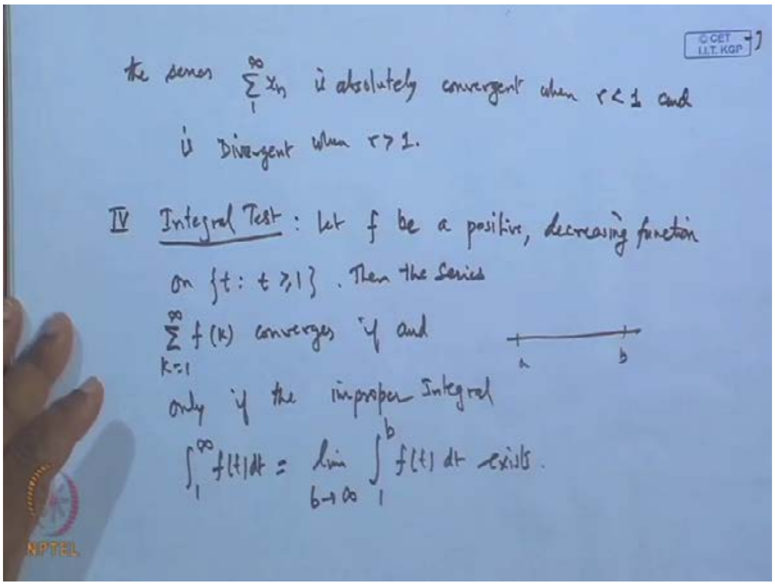

Then the series sigma x n 1 to infinity is absolutely convergent, when r is strictly less than 1, and is divergent, when r is strictly greater than 1, again for r is equal to 1 test fails. So, now, if we look this thing, then for r equal to 1, the testing fails now, there is another test, which is known as the integral test, and which is very powerful test, but of course it requires a knowledge of the Riemann integrable function, but here, we will assume f is an element in the Riemann integrable function; Riemann integrable function we mean, that suppose a b is an interval say here, when we divide this a b into the partition, and then finding the sum of this, then a to b f d can be written as the limit of this sum of this.

So, this we will discuss it, when we go for the Riemann integrable chapter or on Riemann integration, but let us assume that f is an element Riemann class of Riemann integrable functions, where the integral test. So, we are interested in the test right now. So, the third test is or forth is the integral test, which is very powerful test. So, here we assume, let f be a positive now, one more thing is concept of the improper integral. So, here let f is an element let us take, but let f be a positive decreasing function on the set t, where the t is greater than or equal to 1, then the series sigma f k, k is 1 to infinity converges, if and only if the improper integral 1 to infinity f t dt, if the improper integral which is the limit 1 to b f t dt b tends to infinity exist.

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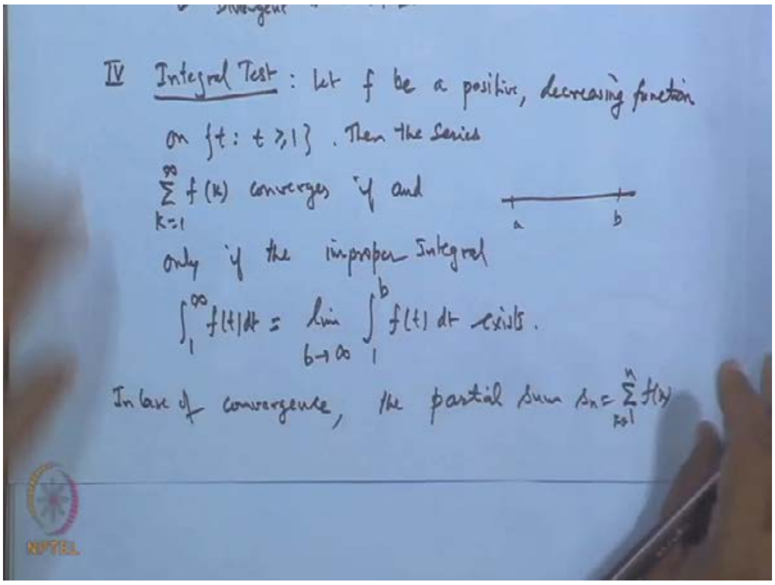

In case of the convergence, the partial sum s n of the series sigma f k, k is 1 to infinity partial sum is k is 1 to n f k and this is k.

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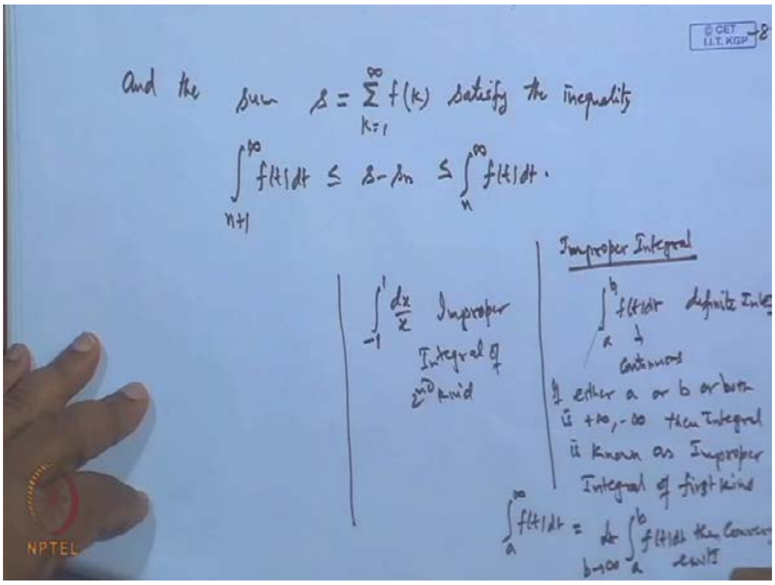

And the integral, and partial sum this and the sum s which is sigma f k k is one to infinity, satisfy the estimates the inequality integral n plus 1 to infinity f t dt f t dt which is less than equal to s minus s n, which is less than equal to n to infinity f t dt. So, what this result says is that, if f be a positive decreasing function on this interval, on this set t, where t is greater than equal to 1, the series sigma f k converges, if and only the corresponding improper integral exists.

So, basically the integral sets test, it connects the convergence of the series infinite series with the improper integral. So, here we are having the two terms one is the improper integrals and other one is the that of course, that Riemann integrable functions now, what is this improper integral, it will before going for the proof, let me see the first the improper integral; improper integral is if suppose the function f, which is defined over a certain interval say a b, then a to b f t dt, if f is a continuous function, then this integral is known as the definite integral, which we know definite integral is it not, if f is a continuous function, then a to b f t dt represent the definite integral known as the definite integral, and it represents the area bounded by the upper bounded by the function y equal to f t, and below y x axis and a to b to ordinates now.

If one of the limits is infinity or minus infinity, then such a integral, we call it as a improper integral with first kind, if either a or b or may be both, one of the limit either a or b is plus infinity minus infinity or may be both, then the integral is known as improper integral of first kind, now once we have one of the integral as improper integral, then that is integral a to infinity f t d t.

So, this is improper integral of the first kind now, whether this integral converges or diverges it depends on the limit of this; what we do is, we consider a to b f t dt and then take the limit b tends to infinity now, if this limit exists, then we say this improper integral converges now, if this limit does not exists, then we say this improper integral diverges similarly, minus infinity alpha, we can in a similar, we can say define the convergence or the divergence of this series, the improper integral of the second kind here, if a and b f t dt is given, but f is not continuous, suppose it has a point of discontinuity over the interval a b, then such a integral is not defined, basically over the whole interval a b say for example, if I take the improper a to, or 0 minus 1 to 1 d x by x now, this integral function f x is 1 by x is not defined at x equal to 0.

So, it is an improper integral of second kind clear. So, these 2 types of improper integral are; one is the improper integral of first kind, another is the improper integral of the second kind now, here what we are assuming is improper integral of the first kind. So, if a series is given, and the function is such, which is of decreasing nature over the interval 1 to infinity, then the nature of the series sigma f k; sigma f k converges is converge, if the corresponding improper integral will exist clear and of course vice versa. So, this relates the convergence of the series with the improper integrals, and that is why it is known as the integral test, it is very powerful test, one can drag the earlier test with the help of this integral test. So, let us see the proof of this result.

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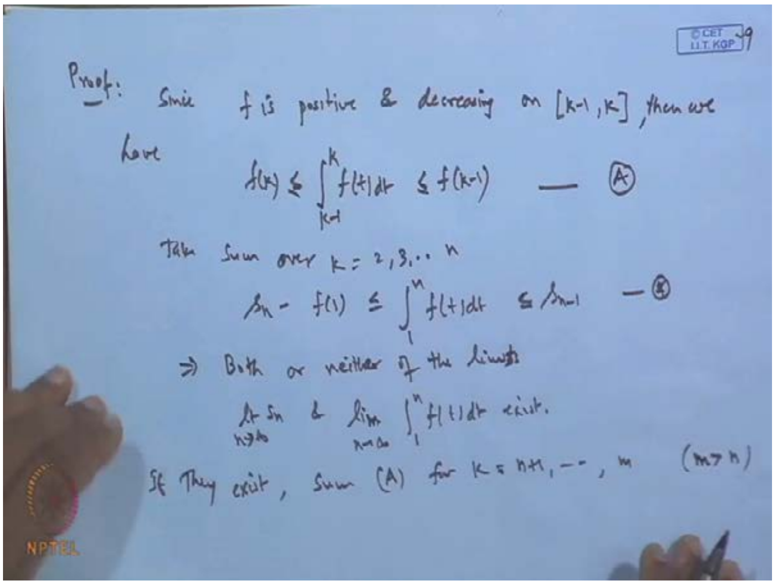

So, we go to the proof of this now, it is given that function is decreasing since, f is positive, and its of decreasing nature, and decreasing on the interval, over the interval t is greater than equal to over this sets. So, we can take the interval say k minus 1 k; this is interval function is decreasing over this interval k k minus 1, then we have then we have; obviously, this relation that f of k is less than equal to integral k minus 1 to k f t dt which is less than or equal to f of k minus 1, since the function is decreasing the largest value will be attended at this point k minus 1, and lowest value attended at the point f k, what is this value the area represented by there. So, basically the functional value between k minus 1 to k. So, this will be the portion in between these two now, let us take the sum. So, take the sum of this, take sum over k, when varies from 2, 3 say up to n, then what happen when you take the sum of this k, then it is sigma k equal to 1 to n is the s n.

So, we can say s n minus the first term f 1 (( )) and this is k equal to 2 3. So, 1 to n f t dt which is less than equal to, when you write the sum of this it is s n minus 1. So, we get this now, from here say star, what we say the nature of this, limit of this integral will depend on the limit of s n and vice versa, if the limit of s n exists, then the limit of this will also exists, if limit of this exists, limit of s n will also exists.

So, this shows that this implies both or neither of the limits exists both or neither of the limits s n as n tends to infinity, and limit as n tends to infinity one to n f t dt both will exist or both will not exist, that is one thing now, if they exist then add them. So, then add. So, this say this part from here again, let it be a then sum is a, for k is equal to n plus 1 to m, where m is greater than n sum of this. So, when you sum up this thing, what happen this is k equal to m? So, s m minus n

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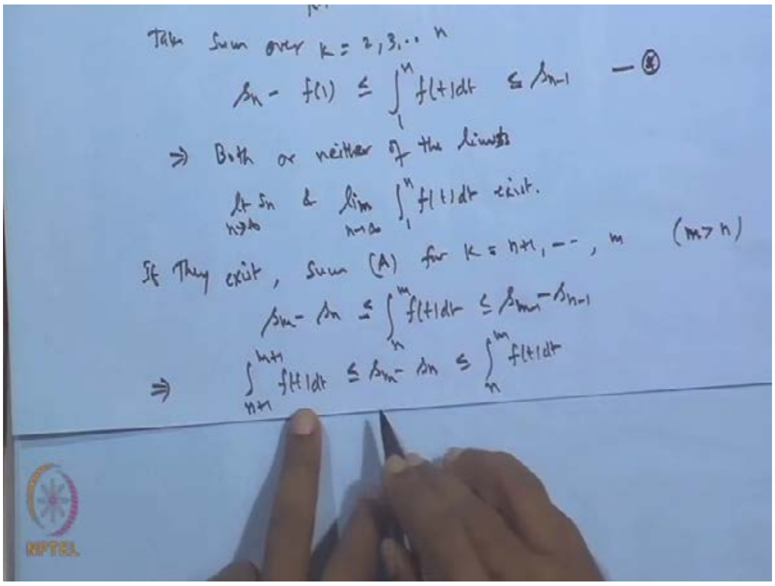

So, this will give s m minus s n, which is less than equal to n to m, and f t dt which is less than equal to again, when you sum up this thing, that will be equal to s m minus 1 minus s n minus 1. So, that will be there now from here. So, this imply that now, if I take m integral of this, say suppose I replace n by n plus 1 m by m plus 1, then what we get n plus 1 to m plus 1 f t dt this is nothing but what, when you write m equal to m plus 1 n equal to n plus 1 this is less than equal to s m minus s n, but s m minus s n is also less than equal to this integral n m f t dt.

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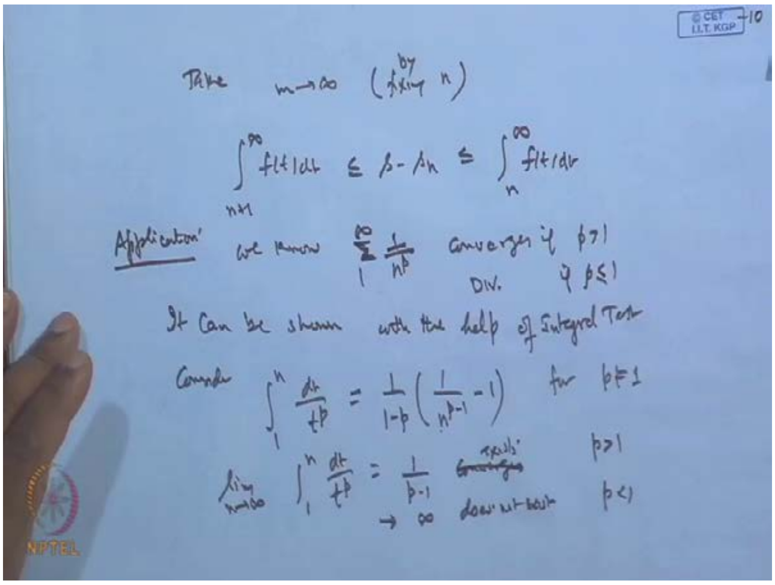

Now, clear now, from here take the limit as n tends to infinity. So, take the limit when m tends to infinity fix n by fixing n; n we are not touching then what happens, when are taking the limit, you are getting n plus 1 to infinity f t dt is less than, when m is sufficiently this will go to s. So, s minus s n this is less than equal to n to infinity f t dt, and that is what the result says the result is this is it not. So, we got the result for. So, this proves that now, let us see application part, suppose I take how to apply of this result let us take this series, we know the series sigma 1 by n to the power p 1 to infinity converges, if p is strictly is less than 1 diverges, if p is less than or equal to 1 because equal to 1 is a harmonic series now, this can be proved, it can be shown with the help of integral test, how let us consider that, what the integral test says, if you look the integral test, the integral test is this, that if you want to find the nature of this write down the corresponding integral f t; f t and then the limit of this exist the corresponding integral will converge. So, this will be taken like this.

So, consider integral 1 to n dt over t to the power p, consider this integral, because we want the sigma of this. So, f t is 1 by t. So, 1 by n p, and then this will give 1 minus 1 by p 1 over n to the power p minus 1 minus 1 for p different from 1 now, if p is greater than 1 if p is strictly greater than 1, what happens, the limit of this as n tends to infinity 1 to n dt over t to the power p, this will, when p is greater than 1, this will remain positive power is positive go to 0, and we are getting basically 1 over p minus 1 a definite number. So, it converges exist finish of this, just exists if p is less than 1, this will go in the numerator.

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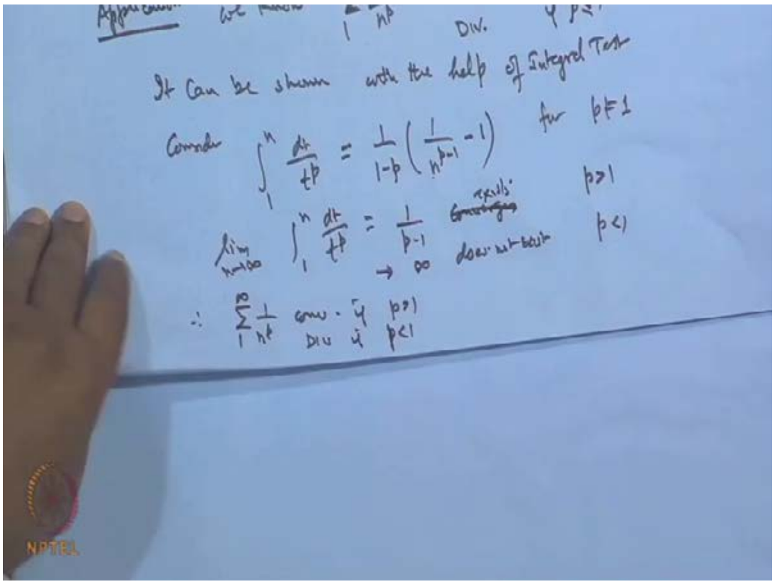

So, it will diverge. So, it will go to infinity, that is does not exist, it means the corresponding series; therefore the series sigma 1 by t to the power p instead of t, we can take n to the power p n is 1 to infinity converges, if p is greater than 1 diverges, if p is less than 1, and for p equal to 1; obviously, it is a harmonic series, so we can get this result. So, that is what, we are getting. Now, there is few more tests which are known as the Raabe’s test, and of course Raabe’s test we will take up in next time.

Then thank you very much.

Thanks.